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Kazım İlarslan, Emilija Nešović

SOME CHARACTERIZATIONS OF OSCULATING CURVES IN THE EUCLIDEAN SPACES

Abstract. In this paper, we give some characterization for a osculating curve in 3-dimensional Euclidean space and we define a osculating curve in the Euclidean 4-space as a curve whose position vector always lies in orthogonal complement $B_1^{\perp}$ of its first binormal vector field B_1. In particular, we study the osculating curves in $\mathbb{E}^4$ and characterize such curves in terms of their curvature functions.

1. Introduction

In the Euclidean space $\mathbb{E}^3$, it is well-known that to each unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ with at least four countinuous derivatives, one can associate three mutually orthogonal unit vector fields T, N and B called respectively the tangent, the principal normal and the binormal vector fields. At each point $\alpha(s)$ of the curve α, the planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known respectively as the osculating plane, the rectifying plane and the normal plane. The curves $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ for which the position vector α always lie in their rectifying plane, are for simplicity called *rectifying curves.* Similarly, the curves for which the position vector always lie in their normal plane, are for simplicity called *normal curves* and finally, the curves for which the position vector α always lie in their osculating plane, are for simplicity called *osculating curves.* By definition, for a rectifying curve, normal curve and osculating curve the position vector α satisfies respectively:

$$\alpha(s) = a_1(s)T(s) + a_2(s)B(s), \tag{1}$$

$$\alpha(s) = b_1(s)N(s) + b_2(s)B(s), \tag{2}$$

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$$\alpha(s) = c_1(s)T(s) + c_2(s)N(s), \tag{3}$$

for some differentiable functions $a_1, a_2, b_1, b_2, c_1, c_2$ of $s \in I \subset \mathbb{R}$.

In the Euclidean 3-space, the rectifying curves are introduced by B. Y. Chen in [1]. The Euclidean rectifying curves are studied in [1, 2]. In particular, it is shown in [2] that there exist a simple relationship between the rectifying curves and the centrodes, which play some important roles in mechanics, kinematics as well as in differential geometry in defining the curves of constant precession.

It is well known that the only normal curves in $\mathbb{E}^3$ are spherical curves (for spherical curves see [6, 7, 8]).

For unit speed plane curves in 2-dimensional Euclidean space, it is well known that the second curvature is $k_2 = 0$. In this case, the first curvature k_1 play an important role for characterization of the curve: if $k_1 = 0$, then the curve is a straight line, if $k_1 = \text{constant} \neq 0$, then the curve is a circle (or a part of the circle) with the radius $r = 1/k_1$ (see [4, 5]).

The following characterizations of circles and straight lines are well-known.

THEOREM 1. *A unit speed plane curve $x(s) : \mathbb{R} \to \mathbb{R}^2$ satisfies $\langle x(s), N(s)\rangle = b$, $(b \in \mathbb{R})$, where $N(s)$ is the unit normal vector, if and only if $x(s)$ is a part of a circle centered at origin or a straight line.*

THEOREM 2. *A unit speed plane curve $x(s) : \mathbb{R} \to \mathbb{R}^2$ defined on the whole line $\mathbb{R}$ satisfies*

$$\langle x(s), T(s) - a\rangle = b, \qquad a \in \mathbb{R}^2, \quad b \in \mathbb{R} \tag{4}$$

if and only if $x(s)$ is a circle or a straight line.

EXAMPLE. Let $x(s)$ be a circle centered at $\beta = (\beta_1, \beta_2)$ of radius ρ with curvature $k_1 = 1/\rho$. Then it is easy to show that $x(s)$ satisfies (4) for $a = (1/\rho)(\beta_2, -\beta_1)$ and $b = 0$. Obviously, each straight line satisfies (4).

In this paper, we give some characterizations for osculating curves in the Euclidean space $\mathbb{E}^3$, then we define the osculating curve in the Euclidean space $\mathbb{E}^4$ as a curve whose position vector always lies in the orthogonal complement $B_1^{\perp}$ of its first binormal vector field B_1. Consequently, $B_1^{\perp}$ is given by

$$B_1^{\perp} = \{W \in \mathbb{E}^4 \mid \langle W, B_1\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{E}^4$. Hence $B_1^{\perp}$ is a 3-dimensional subspace of $\mathbb{E}^4$, spanned by the tangent, the principal normal, and the second binormal vector fields T, N and B_2 respectively. Therefore, the position vector with respect to some chosen origin, of a osculating curve

α in $\mathbb{E}^4$, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)N(s) + \nu(s)B_2(s), \tag{5}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ in arclength function s. Next, we characterize osculating curves in terms of their curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ and give the necessary and the sufficient conditions for arbitrary curve in $\mathbb{E}^4$ to be a osculating. Moreover, we obtain an explicit equation of a osculating curve in $\mathbb{E}^4$.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be arbitrary curve in the Euclidean space $\mathbb{E}^4$. Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $\langle \alpha'(s), \alpha'(s)\rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product of $\mathbb{E}^4$ given by

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

for each $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in \mathbb{E}^4$. In particular, the norm of a vector $X \in \mathbb{E}^4$ is given by $||X|| = \sqrt{\langle X, X\rangle}$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along the unit speed curve α, where T, N, B_1 and B_2 denote respectively the tangent, the principal normal, the first binormal and the second binormal vector fields. Then the Frenet formulas are given by (see [3, 4]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{6}$$

The functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ are called respectively the first, the second and the third curvature of the curve α. If $k_3(s) \neq 0$ for each $s \in I \subset \mathbb{R}$, the curve α lies fully in $\mathbb{E}^4$.

3. Osculating curves in $\mathbb{E}^3$

In 3-dimensional Euclidean space, the osculating curves, which their position vector satisfy the equation (3), we have the following well-known result.

THEOREM 3.1. *Let $\alpha(s)$ be a unit speed regular curve lying fully in $\mathbb{E}^3$. Then, α is a osculating curve, if and only if α is a straight line or a planar curve.*

THEOREM 3.2. *Let $\alpha(s)$ be a unit speed osculating curve lying fully in $\mathbb{E}^3$ then tangential component a (i.e., $\langle \alpha(s), T(s)\rangle = a$) and the principal normal component b (i.e., $\langle \alpha(s), N(s)\rangle = b$) of the position vector of the curve* *satisfy the following equation*

$$a^2(s) + b^2(s) = 2\int a(s)ds. \tag{7}$$

Conversely, if the tangential component a (i.e., $\langle \alpha(s), T(s)\rangle = a$) and the principal normal component b (i.e., $\langle \alpha(s), N(s)\rangle = b$) of the position vector of a unit speed curve $\alpha(s)$ in E^3 satisfy the equation (7) then α is a osculating curve, or a rectifying curve.

Proof. The first part of the proof is clear from Theorem 3.1.

We assume that the tangential component a (i.e., $\langle \alpha(s), T(s)\rangle = a$) and the principal normal component b (i.e., $\langle \alpha(s), N(s)\rangle = b$) of the position vector of a unit speed curve $\alpha(s)$ in E^3 satisfy the equation (7). Then we get from (7)

$$aa' + bb' = a, \tag{8}$$

where $a' = \frac{d}{ds}\langle \alpha(s), T(s)\rangle$ and $b' = \frac{d}{ds}\langle \alpha(s), N(s)\rangle$. By using (6), we obtain from (8)

$$k_2\langle \alpha(s), N(s)\rangle\langle \alpha(s), B(s)\rangle = 0. \tag{9}$$

From (9), we have $k_2 = 0$, or $\langle \alpha(s), N(s)\rangle = 0$, or $\langle \alpha(s), B(s)\rangle = 0$. If $k_2 = 0$, which means that $\alpha(s)$ is a planar curve. According to the theorem 3.1., it is a osculating curve. If $\langle \alpha(s), N(s)\rangle = 0$, then $\alpha(s)$ is a rectifying curve(i.e., the position vector always lies in its rectifying plane). If $\langle \alpha(s), B(s)\rangle = 0$, then $\alpha(s)$ is a osculating curve. ∎

From Theorem 3.2., we get the following corollary.

COROLLARY 3.1. *Let α be a osculating curve lying fully in $\mathbb{E}^3$ with the tangential component a (i.e., $\langle \alpha(s), T(s)\rangle = a$) and the principal normal component b (i.e., $\langle \alpha(s), N(s)\rangle = b$).*

(i) *if tangential component a is zero, then α is a circle,*

(ii) *if principal normal component b is zero, then α is a circle or a straight line.*

4. Osculating curves in $\mathbb{E}^4$

In this section, we firstly characterize the osculating curves in $\mathbb{E}^4$ in terms of their curvatures. Let $\alpha = \alpha(s)$ be a unit speed osculating curve in $\mathbb{E}^4$, with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. By definition, the position vector of the curve α satisfies the equation (5), for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$. Differentiating the equation (5) with respect to s and using the Frenet equations (6), we obtain

$$T = (\lambda' - \mu k_1)T + (\lambda k_1 + \mu')N + (\mu k_2 - \nu k_3)B_1 + \nu' B_2.$$

It follows that

$$(10)\qquad \begin{aligned} \lambda' - \mu k_1 &= 1, \\ \lambda k_1 + \mu' &= 0, \\ \mu k_2 - \nu k_3 &= 0, \\ \nu' &= 0, \end{aligned}$$

and therefore

$$(11)\qquad \begin{aligned} \lambda(s) &= -c\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)', \\ \mu(s) &= c\frac{k_3}{k_2}, \\ \nu(s) &= c, \end{aligned}$$

where $c \in \mathbb{R}_0$. In this way, the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are expressed in terms of the curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ of the curve α. Moreover, by using the first equation in (10) and relation (11), we easily find that the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ satisfy the equation

$$(12)\qquad \left(\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right)' + \frac{k_1 k_3}{k_2} = -1/c, \qquad c \in \mathbb{R}_0.$$

Conversely, assume that the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$, of an arbitrary unit speed curve α in $\mathbb{E}^4$, satisfy the equation (12). Let us consider the vector $X \in \mathbb{E}^4$ given by

$$X(s) = \alpha(s) + c\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)' T(s) - c\frac{k_3}{k_2}N(s) - cB_2(s).$$

By using the relations (6) and (12), we easily find $X'(s) = 0$, which means that X is a constant vector. This implies that α is congruent to a osculating curve. In this way, the following theorem is proved.

THEOREM 4.1. *Let $\alpha(s)$ be unit speed curve in $\mathbb{E}^4$, with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Then α is congruent to a osculating curve if and only if*

$$\left(\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right)' + \frac{k_1 k_3}{k_2} = -1/c, \qquad c \in \mathbb{R}_0.$$

Recall that arbitrary curve α in $\mathbb{E}^4$ is called a W-curve if it has constant curvature functions (see [4]). It is clear that such curves holds Theorem 4.1.

The following theorem gives the characterization of a W-curves in $\mathbb{E}^4$, in terms of osculating curves.

THEOREM 4.2. *Every unit speed W-curve, with non-zero curvatures k_1, k_2 and k_3 in $\mathbb{E}^4$, is congruent to a osculating curve.*

Proof. It is clear from Theorem 4.1. ∎

EXAMPLE.

$$\alpha(s) = \left(a\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), a\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right),\right.$$
$$\left. b\cos\left(\frac{1}{\sqrt{a^2r^2+b^2}}s\right), b\sin\left(\frac{1}{\sqrt{a^2r^2+b^2}}s\right)\right)$$

is a unit speed curve in $\mathbb{E}^4$. It is easily obtain the Frenet vectors and curvatures as follows:

$$T(s) = \begin{pmatrix} \frac{-ar}{\sqrt{a^2r^2+b^2}}\sin(\frac{r}{\sqrt{a^2r^2+b^2}}s), \frac{ar}{\sqrt{a^2r^2+b^2}}\cos(\frac{r}{\sqrt{a^2r^2+b^2}}s), \\ \frac{-b}{\sqrt{a^2r^2+b^2}}\sin(\frac{r}{\sqrt{a^2r^2+b^2}}s), \frac{b}{\sqrt{a^2r^2+b^2}}\cos(\frac{r}{\sqrt{a^2r^2+b^2}}s) \end{pmatrix},$$

$$k_1(s) = \frac{\sqrt{a^2r^4+b^2}}{a^2r^2+b^2},$$

$$N(s) = \begin{pmatrix} \frac{-ar^2}{\sqrt{a^2r^4+b^2}}\cos(\frac{r}{\sqrt{a^2r^2+b^2}}s), \frac{-ar^2}{\sqrt{a^2r^4+b^2}}\sin(\frac{r}{\sqrt{a^2r^2+b^2}}s), \\ \frac{-b}{\sqrt{a^2r^4+b^2}}\cos(\frac{r}{\sqrt{a^2r^2+b^2}}s), \frac{-b}{\sqrt{a^2r^4+b^2}}\sin(\frac{r}{\sqrt{a^2r^2+b^2}}s) \end{pmatrix},$$

$$k_2(s) = \frac{abr(r^2-1)}{(a^2r^2+b^2)\sqrt{a^2r^4+b^2}}$$

$$B_1(s) = \begin{pmatrix} \frac{b}{\sqrt{a^2r^2+b^2}}\sin(\frac{r}{\sqrt{a^2r^2+b^2}}s), \frac{-b}{\sqrt{a^2r^2+b^2}}\cos(\frac{r}{\sqrt{a^2r^2+b^2}}s), \\ \frac{-ar}{\sqrt{a^2r^2+b^2}}\sin(\frac{r}{\sqrt{a^2r^2+b^2}}s), \frac{ar}{\sqrt{a^2r^2+b^2}}\cos(\frac{r}{\sqrt{a^2r^2+b^2}}s) \end{pmatrix}$$

$$k_3(s) = \frac{r}{\sqrt{a^2r^4+b^2}},$$

$$B_2(s) = \begin{pmatrix} \frac{b}{\sqrt{a^2r^4+b^2}}\cos(\frac{r}{\sqrt{a^2r^2+b^2}}s), \frac{b}{\sqrt{a^2r^4+b^2}}\sin(\frac{r}{\sqrt{a^2r^2+b^2}}s), \\ \frac{-ar^2}{\sqrt{a^2r^4+b^2}}\cos(\frac{r}{\sqrt{a^2r^2+b^2}}s), \frac{-ar^2}{\sqrt{a^2r^4+b^2}}\sin(\frac{r}{\sqrt{a^2r^2+b^2}}s) \end{pmatrix}.$$

Since $k_1(s) =$ constant, $k_2(s) =$ constant and $k_3(s) =$ constant, $\alpha(s)$ is a W-curve in $\mathbb{E}^4$. According to Theorem 4.2, α is congruent to a osculating curve in $\mathbb{E}^4$.

Thus, we get the position vector of the curve given in example as follows: From equation (12) we get $c = \frac{-k_2}{k_1k_3}$ and $c = \frac{-ab(r^2-1)}{\sqrt{a^2r^4+b^2}}$. From equation (10), we find

$$\lambda(s) = -c\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)' = 0, \qquad \mu(s) = c\frac{k_3}{k_2} = \frac{-1}{k_1} = \frac{-(a^2r^2+b^2)}{\sqrt{a^2r^4+b^2}},$$
$$\nu(s) = c = \frac{-ab(r^2-1)}{\sqrt{a^2r^4+b^2}}.$$

From equation (5), we write the position vector of the curve as:

$$\alpha(s)=\frac{-(a^2r^2+b^2)}{\sqrt{a^2r^4+b^2}}N(s)+\frac{-ab(r^2-1)}{\sqrt{a^2r^4+b^2}}B_2(s).$$

THEOREM 4.3. *Let $\alpha(s)$ be unit speed osculating curve in $\mathbb{E}^4$, with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Then the following statements hold:*

(i) *The curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ satisfy the following equality*

$$\frac{k_3(s)}{k_2(s)}=\left(\frac{1}{c}\int\sin\left(\int k_1(s)ds\right)ds+c_1\right)\cos\left(\int k_1(s)ds\right)+\left(\frac{-1}{c}\int\cos\left(\int k_1(s)ds\right)ds+c_2\right)\sin\left(\int k_1(s)ds\right)\tag{13}$$

where $c\in\mathbb{R}_0$ and $c_1,c_2\in\mathbb{R}$.

(ii) *The tangential component and the principal normal component of the position vector of the curve are respectively given by*

$$\langle\alpha(s),T(s)\rangle=-c\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)',\qquad\langle\alpha(s),N(s)\rangle=c\frac{k_3}{k_2},\qquad c\in\mathbb{R}_0.\tag{14}$$

(iii) *The second binormal component of the position vector of the curve is non-zero constant.*

Conversely, if $\alpha(s)$ is a unit speed curve in $\mathbb{E}^4$ with non-zero curvatures $k_1(s)$, $k_2(s)$, $k_3(s)$ and one of the statements (i), (ii) or (iii) holds, then α is a osculating curve or congruent to a osculating curve.

Proof. Let us first suppose that $\alpha(s)$ is a unit speed osculating curve in $\mathbb{E}^4$ with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. The position vector of the curve α satisfies the equation (12).

$$\left(\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right)'+\frac{k_1k_3}{k_2}=-1/c,\quad c\in\mathbb{R}_0.$$

Here if we express $y(s)=\frac{k_3(s)}{k_2(s)}$ and $p(s)=\frac{1}{k_1(s)}$, then equation (12) can be written as

$$\frac{d}{ds}\left(p(s)\frac{dy}{ds}\right)+\frac{y(s)}{p(s)}=-1/c,\quad c\in\mathbb{R}_0.$$

If we change variables in above equation as $t=\int\frac{1}{p(s)}ds$, then we get

$$\frac{d^2y}{dt^2}+y=\frac{-1}{ck_1},\quad c\in\mathbb{R}_0,$$

the solution of this differential equation is

$$y=\left(\frac{1}{c}\int\frac{\sin t}{k_1}dt+c_1\right)\cos t+\left(\frac{-1}{c}\int\frac{\cos t}{k_1}dt+c_2\right)\sin t,$$

where $c \in \mathbb{R}_0, c_1, c_2 \in \mathbb{R}$. Here, if we put $y(s) = \frac{k_3(s)}{k_2(s)}$ and $dt = k_1 ds$, we get

$$\frac{k_3(s)}{k_2(s)} = \left(\frac{1}{c}\int \sin\left(\int k_1(s)ds\right)ds + c_1\right)\cos\left(\int k_1(s)ds\right) + \left(\frac{-1}{c}\int \cos\left(\int k_1(s)ds\right)ds + c_2\right)\sin\left(\int k_1(s)ds\right).$$

Thus we prove the statement (i).

By using the relations (5) and (11), we can write the position vector of the curve as follows:

$$\alpha(s) = -c\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)' T(s) + c\frac{k_3}{k_2}N(s) + cB_2(s). \tag{15}$$

From (15), we have $\langle\alpha(s), T(s)\rangle = -c\frac{1}{k_1}(\frac{k_3}{k_2})'$, $\langle\alpha(s), N(s)\rangle = c\frac{k_3}{k_2}$ and $\langle\alpha(s), B_2(s)\rangle = c$, $c \in \mathbb{R}_0$.

Thus we proved the statements (ii) and (iii).

Conversely, assume that statement (i) holds. Then the curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ satisfy the equality

$$\frac{k_3(s)}{k_2(s)} = \left(\frac{1}{c}\int \sin\left(\int k_1(s)ds\right)ds + c_1\right)\cos\left(\int k_1(s)ds\right) + \left(\frac{-1}{c}\int \cos\left(\int k_1(s)ds\right)ds + c_2\right)\sin\left(\int k_1(s)ds\right).$$

Differentiating the previous equation two times with respect to s we get,

$$\left(\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right)' + \frac{k_1 k_3}{k_2} = -1/c, \qquad c \in \mathbb{R}_0,$$

which means that according to the theorem 3.1. α is congruent to osculating curve. Next assume that statements (ii) holds. By taking derivative of $\langle\alpha(s), N(s)\rangle = c\frac{k_3}{k_2}$ with respect to s and using (6) we get,

$$-k_1\langle\alpha, T\rangle + k_2\langle\alpha, B_1\rangle = c\left(\frac{k_3}{k_2}\right)'.$$

By using $\langle\alpha(s), T(s)\rangle = -c\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'$ and $k_2 \neq 0$, we get $\langle\alpha, B_1\rangle = 0$, which means that α is a osculating curve.

If statement (iii) holds, then we have $< \alpha, B_2 >= c$, $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using (6), we find

$$-k_3\langle\alpha, B_1\rangle = 0.$$

It follows that $\langle\alpha, B_1\rangle = 0$ and hence the curve α is a osculating curve. This proves the theorem. ∎

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References

[1] B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Monthly **110** (2003), 147–152.
[2] B. Y. Chen, F. Dillen, *Rectifying curves as centrodes and extremal curves*, Bull. Inst. Math. Academia Sinica **33**, No. 2 (2005), 77–90.
[3] H. Gluck, *Higher curvatures of curves in Euclidean space*, Amer. Math. Monthly **73** (1966), 699–704.
[4] W. Kuhnel, *Differential Geometry: Curves-Surfaces-Manifolds*, Braunschweig, Wiesbaden 1999.
[5] R. S. Millman, G. D. Parker, *Elements of Differential Geometry*, Prentice-Hall, New Jersey 1977.
[6] D. J. Struik, *Differential Geometry*, second ed., Addison-Wesley, Reading, Massachusetts 1961.
[7] Y. C. Wong, *On a explicit characterization of spherical curves*, Proc. Amer. Math. Soc. **34** (1972), 239–242.
[8] Y. C. Wong, *A global formulation of the condition for a curve to lie in a sphere*, Monats. Math. **67** (1963), 363–365.

K. İlarslan
KIRIKKALE UNIVERSITY
FACULTY OF SCIENCES AND ARTS
DEPARTMENT OF MATHEMATICS
KIRIKKALE, TURKEY
e-mail: kilarslan@yahoo.com

E. Nešović
UNIVERSITY OF KRAGUJEVAC
FACULTY OF SCIENCE
KRAGUJEVAC, SERBIA
e-mail: emines@ptt.yu

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