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A NOTE ON GÂTEAUX DIFFERENTIALS OF HERMITIAN ELEMENTS IN BANACH ALGEBRAS

Abstract. Let $\mathcal{A}$ be a complex unital Banach algebra with unit $\mathbf{1}$. If $a \in \mathcal{A}$ is hermitian then we show that

$$\|a\|^2 = \lim_{t \rightarrow 0+} t^{-1}(\|\mathbf{1} + ta^2\| - 1),$$

and we give a proof of an inequality due to J. Nieto.

1. Terminology and results

In this note let $\mathcal{A}$ be a complex Banach algebra, with a unit element $\mathbf{1}$ of norm one. The *spectrum* of $a \in \mathcal{A}$ is denoted by $\sigma(a)$, and $r(a)$ denotes its *spectral radius*. Let $\mathcal{A}'$ denote the dual space of $\mathcal{A}$. The set of *normalized states* on $\mathcal{A}$ is given by

$$D(\mathcal{A}; \mathbf{1}) = \{\varphi \in \mathcal{A}' : \varphi(\mathbf{1}) = 1 = \|\varphi\|\}.$$

The *numerical range* of $a \in \mathcal{A}$ is defined by

$$V(\mathcal{A}; a) = \{\varphi(a) : \varphi \in D(\mathcal{A}; \mathbf{1})\}.$$

The set $V(\mathcal{A}; a)$ is compact and convex. We say that $a \in \mathcal{A}$ is *hermitian* if $V(\mathcal{A}; a) \subseteq \mathbb{R}$. $H(\mathcal{A})$ denotes the set of all hermitian elements in $\mathcal{A}$.

The Gâteaux differentials $n_+[\cdot]$, $n_-[\cdot] : \mathcal{A} \rightarrow \mathbb{R}$ are defined by

$$n_+[a] = \lim_{t \rightarrow 0+} \frac{\|\mathbf{1} + ta\| - 1}{t} \quad \text{and} \quad n_-[a] = \lim_{t \rightarrow 0-} \frac{\|\mathbf{1} + ta\| - 1}{t}.$$

The main results of this note read as follows:

THEOREM 1.1. *Given $a, b \in H(\mathcal{A})$ and $\varphi \in D(\mathcal{A}; \mathbf{1})$, then*

$$(\operatorname{Re} \varphi(ab))^2 \leq \operatorname{Re} \varphi(a^2) \operatorname{Re} \varphi(b^2).$$

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THEOREM 1.2. *If $a \in H(\mathcal{A})$, then*

$$n_+[a^2] = \|a\|^2.$$

REMARK. If $a \in H(\mathcal{A})$, then in general it does not follow that $a^2 \in H(\mathcal{A})$ (see [2, Example 6.1]).

2. Proofs

Note the following properties of n_+ and n_- : the function $a \mapsto n_+[a]$ is sublinear,

$$n_-[a] \leq n_+[a], \quad n_-[-a] = -n_+[a]$$

and

$$n_{\pm}[a + \lambda \mathbf{1}] = n_{\pm}[a] + \lambda$$

for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{R}$.

Furthermore we have

$$\sigma(a) \subseteq V(\mathcal{A}; a)$$

([2, Theorem 2.6]) and

$$(2.1) \quad \{\operatorname{Re} \lambda : \lambda \in V(\mathcal{A}; a)\} = [n_-[a], n_+[a]]$$

(see [2] or [3] or [4]).

PROPOSITION 2.1. *Given $a \in \mathcal{A}$, the following statements are equivalent:*

- (1) $a \in H(\mathcal{A})$;
- (2) $n_+[ia] = n_-[ia] = 0$;
- (3) $\|e^{ita}\| = 1$ for all $t \in \mathbb{R}$;
- (4) $\sigma(a) \subseteq \mathbb{R}$ and $r(\alpha \mathbf{1} + \beta a) = \|\alpha \mathbf{1} + \beta a\|$ ($\alpha, \beta \in \mathbb{C}$).

Proof. [2, Lemma 5.2] and [1]. ■

PROPOSITION 2.2.

- (1) *If $a, b \in H(\mathcal{A})$, then $i(ab - ba) \in H(\mathcal{A})$;*
- (2) *$H(\mathcal{A})$ is a closed real subspace of $\mathcal{A}$ and*

$$H(\mathcal{A}) \cap iH(\mathcal{A}) = \{0\};$$

- (3) *$V(\mathcal{A}; a) = \operatorname{co} \sigma(a)$ for all $a \in H(\mathcal{A})$ (where co denotes the convex hull).*

Proof. [2, Chapter 5]. ■

Note that by (2.1),

$$(2.2) \quad V(\mathcal{A}; a) = [n_-[a], n_+[a]] \quad \text{if } a \in H(\mathcal{A}).$$

PROPOSITION 2.3. *Let $a, b \in H(\mathcal{A})$, and $\varphi \in D(\mathcal{A}, 1)$. Then*

$$\operatorname{Re} \varphi(ab) = \operatorname{Re} \varphi(ba)$$

and

$$\operatorname{Re} \varphi(a^2) \geq 0,$$

thus $0 \leq n_-[a^2] \leq n_+[a^2]$.

Proof. Since $i(ab - ba) \in H(\mathcal{A})$,

$$i(\varphi(ab) - \varphi(ba)) = \varphi(i(ab - ba)) \in \mathbb{R},$$

hence $\operatorname{Re}(\varphi(ab) - \varphi(ba)) = 0$.

We have

$$\frac{a^2}{2} = \lim_{t \rightarrow 0} \frac{1 + ita - e^{ita}}{t^2}.$$

Since $\varphi(1) = 1$ and $\varphi(ta) \in \mathbb{R}$,

$$\operatorname{Re} \varphi\left(\frac{a^2}{2}\right) = \lim_{t \rightarrow 0} \frac{1 - \operatorname{Re} \varphi(e^{ita})}{t^2}.$$

We have $\operatorname{Re} \varphi(e^{ita}) \leq |\varphi(e^{ita})| \leq \|\varphi\| \|e^{ita}\| = 1$, therefore $\operatorname{Re} \varphi\left(\frac{a^2}{2}\right) \geq 0$. ■

Proof of Theorem 1.1. Let $t \in \mathbb{R}$. Then $a + tb \in H(\mathcal{A})$, thus, by Proposition 2.3, we have $\operatorname{Re} \varphi((a + tb)^2) \geq 0$ and $\operatorname{Re} \varphi(ab) = \operatorname{Re} \varphi(ba)$. Hence

$$0 \leq \operatorname{Re} \varphi(a^2) + 2t \operatorname{Re} \varphi(ab) + t^2 \operatorname{Re} \varphi(b^2) \quad (t \in \mathbb{R})$$

with $\alpha = \operatorname{Re} \varphi(a^2)$, $\beta = \operatorname{Re} \varphi(ab)$ and $\gamma = \operatorname{Re} \varphi(b^2)$ we have $0 \leq \alpha + 2\beta t + \gamma t^2$ for all $t \in \mathbb{R}$.

If $\gamma = 0$, then $\alpha + 2\beta t \geq 0$ for all $t \in \mathbb{R}$, hence $\beta = 0$, and the result follows. Now assume that $\gamma \neq 0$. Thus $\gamma = \operatorname{Re} \varphi(b^2) > 0$. Since the quadratic equation $0 = \alpha + 2\beta t + \gamma t^2$ has at most one solution, we get $\beta^2 \leq \alpha\gamma$. ■

Proof of Theorem 1.2. From Theorem 1.1 and (2.1) we see that

$$(n_+[ab])^2 \leq n_+[a^2]n_+[b^2] \quad (a, b \in H(\mathcal{A})).$$

Hence, with $b = 1$,

$$(2.3) \quad n_+[a]^2 \leq n_+[a^2].$$

The definition of $n_+[\cdot]$ gives $n_+[a^2] \leq \|a^2\|$. Proposition 2.1 (4) shows that $\|a^2\| = \|a\|^2$ hence

$$n_+[a^2] \leq \|a\|^2.$$

It remains to show that $\|a\|^2 \leq n_+[a^2]$.

Case 1: $n_+[a] \geq -n_-[a]$. By Proposition 2.2 (3),

$$n_+[a] = r(a) = \|a\|,$$

thus, by (2.3)

$$\|a\|^2 = n_+[a]^2 \leq n_+[a^2].$$

Case 2: $n_+[a] < -n_-[a]$. Let $b = -a$, then $b \in H(\mathcal{A})$ and

$$n_+[b] = -n_-[-b] = -n_-[a] > n_+[a] = -n_-[-a] = -n_-[b].$$

So Case 1 applies to b . Therefore

$$\|a\|^2 = \|b\|^2 = n_+[b^2] = n_+[a^2]. \blacksquare$$

In [5] the following subset of $\mathcal{A}$ is introduced:

$$\mathcal{A}_0 = \{x \in \mathcal{A} : n_+[x] = n_-[x]\}.$$

It is easy to see (use Proposition 2.1 (2)) that $x \in \mathcal{A}_0 \Leftrightarrow i(x - \alpha 1) \in H(\mathcal{A})$ for some $\alpha \in \mathbb{R}$.

In [5] J. Nieto has shown

$$(2.4) \quad \|x\| = r(x) \leq (2n_+[x]^2 + n_+[-x^2])^{1/2} \quad (x \in \mathcal{A}_0).$$

Now let $a \in H(\mathcal{A})$, then $ia \in \mathcal{A}_0$, hence $n_+[ia] = 0$ and $-(ia)^2 = a^2$, thus by (2.4),

$$\|a\|^2 \leq n_+[a^2].$$

Hence (2.4) implies Theorem 1.2. On the other hand Theorem 1.2 implies (2.4):

Let $x \in \mathcal{A}_0$, hence there is $\alpha \in \mathbb{R}$ such that $a := i(x - \alpha 1) \in H(\mathcal{A})$. By Proposition 2.1, $\|x\| = r(x)$. We have $-x^2 = -\alpha^2 1 + 2i\alpha a + a^2$, hence for $\varphi \in D(\mathcal{A}; 1)$

$$\operatorname{Re}\varphi(-x^2) = -\alpha^2 + 2\operatorname{Re}\varphi(i\alpha a) + \operatorname{Re}\varphi(a^2) = \operatorname{Re}\varphi(a^2) - \alpha^2.$$

Consequently

$$n_+[-x^2] = n_+[a^2] - \alpha^2 = \|a\|^2 - \alpha^2.$$

Since $n_+[x] = \alpha$, we have

$$(2.5) \quad 2n_+[x]^2 + n_+[-x^2] = \|a\|^2 + \alpha^2.$$

Now take $\lambda \in \sigma(x) = \sigma(\alpha 1 - ia)$. Then $\lambda = \alpha - i\beta$ with $\beta \in \sigma(a) \subseteq \mathbb{R}$. It follows that

$$|\lambda|^2 = \alpha^2 + \beta^2 \leq \alpha^2 + \|a\|^2,$$

therefore $r(x) \leq \alpha^2 + \|a\|^2$. Now use (2.5) to see that (2.4) holds.

References

- [1] M. Baillel, *Sur les éléments hermitiens d'une algèbre de Banach et les dilatations de certains opérateurs normaloides*, C. R. Acad. Sci., Paris, Ser. A **281** (1975), 1039–1042.
- [2] F. F. Bonsall, J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*. London, Cambridge University Press, 1971.
- [3] R. H. Martin, *Nonlinear operators and differential equations in Banach spaces*, Pure and Applied Mathematics. New York etc., John Wiley & Sons, 1976.
- [4] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Stud. Math. **4** (1933), 70–84.
- [5] J. Nieto, *Gâteaux Differentials in Banach Algebras*, Math. Z. **139** (1974), 23–34.

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