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COMPLETE LIFT OF $F_a(K,1)$ STRUCTURE IN THE TANGENT BUNDLE

Abstract. Prasad and Gupta have obtained the integrability conditions of a-structure. In the present paper I have studied the complete lift of - structure in the tangent bundle.

1. Introduction

Let be an M^n n-dimensional differentiable manifold of class C^∞ equipped with a non null tensor field F of type $(1,1)$ and of class satisfying

(1) $$F^K - a^2F = 0$$

where K is a positive integer $\geqslant 2$ and a is complex number not equal to zero. Let us define the operators l and m as follows

(2) $$l = \frac{F^{K-1}}{a^2}, \quad m = I - \frac{F^{K-1}}{a^2},$$

where I denote the unit tensor. Then we have [1]

(3) $$l + m = I, \quad lm = ml, \quad l^2 = l.$$

Thus if there is given a $(1,1)$ tensor field $F \neq 0$ satisfying (1), then there exist two complementary distributions L and M corresponding to the projection operators l and m respectively. When the rank of F is constant and is equal to r everywhere then the dimensions of L and M are r and $(n-r)$ respectively. Such a structure is called $F_a\,(K,1)$ a - structure of rank r and the manifold M^n is called a structure manifold [1].

The relations [1]

(4) $$\begin{gathered} Fl = lF = F, \quad Fm = mF = 0, \\ F^2l = lF^2 = F^2,\ F^2m = mF^2 = 0, \\ lF^{K-1} = F^{K-1}l = a^2l \\ mF^{K-1} = F^{K-1}m = 0 \end{gathered}$$

show that F^{K-1} acts on L as GF-structure operator and on M as a null operator. If the rank of F is maximal that is $r = n$ then F satisfies

$$I - \frac{F^{K-1}}{a^2} = 0$$

or

$$F^{K-1} - a^2 I = 0.$$

It is proved that $F_a\,(K, 1)$ structure of maximal rank is a GF-structure.

2. Complete lift of $F_a(K, 1)$ structure in tangent bundle

Let $T_s^r(M^n)$ be the set of tensor fields of class C^∞ and of the type (r, s) in M^n and $T(M^n)$ be the tangent bundle over M. Let $F\epsilon(M^n)$ and F have local components F_i^h in a coordinate neighborhood U of M^n. Then the complete lift F^C of F will have the components of the form [2].

$$F^C : \begin{bmatrix} F_i^h & 0 \\ \delta F_i^h & F_i^h \end{bmatrix}. \tag{5}$$

Now we shall prove the following theorem.

THEOREM 2.1. *Let $F\epsilon(M^n)$ then the complete lift F^C of F is a $F_a\,(K, 1)$ structure in $T(M^n)$ if and only if so is F. Thus F is of rank r if and only if F^C is of rank $2r$.*

Proof. For any $F, G \in (M^n)$, we have [2]

$$(FG)^C = F^C G^C. \tag{6}$$

Replacing G by F in (6), we get

$$(FF)^C = (F^2)^C = (F^C)^2. \tag{7}$$

Since G is a $(1, 1)$ tensor field, therefore as a consequence of equation (6), we have

$$(FF^2)^C = F^C(F^2)^C = F^C(F^C)^2$$

which in view of equation (7) gives

$$(F^3)^C = (F^C)^3. \tag{8}$$

Again repeating the same process, we get

$$(F^5)^C = (F^C)^5 = (F^K)^C = (F^C)^K. \tag{9}$$

Taking complete lift of both sides of equation (1), we get

$$(F^K)^C - (a^2 F)^C = 0$$

which with the help of equation (9) implies that

$$(F^C)^K - a^2 F^C = 0.$$

Thus $F^K - a^2F = 0$ and $(F^C)^K - a^2F^C = 0$, are equivalent. The remaining part of the theorem follows from equation (5).

Let F be a $F_a\,(K,1)$ structure of rank r in M^n. Then the complete lift l^c of l and m^c of m are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions L^C and M^C determined by l^c and m^c respectively.

3. Integrability conditions of $F_a\,(K,1)$ -structure in tangent bundle

Let F be a $F_a\,(K,1)$ -structure of rank r in M^n. Then the Nijenhuis tensor $N(X,Y)$ of F is given in [2]

(10) $$N(X,Y) = [FX,FY] - F[FX,FY] - F[X,FY] + F^2[X,Y],$$

for any, $X, Y\epsilon(M^n)$ and $F\epsilon(M^n)$. We have [2]

(11) $$F^CX^C = (FX)^C$$
$$[X^C,Y^C] = [X,Y]^C$$

and

$$(X+Y)^C = X^C + Y^C.$$

From equation (4) and equation (11), we obtain

(12) $$F^Cm^C = (Fm)^C = 0.$$

THEOREM 3.1. *The complete lift M^C of a distribution M in M^n is integrable if and only if M is integrable in M^n.*

Proof. The distribution M is integrable if and only if [1]

(13) $$l[mX,mY] = 0,$$

for any $X, Y\epsilon(M^n)$. Taking complete lift of both sides (13), we get

(14) $$l^C[m^CX, m^CY^C] = 0,$$

where $l^C = (I-m)^C = I - m^C$, is the projection tensor complementary m^C. Thus the condition (13) and (14) are equivalent. Hence the result follows.

THEOREM 3.2. *For any $X, Y\epsilon(M^n)$ let the distribution M be integrable in M^n, that is $N(mX,mY) = 0$, then the distribution M^C is integrable in $T(M^n)$ if and only if*

$$N^C(m^CX^C, m^CY^C) = 0.$$

Proof. Let N^C be the Nijenhuis tensor of F^C in $T(M^n)$ of F in M^n. Then, we have

(15) $$N^C(X^C,Y^C) = [F^CX^C,F^CY^C] - F^C[F^CX^C,F^CY^C] - F^C[X^C,F^CY^C] + (F^2)^C[X^C,Y^C],$$

$$(16)\quad N^C(m^CX^C, m^CY^C) = [F^Cm^CX^C, F^Cm^CY^C] - F^C[F^Cm^CX^C, F^Cm^CY^C] - F^C[m^CX^C, F^Cm^CY^C] + (F^2)^C[m^CX^C, m^CY^C].$$

Equation (16) with the help of equation (12) gives

$$N^C(m^CX^C, m^CY^C) = (F^2)^C[m^CX^C, m^CY^C].$$

This in view of equation (4) gives

$$N^C(m^CX^C, m^CY^C) = (F^2l)^C[m^CX^C, m^CY^C]$$

or

$$N^C(m^CX^C, m^CY^C) = (F^2)^Cl^C[m^CX^C, m^CY^C].$$

Using the equation (14), we have

$$N^C(m^CX^C, m^CY^C) = 0.$$

This proves the theorem.

THEOREM 3.3. *For any $X, Y \epsilon (M^n)$, let the distribution L be integrable in M^n, that is $mN(X, Y) = 0$. Then the distribution L^C is integrable in $T(M^n)$, if and only if*

$$m^CN^C(X^C, Y^C) = 0.$$

Proof. In view of equations (15), (12) and (11), we obtain

$$m^CN^C(X^C, Y^C) = m^C[F^CX^C, F^CY^C] = m^C[FX, FY]^C = 0.$$

This completes the proof of the theorem.

When the distribution L is integrable, then F operates on each integrable manifold of L as a GF-structure operator F_* such as

$$F_*X_1 = FX_1,$$

where X_1 is an arbitrary vector field tangent to the integral manifold of L.

When the distribution L is integrable and the GF-structure F_* induced from F on each integral manifold of L is also integrable, then c-structure is said to be partially integrable.

THEOREM 3.4. *For any $X, Y \epsilon (M^n)$, let $F_a\,(K,1)$ -structure be partially integrable in M^n, that is $N(lX, lY) = 0$. Then $F_a\,(K,1)$ -structure is partially integrable in $T(M^n)$, if and only if*

$$N^C(l^CX^C, l^CY^C) = 0.$$

Proof. From equations (15) and (11), we have

$$N^C(l^CX^C,l^CY^C) = N^C(l^CX^C,l^CY^C) - F^C[F^Cl^CX^C,F^Cl^CY^C] - F^C[l^CX^C,F^Cl^CY^C] + (F^2)^C[l^CX^C,l^CY^C]$$
$$= [FlX,FlY]^C - F^C[FlX,Y]^C - F^C[X,FlY]^C + (F^2)^C[lX,lY]^C$$
$$= (N(lX,lY))^C = 0$$

since the $F_a\,(K,1)$ -structure is partially integrable in M^n.

When both distribution L and M are integrable, we can choose a local coordinate system such that all L are represented by putting $(n-r)$ local coordinate constant and all M by putting the other r coordinate system is called an adapted coordinate system. In an adapted coordinate system, the projection operators and have the components of the form [2]

$$l = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad m = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

respectively, where I_r is a unit matrix of order r and I_{n-r} is of order $n-r$. Since F satisfies equation (4), the tensor F has components of the form

$$F = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

in an adapted coordinate system, whose F_r is a square matrix of order $r \times r$.

Definition 3.1. The $F_a\,(K,1)$ structure is said to be integrable if

1. The $F_a\,(K,1)$ structure is partially integrable,
2. The distribution M is integrable,
3. The components of the $F_a\,(K,1)$ structure are independent of the coordinates, which are constant along the integral manifold of L in an adapted coordinate system.

Theorem 3.5. *For any vector fields X and Y, let the structure $F_a\,(K,1)$ be integrable in M^n, that is*

$$N(X,Y) = 0.$$

Then the structure $F_a\,(K,1)$ is integrable in $T(M^n)$ if and only if

$$N^C(X^C,Y^C) = 0.$$

Proof. From the equation (13), we have

$$N^C(X^C,Y^C) = [F^CX^C,F^CY^C] - F^C[F^CX^C,F^CY^C] - F^C[X^C,F^CY^C] + (F^2)^C[X^C,Y^C].$$

In view of equations (11) and (10), we have

$$N^C(X^C,Y^C) = (N^C(X,Y))^C = 0.$$

since the structure $F_a\,(K,1)$ is integrable in M^n.

References

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