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CR-SUBMANIFOLDS OF A NEARLY TRANS-HYPERBOLIC SASAKIAN MANIFOLD

Abstract. In the present paper we have investigated CR-submanifold of a nearly trans- hyperbolic Sasakian manifold. We have also studied parallel distribution relating to ξ-vertical CR-submanifold of a nearly trans- hyperbolic Sasakian manifold.

1. Introduction

In 1978, Bejancu introduced the notion of CR- submanifold of a Kaehler manifold [2]. Since then several papers on CR- submanifolds of Kaehler manifold have been published. On the other hand, CR- submanifold have been studied by Kobayashi[10], Shahid et al. [11, 12], Yano and Kon [14] and others. Upadhyay and Dube [13] have studied almost contact hyperbolic (f, g, η, ξ)-structure, Dube and Mishra [5] have considered Hypersurfaces immersed in an almost hyperbolic Hermitian manifold also Dube and Niwas [6] worked with almost r-contact hyperbolic structure in a product manifold. Later Bhatt and Dube [3] studied on CR-submanifolds of trans- hyperbolic Sasakian manifold. Joshi and Dube [9] studied on Semi-invariant submanifold of an almost r-contact hyperbolic metric manifold. Gill and Dube have also worked on CR submanifolds of trans-hyperbolic Sasakian manifolds [8]. Gherghe studied on harmonicity on nearly trans-Sasaki manifolds [7].

2. Preliminaries

Let $\overline{M}$ be an n dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ, ξ, η, g) where a tensor ϕ of type $(1,1)$, a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the following

$$(2.1) \qquad \phi^2 X = X - \eta(X)\xi \qquad g(X,\xi) = \eta(X)$$

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$$\eta(\xi) = -1 \qquad \phi(\xi) = 0 \qquad \eta \circ \phi = 0 \tag{2.2}$$

$$g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any X, Y tangent to $\overline{M}$ [4]. In this case

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.4}$$

An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called trans-hyperbolic Sasakian [8] if and only if

$$(\overline{\nabla}_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for all X, Y tangent to $\overline{M}$, α, β are functions on $\overline{M}$. On a trans-hyperbolic Sasakian manifold $\overline{M}$, we have

$$\overline{\nabla}_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi) \tag{2.6}$$

a Riemannian metric g and Riemannian connection $\overline{\nabla}$.

Further, an almost contact metric manifold $\overline{M}$ on (ϕ, ξ, η, g) is called nearly trans-hyperbolic Sasakian if [5]

$$\begin{aligned}(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y)\\ - \beta(\eta(X)\phi Y + \eta(Y)\phi X).\end{aligned} \tag{2.7}$$

Now, let M be a submanifold immersed in $\overline{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^\perp M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.8}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{2.9}$$

for any $X, Y \in TM$ and $V \in T^\perp M$, where $\nabla^\perp$ is the connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X,Y), V). \tag{2.10}$$

For any $x \in M$ and $X \in T_X M$, we write

$$X = PX + QX \tag{2.11}$$

where $PX \in D$ and $QX \in D^\perp$.

Similarly for N normal to M, we have

$$\phi N = BN + CN \tag{2.12}$$

where BN (resp. CN) is the tangential component (resp. normal component) of ϕN.

DEFINITION 1. An m dimensional Riemannian submanifold M of $\overline{M}$ is called a *CR-submanifold of* $\overline{M}$ if there exists a differentiable distribution $D : x \to D_X$ on M satisfying the following conditions:

(i) D is invariant, that is $\phi D_X \subset D_X$ for each $x \in M$,

(ii) The complementary orthogonal distribution $D^\perp : X \to D_X^\perp \subset T_X M$ of D is anti-invariant, that is, $\phi D_X^\perp \subset T_X^\perp M$ for each $x \in M$. If $\dim D_X^\perp = 0$ (resp., $\dim D_x = 0$), then the CR-submanifold is called an *invariant* (resp., *anti-invariant*) submanifold. The distribution D (resp., $D^\perp$) is called the *horizontal* (resp., *vertical*) *distribution.* Also, the pair $(D, D^\perp)$ is called ξ*-horizontal* (resp., *vertical*) if $\xi_X \in D_X$ (resp., $\xi_X \in D_X^\perp$).

3. Some basic lemmas

LEMMA 1. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$. Then*

$$\begin{aligned}(3.1)\quad & P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QX} Y) - P(A_{\phi QY} X) \\ &\quad = \phi P \nabla_X Y + \phi P \nabla_Y X + 2\alpha g(X,Y) P\xi \\ &\qquad - \alpha\eta(Y)\phi PX - \alpha\eta(X)\phi PY - \beta\eta(Y)\phi PX - \beta\eta(X)\phi PY,\end{aligned}$$

$$\begin{aligned}(3.2)\quad & Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QX} Y) - Q(A_{\phi QY} X) \\ &\quad = 2Bh(X,Y) + 2\alpha g(X,Y) Q\xi - \alpha\eta(Y)\phi QX - \alpha\eta(X)\phi QY\end{aligned}$$

$$\begin{aligned}(3.3)\quad & h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^\perp \phi QY + \nabla_y^\perp \phi QX \\ &\quad = \phi Q \nabla_Y X + \phi Q \nabla_X Y + 2\,\mathrm{Ch}(X,Y) - \beta\eta(Y)\phi QX - \beta\eta(X)\phi QY\end{aligned}$$

for $X, Y \in TM$.

Proof. Using (2.4), (2.5), and (2.6), in (2.7)we get

$$\begin{aligned}&\nabla_X \phi PY + h(X, \phi PY) - A_{\phi QY} X + \nabla_X^\perp \phi QY - \phi(\nabla_X Y + h(X,Y)) \\ &\quad + \nabla_Y \phi PX + h(Y, \phi PX) - A_{\phi QX} Y - \nabla_y^\perp \phi QX - \phi(\nabla_Y X) + h(X,Y) \\ &\qquad = \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X).\end{aligned}$$

Again using (2.11) we get

$$\begin{aligned}(3.4)\quad & P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QX} Y) - P(A_{\phi QY} X) - \phi P \nabla_X Y \\ &\quad - \phi P \nabla_Y X + Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QX} Y) \\ &\quad - Q(A_{\phi QY} X) - 2Bh(X,Y) + h(X, \phi PY) + h(Y, \phi PX) \\ &\quad + \nabla_X^\perp \phi QY + \nabla_y^\perp \phi QX - \phi Q \nabla_Y X - \phi Q \nabla_X Y - 2\,\mathrm{Ch}(X,Y)\end{aligned}$$

$$
\begin{aligned}
&=\beta\eta(Y)\phi QX - \beta\eta(X)\phi QY + 2\alpha g(X,Y)Q\xi - \alpha\eta(Y)\phi QX \\
&\quad - \alpha\eta(X)\phi QY + 2\alpha g(X,Y)P\xi - \alpha\eta(Y)\phi PX - \alpha\eta(X)\phi PY \\
&\quad - \beta\eta(Y)\phi PX - \beta\eta(X)\phi PY
\end{aligned}
$$

for any $X, Y \in TM$.

Now equating horizontal, vertical, and normal components in (3.4), we get the desired result.

LEMMA 2. *Let M be a CR-submanifold of a nearly trans- hyperbolic Sasakian manifold $\overline{M}$. Then*

$$
\begin{aligned}
(3.5)\quad 2(\overline{\nabla}_X\phi)(Y) &= \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] \\
&\quad + \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\}
\end{aligned}
$$

$$
\begin{aligned}
(3.6)\quad 2(\overline{\nabla}_Y\phi)(X) &= \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) \\
&\quad - \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) + \beta(\eta(X)\phi Y + \eta(Y)\phi X)
\end{aligned}
$$

for any $X, Y \in D$.

Proof. From Gauss formula (2.8), we have

$$
(3.7)\qquad \overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X).
$$

Also, we have

$$
(3.8)\qquad \overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)(Y) - (\overline{\nabla}_Y\phi)(X) + \phi[X,Y].
$$

From (3.6) and (3.7), we get

$$
\begin{aligned}
(3.9)\quad &(\overline{\nabla}_X\phi)(Y) - (\overline{\nabla}_Y\phi)(X) \\
&\qquad = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y].
\end{aligned}
$$

Also for nearly trans-hyperbolic Sasakian manifolds, we have

$$
\begin{aligned}
(3.10)\quad &(\overline{\nabla}_X\phi)(Y) + (\overline{\nabla}_Y\phi)(X) \\
&\qquad = \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X).
\end{aligned}
$$

Adding (3.9) and (3.10), we get

$$
\begin{aligned}
2(\overline{\nabla}_X\phi)(Y) &= \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] \\
&\quad + \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\}.
\end{aligned}
$$

Subtracting (3.9) from (3.10) we get

$$
\begin{aligned}
2(\overline{\nabla}_Y\phi)(X) &= \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) \\
&\quad - \nabla_X\phi Y - h(X,\phi Y) + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y].
\end{aligned}
$$

Hence Lemma is proved.

LEMMA 3. *Let M be a CR-submanifold of a nearly trans- hyperbolic Sasakian manifold $\overline{M}$, then*

$$2(\overline{\nabla}_Y\phi)(Z) = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z] + \alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\}$$

$$2(\overline{\nabla}_Z\phi)(Y) = \alpha(2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y) - A_{\phi Y}Z + A_{\phi Z}Y - \nabla_y^{\perp}\phi Z + \nabla_Z^{\perp}\phi Y - \phi[Y,Z]$$

for any $Y, Z \in D^{\perp}$.

Proof. From Weingarten formula (2.9), we have

$$\overline{\nabla}_Z\phi Y - \overline{\nabla}_Y\phi Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y. \tag{3.11}$$

Also, we have

$$\overline{\nabla}_Z\phi Y - \overline{\nabla}_Y\phi Z = (\overline{\nabla}_Y\phi)(Z) - (\overline{\nabla}_Z\phi)(Y) + \phi[Y,Z]. \tag{3.12}$$

From (3.11) and (3.12), we get

$$(\overline{\nabla}_Y\phi)(Z) - (\overline{\nabla}_Z\phi)(Y) = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_y^{\perp}Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z]. \tag{3.13}$$

Also for nearly trans-hyperbolic Sasakian manifolds, we have

$$(\overline{\nabla}_Y\phi)(Z) + (\overline{\nabla}_Z\phi)(Y) = \alpha(2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y). \tag{3.14}$$

Adding (3.13) and (3.14), we get

$$2(\overline{\nabla}_Y\phi)(Z) = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z] + \alpha(2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y).$$

Subtracting (3.13) from (3.14) we get

$$2(\overline{\nabla}_Z\phi)(Y) = \alpha(2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y) - A_{\phi Y}Z + A_{\phi Z}Y - \nabla_y^{\perp}\phi Z + \nabla_Z^{\perp}\phi Y + \phi[Y,Z].$$

This proves our assertions.

LEMMA 4. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$, then*

$$2(\overline{\nabla}_X\phi)(Y) = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} - A_{\phi Y}X + -\nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y],$$

$$2(\overline{\nabla}_Y\phi)(X) = \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) + A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y]$$

for any $X \in D$ and $Y \in D^{\perp}$.

Proof. By using Gauss equation and Weingarten equation for $X \in D$ and $Y \in D^{\perp}$ respectively we get

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y, \phi X). \tag{3.15}$$

Also, we have

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) + \phi[X, Y]. \tag{3.16}$$

From (3.6) and (3.7), we get

$$\begin{aligned}(\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) \\ = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y].\end{aligned} \tag{3.17}$$

Also for nearly trans-hyperbolic Sasakian manifolds, we have

$$\begin{aligned}(\overline{\nabla}_X \phi)(Y) + (\overline{\nabla}_Y \phi)(X) = \alpha(2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) \\ -\beta(\eta(X)\phi Y + \eta(Y)\phi X).\end{aligned} \tag{3.18}$$

Adding (3.17) and (3.18), we get

$$\begin{aligned}2(\overline{\nabla}_X \phi)(Y) = \alpha(2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) \\ - A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y].\end{aligned}$$

Subtracting (3.9) from (3.10) we get

$$\begin{aligned}2(\overline{\nabla}_Y \phi)(X) = \alpha(2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) \\ + A_{\phi Y} X - \nabla^{\perp}_X \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y].\end{aligned}$$

Hence Lemma is proved.

4. Parallel distributions

DEFINITION 2. The horizontal (resp., vertical) distribution D (resp., $D^{\perp}$)is said to be parallel [1] with respect to the connection on M if $\nabla_X Y \in D$ (resp., $\nabla_Z W \in D^{\perp}$) for any vector field $X, Y \in D$ (resp., $W, Z \in D^{\perp}$).

PROPOSITION 1. *Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$. If the horizontal distribution D is parallel, then*

$$h(X, \phi Y) = h(Y, \phi X) \tag{4.1}$$

for all $X, Y \in D$.

Proof. Using parallelism of horizontal distribution D, we have

$$\nabla_X \phi Y \in D, \ \nabla_Y \phi X \in D \quad \text{for any } X, Y \in D. \tag{4.2}$$

Thus using the fact that $X = QY = 0$ for $Y \in D$, (3.2) gives

$$Bh(X, Y) = g(X, Y)Q\xi \quad \text{for any } X, Y \in D. \tag{4.3}$$

Also, since

$$\phi h(X,Y) = Bh(X,Y) + \mathrm{Ch}(X,Y), \tag{4.4}$$

then

$$\phi h(X,Y) = g(X,Y)Q\xi + \mathrm{Ch}(X,Y) \quad \text{for any } X,Y \in D. \tag{4.5}$$

Next from (3.3), we have

$$h(X,\phi Y) + h(Y,\phi X) = 2\,\mathrm{Ch}(X,Y) = 2\phi h(X,Y) - 2g(X,Y)Q\xi, \tag{4.6}$$

for any $X, Y \in D$. Putting $X = \phi X \in D$ in (4.6), we get

$$h(\phi X,\phi Y) + h(Y,\phi^2 X) = 2\phi h(\phi X,Y) - 2g(\phi X,Y)Q\xi \tag{4.7}$$

or

$$h(\phi X,\phi Y) - h(Y,X) = 2\phi h(\phi X,Y) - 2g(\phi X,Y)Q\xi. \tag{4.8}$$

Similarly, putting $Y = \phi Y \in D$ in (4.6), we get

$$h(\phi Y,\phi X) - h(X,Y) = 2\phi h(X,\phi Y) - 2g(X,\phi Y)Q\xi. \tag{4.9}$$

Hence from (4.8) and (4.9), we have

$$\phi h(X,\phi Y) - \phi h(Y,\phi X) = g(X,\phi Y)Q\xi - g(\phi X,Y)Q\xi. \tag{4.10}$$

Operating ϕ on both sides of (4.10) and using $\phi\xi = 0$, we get

$$h(X,\phi Y) = h(Y,\phi X) \tag{4.11}$$

for all $X, Y \in D$.

Now, for the distribution $D^\perp$, we prove the following proposition.

PROPOSITION 2. *Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\overline{M}$. If the distribution $D^\perp$ is parallel with respect to the connection on M, then*

$$A_{\phi Y}Z + A_{\phi Z}Y \in D^\perp \quad \textit{for any } Y, Z \in D^\perp. \tag{4.12}$$

Proof. Let $Y, Z \in D^\perp$, then using Gauss and Weingarten formula (2.6), we obtain

$$\begin{aligned} -A_{\phi Z}Y + \nabla^\perp_Y \phi Z - A_{\phi Y}Z + \nabla^\perp_Z \phi Y &= \phi\nabla_Y Z + \phi\nabla_Z Y + 2\phi h(Y,Z) \\ &\quad + \alpha(2g(X,Y)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - \beta[\eta(Z)\phi Y + \eta(Y)\phi Z] \end{aligned} \tag{4.13}$$

for any $Y, Z \in D^\perp$. Taking inner product with $X \in D$ in (4.13), we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = g(\nabla_Y Z, \phi X) + g(\nabla_Z Y, \phi X). \tag{4.14}$$

If the distribution $D^\perp$ is parallel, then $\nabla_Y Z \in D^\perp$ and $\nabla_Z Y \in D^\perp$, for any $Y, Z \in D^\perp$.

So from (4.14) we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = 0 \quad \text{or} \quad g(A_{\phi Y}Z + A_{\phi Z}Y, X) = 0 \tag{4.15}$$

which is equivalent to

(4.16) $$A_{\phi Y}Z + A_{\phi Z}Y \in D^{\perp} \quad \text{for any } Y, Z \in D^{\perp},$$

and this completes the proof.

DEFINITION 3. A CR-submanifold is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X \in D$ and $Z \in D^{\perp}$.

The following lemma is an easy consequence of (2.10).

LEMMA 5. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.*

DEFINITION 4. A normal vector field $N \neq 0$ is called D-parallel normal section if $\nabla^{\perp}_X N = 0$ for all $X \in D$.

PROPOSITION 3. *Let M be a mixed totally geodesic ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$. Then the normal section $N \in \phi D^{\perp}$ is D-parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.*

Proof. Let $N \in \phi D^{\perp}$. Then from (3.2) we have

(4.17) $$Q(\nabla_Y \phi X) = 0 \quad \text{for any } X \in D,\ Y \in D^{\perp}.$$

In particular, we have $Q(\nabla_Y X) = 0$. By using it in (3.3), we get

(4.18) $$\nabla^{\perp}_X \phi QY = \phi Q \nabla_X Y \quad \text{or} \quad \nabla^{\perp}_X N = -\phi Q \nabla_X \phi N.$$

Thus, if the normal section $N \neq 0$ is D-parallel, then using Definition 4 and (4.18), we get

(4.19) $$\phi Q(\nabla_X \phi N) = 0$$

which is equivalent to $\nabla_X \phi N \in D$ for all X $\in D$. The converse part easily follows from (4.18). This completes the proof of the proposition.

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