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GENERALIZED THREE-STEP ITERATION SCHEMES AND COMMON FIXED POINTS OF THREE ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

Abstract. In this paper, we study strong convergence theorems for a generalized three-step iterative scheme with errors to approximate common fixed points of three asymptotically quasi-nonexpansive mappings in real Banach spaces. Our results generalize and improve upon the corresponding results in [1], [2], [3], [4], [5], [6], [7], [9], [13] and [16]. As an application of our results, we give and prove strong convergence theorems in uniformly convex Banach spaces.

1. Introduction and preliminaries

Let C be a nonempty subset of a real Banach space E. A mapping $T : C \to C$ is called uniformly L-Lipschitzian if there exists a positive constant L such that $\|T^n x - T^n y\| \leq L\|x-y\|$, for all $x, y \in C$ and for all $n \geq 1$.

A mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - T^n y\| \leq (1+k_n)\|x-y\|$, for all $x, y \in C$ and for all $n \geq 1$.

Let $F(T)$ denote the set of all fixed points of a mapping T. If $F(T) \neq \emptyset$, then T is called asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - p\| \leq (1+k_n)\|x-p\|$, for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Clearly, an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive as well as uniformly L-Lipschitzian with the uniform Lipschitz constant $L = sup\{1+k_n : n \geq 1\}$, but the converse is not always true.

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In 1995, *Lui* [8] introduced the concept of *Ishikawa iteration process with errors* by the sequence $\{x_n\}_{n=1}^{\infty}$ defined as follows:

$$
\begin{aligned}
&x_1 \in X \\
(1.1)\qquad &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \qquad n = 1, 2, \dots \\
&y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n
\end{aligned}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1] and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in E satisfying the following conditions

$$(1.2)\qquad \sum_{n=0}^{\infty} \|u_n\| < \infty,$$

$$(1.3)\qquad \sum_{n=0}^{\infty} \|v_n\| < \infty.$$

If $\beta_n = 0$, $n \geq 0$ and $v_n = 0$, $n \geq 0$ then the Ishikawa iteration process with errors (1.1) reduces to the *Mann iteration procedure with errors in the sense of Liu* which is defined recursively as follows

$$
\begin{aligned}
&x_1 \in X \\
(1.4)\qquad &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \qquad n = 1, 2, \dots
\end{aligned}
$$

with $\{\alpha_n\} \subset [0,1]$ satisfying appropriate conditions and $\{u_n\}$ satisfying condition (1.2).

The Ishikawa iteration process with errors (1.1) with null sequences $\{u_n\}$ and $\{v_n\}$ clearly reduces to the usual Ishikawa iteration procedure and similarly the Mann iteration procedure with errors (1.4) with a null sequence $\{u_n\}$ reduces to the usual Mann iteration procedure.

A more satisfactory concept of Ishikawa and Mann iterative processes with errors was given by Y. G. Xu [15] as follows:

Let C be a nonempty convex subset of a Banach space E and $T : C \to C$ a mapping. The sequence $\{x_n\}_{n=1}^{\infty}$ defined iteratively by

$$
\begin{aligned}
&x_1 \in C \\
(1.5)\qquad &x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \qquad n \geq 1 \\
&y_n = \acute{\alpha}_n x_n + \acute{\beta}_n T x_n + \acute{\gamma}_n v_n
\end{aligned}
$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\acute{\alpha}_n\}$, $\{\acute{\beta}_n\}$ and $\{\acute{\gamma}_n\}$ are sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n = \acute{\alpha}_n + \acute{\beta}_n + \acute{\gamma}_n = 1$, $n \geq 1$ is called the *Ishikawa iteration sequence with errors.* If $\acute{\alpha}_n = 1$, $n \geq 1$ then the Ishikawa iteration with errors (1.5) reduces to the Mann iteration with errors defined by the following scheme

$$
\begin{aligned}
&x_1 \in C \\
(1.6)\qquad &x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \qquad n \geq 1
\end{aligned}
$$

with $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$, $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ and bounded sequences $\{u_n\} \subset C$ in C.

In 2004, H. Fukhar-ud-din and S. H. Khan [2] studied an iterative process with errors in the sense of Liu for two asymptotically nonexpansive mappings in a uniformly convex Banach space.

In 2006, J. U. Jeong and S. H. Kim [5] studied the Ishikawa iterative scheme with error members for a pair of asymptotically nonexpansive mappings S, T defined as follows:

$$
\begin{aligned}
&x_1 \in X \\
(1.7)\qquad &x_{n+1} = a_n S^n y_n + b_n x_n + c_n u_n, \qquad n \geq 1, \\
&y_n = \acute{a}_n T^n x_n + \acute{b}_n x_n + \acute{c}_n v_n,
\end{aligned}
$$

where $\{a_n\}$, $\{b_n\}$,$\{c_n\}$, $\{\acute{a}_n\}$, $\{\acute{b}_n\}$ and $\{\acute{c}_n\}$ are sequences in [0,1] with $0 < \delta \leq a_n, \acute{a}_n \leq (1-\delta) < 1$, $a_n + b_n + c_n = \acute{a}_n + \acute{b}_n + \acute{c}_n = 1$ and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C.

Recently, H. Fukhar-ud-din and S. H. Khan [3] studied the iterative scheme (1.7) for two asymptotically quasi-nonexpansive mappings in real Banach spaces.

Motivated and inspired by the previous studies, we introduce a new three-step iterative scheme $\{x_n\}_{n=1}^{\infty}$ associated with three asymptotically quasi-nonexpasnsive mappings $S, T, R : C \to C$ as follows:

$$
\begin{aligned}
&x_1 \in X \\
(1.8)\qquad &x_{n+1} = \alpha_n S^n y_n + \beta_n x_n + \gamma_n u_n, \qquad n = 1, 2, \ldots \\
&y_n = \lambda_n T^n z_n + \mu_n x_n + \nu_n v_n, \\
&z_n = \xi_n R^n x_n + \eta_n x_n + \zeta_n w_n,
\end{aligned}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ and $\{\zeta_n\}$ are sequences in [0,1] such that $\alpha_n+\beta_n+\gamma_n = 1$, $\lambda_n+\mu_n+\nu_n = 1$, $\xi_n+\eta_n+\zeta_n = 1$, $n \geq 1$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C.

If S, T and R are asymptotically nonexpansive mappings such that $S = T = R$, then the iterative procedure (1.8) reduces to the one introduced by Cho and Zhou [1]. If S, T and R are asymptotically nonexpansive mappings, $S = T = R$ and $u_n = v_n = w_n = 0$, $n \geq 1$, then scheme (1.8) reduces to the three-step iteration defined by Xu and Noor [16].

In this paper, we prove weak and strong convergence of the procedure (1.8) to a common fixed point of S,T and R under certain restrictions. Our results generalize and improve upon the corresponding results in [1], [2], [3], [4], [5], [6], [7], [9], [13] and [16].

In the sequal, we need of the following definitions and lemmas.

A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) (see [12]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T))),$$

for all $x \in C$, where

$$d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}.$$

Khan and Fukhar ud-din [7] modified condition (A) for two mappings as follows:

Two mappings $S, T : C \to C$ are said to satify condition $(\acute{A})$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, F)),$$

for all $x \in C$, where

$$d(x, F) = \inf\{\|x - x^*\| : x^* \in F\}, \quad F = F(S) \cap F(T).$$

Note that condition (A') reduces to condition (A) when $S = T$.

We modify the latter for three mappings $S, T, R : C \to C$ as follows:

Three mappings $S, T, R : C \to C$ are said to satisfy condition (A'') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that, for all $x \in C$, either $\|x - Sx\| \geq f(d(x, F))$ or $\|x - Tx\| \geq f(d(x, F))$ or $\|x - Rx\| \geq f(d(x, F))$, where,

$$d(x, F) = \inf\{\|x - x^*\| : x^* \in F\}, \quad F = F(S) \cap F(T) \cap F(R).$$

Remark 1.1. If $R = I$ or $R = S$ or $R = T$ then condition (A'') obviously reduces to condition (A') introduced by Khan and Fukhar ud-din [3] which makes condition (A'') more general than condition (A').

The following lemmas are useful in proving our main results.

Lemma 1.1 ([11]). *Let X be a uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, and $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq l$, $\limsup_{n\to\infty} \|y_n\| \leq l$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = l$ for some $l \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.2 ([14]). *Let $\{\omega_n\}$, $\{\rho_n\}$, and $\{\sigma_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$\omega_{n+1} \leq (1 + \rho_n)\omega_n + \sigma_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} \rho_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \omega_n$ exists.

2. Main results

Throughout this paper, F will denote the set of common fixed points of S, T and R, i.e, $F = F(S) \cap F(T) \cap F(R)$. We begin with giving the following proposition.

PROPOSITION 2.1. *If C is a nonempty convex subset of a real uniformly convex Banach space E and $S, T, R : C \to C$ are asymptotically quasi-nonexpansive mappings with sequences $\{l_n\}$, $\{l'_n\}$, $\{l''_n\}$ in $[0, \infty)$ such that $lim_{n\to\infty} l_n = lim_{n\to\infty} l'_n = lim_{n\to\infty} l''_n = 0$. Then S, T and R are asymptotically quasi-nonexpansive each with a sequence $\{k_n\}_{n=1}^{\infty}$ such that $k_n = \max\{l_n, l'_n, l''_n\}$.*

Proof. Clearly, $\{k_n\}_{n=1}^{\infty}$ is in $[0, \infty)$, furthermore $\lim_{n\to\infty} k_n = 0$. ∎

LEMMA 2.2. *Let C be a nonempty convex subset of a normed space E. Let $S, T, R : C \to C$ be asymptotically quasi-nonexpansive mappings such that for all $x \in C$, $p \in F$ and for all $n \geq 1$*

$$\|S^n x - p\| \leq (1 + l_n)\|x - p\|,$$
$$\|T^n x - p\| \leq (1 + l'_n)\|x - p\|,$$
$$\|R^n x - p\| \leq (1 + l''_n)\|x - p\|,$$

where $\{l_n\}, \{l'_n\}, \{l''_n\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} l_n < \infty$, $\sum_{n=1}^{\infty} l'_n < \infty$ and $\sum_{n=1}^{\infty} l''_n < \infty$. Let $\{x_n\}$ be the sequence defined in (1.8) with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \zeta_n < \infty$. If $F \neq \emptyset$ then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.

Proof. By Proposition 2.1, there exists a sequence $\{k_n\}_{n=1}^{\infty}$ in $[0, \infty)$ with $\sum_{n=1}^{\infty} k_n < \infty$ such that for all $x \in C$, $p \in F$ and all $n \geq 1$

$$\|S^n x - p\| \leq (1 + k_n)\|x - p\|,$$
$$\|T^n x - p\| \leq (1 + k_n)\|x - p\|,$$
$$\|R^n x - p\| \leq (1 + k_n)\|x - p\|.$$

Since $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are bounded sequences in C then there exists, $0 < M < \infty$, such that

$$M = \max\{\sup_{n\geq 1} \|u_n - p\|, \sup_{n\geq 1} \|v_n - p\|, \sup_{n\geq 1} \|w_n - p\|\}.$$

Now, for any $p \in F$ we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n S^n y_n + \beta_n x_n + \gamma_n u_n - p\| \\
&\leq \alpha_n \|S^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\leq (1 + k_n)\alpha_n \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n M \\
&= (1 + k_n)\alpha_n \|\lambda_n T^n z_n + \mu_n x_n + \nu_n v_n - p\| + \beta_n \|x_n - p\| + \gamma_n M
\end{aligned}$$

$$\begin{aligned}
&\leq (1+k_n)\alpha_n\lambda_n\|T^n z_n - p\| + (1+k_n)\alpha_n\mu_n\|x_n - p\| + \beta_n\|x_n - p\| \\
&\quad + (1+k_n)\alpha_n\nu_n\|v_n - p\| + \gamma_n M \\
&\leq (1+k_n)^2\alpha_n\lambda_n\|z_n - p\| + [(1+k_n)\alpha_n\mu_n + \beta_n]\|x_n - p\| \\
&\quad + (1+k_n)\alpha_n\nu_n M + \gamma_n M \\
&\leq (1+k_n)^2\alpha_n\lambda_n\xi_n\|R^n x_n - p\| \\
&\quad + (1+k_n)^2\alpha_n\lambda_n\zeta_n M + (1+k_n)\alpha_n\nu_n M \\
&\quad + \gamma_n M + [\beta_n + (1+k_n)\alpha_n\mu_n + (1+k_n)^2\alpha_n\lambda_n\eta_n]\|x_n - p\| \\
&\leq [\beta_n + (1+k_n)\alpha_n\mu_n + (1+k_n)^2\alpha_n\lambda_n\eta_n + (1+k_n)^3\alpha_n\lambda_n\xi_n]\|x_n - p\| \\
&\quad + (1+k_n)^2\alpha_n\lambda_n\zeta_n M + (1+k_n)\alpha_n\nu_n M + \gamma_n M \\
&\leq (\alpha_n\lambda_n\xi_n + \alpha_n\lambda_n\eta_n + \alpha_n\mu_n + \beta_n)(1+k_n)^3\|x_n - p\| \\
&\quad + [(1+k_n)^2\alpha_n\lambda_n\zeta_n + (1+k_n)\alpha_n\nu_n + \gamma_n]M \\
&\leq (1+k_n)^3\|x_n - p\| + [(1+k_n)^2\zeta_n + (1+k_n)\nu_n + \gamma_n]M.
\end{aligned}$$

Since $\{k_n\}$ is a bounded sequence, then there exists $h > 0$ such that $k_n \leq h$, $n \geq 1$. Therefore,

$$\|x_{n+1} - p\| \leq (1+k_n)^3\|x_n - p\| + [(1+h)^2\zeta_n + (1+h)\nu_n + \gamma_n]M.$$

Using that $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\nu_n < \infty$, $\sum_{n=1}^{\infty}\zeta_n < \infty$ and $\sum_{n=1}^{\infty}k_n < \infty$ and applying Lemma 1.2, we deduce that $\lim_{n\to\infty}\|x_{n+1} - p\|$ exists for all $p \in F$. ∎

LEMMA 2.3. *Let C be a nonempty convex subset of a normed space E. Let $S, T, R : C \to C$ be uniformly L-Lipschitzian. Let $\{x_n\}$ be the iterative sequence defined in* (1.8) *with $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\zeta_n < \infty$. If*

$$\lim_{n\to\infty}\|x_n - S^n x_n\| = \lim_{n\to\infty}\|x_n - T^n x_n\| = \lim_{n\to\infty}\|x_n - R^n x_n\| = 0.$$

Then

$$\lim_{n\to\infty}\|x_n - Sx_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - Rx_n\| = 0.$$

Proof. Set

$$\begin{aligned}
a_n &= \|x_n - S^n x_n\|, \\
b_n &= \|x_n - T^n x_n\|, \\
c_n &= \|x_n - R^n x_n\|.
\end{aligned}$$

We have

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - S^n x_n\| + \|S^n x_n - Sx_n\| \\
&\leq a_n + L\|S^{n-1}x_n - x_n\|
\end{aligned}$$

$$
\begin{aligned}
&= a_n + L(\|x_n - x_{n-1}\| + \|x_{n-1} - S^{n-1}x_{n-1}\| + \|S^{n-1}x_{n-1} - S^{n-1}x_n\|) \\
&\leq a_n + La_{n-1} + L\|x_n - x_{n-1}\| + L^2\|x_{n-1} - x_n\|.
\end{aligned}
$$

Hence, we get

$$\|x_n - Sx_n\| \leq a_n + La_{n-1} + L(L+1)\|x_n - x_{n-1}\|. \tag{2.1}$$

We can similarly show that

$$\|x_n - Tx_n\| \leq b_n + Lb_{n-1} + L(L+1)\|x_n - x_{n-1}\|, \tag{2.2}$$

and

$$\|x_n - Rx_n\| \leq c_n + Lc_{n-1} + L(L+1)\|x_n - x_{n-1}\|. \tag{2.3}$$

Since $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{x_n\}$ are bounded sequences in C then there exists a positive real number M' such that

$$\max\{\sup_{n\geq 1}\|u_n - x_n\|, \sup_{n\geq 1}\|v_n - x_n\|, \sup_{n\geq 1}\|w_n - x_n\|\} \leq M'.$$

Now, we have

$$
\begin{aligned}
&\|x_n - x_{n-1}\| \leq \|\alpha_{n-1}S^{n-1}y_{n-1} + \beta_{n-1}x_{n-1} + \gamma_{n-1}u_{n-1} - x_{n-1}\| \\
&\leq \alpha_{n-1}\|S^{n-1}y_{n-1} - x_{n-1}\| + \gamma_{n-1}\|u_{n-1} - x_{n-1}\| \\
&\leq \alpha_{n-1}\|S^{n-1}y_{n-1} - S^{n-1}x_{n-1}\| + \alpha_{n-1}\|S^{n-1}x_{n-1} - x_{n-1}\| + \gamma_{n-1}M' \\
&\leq \alpha_{n-1}L\|y_{n-1} - x_{n-1}\| + \alpha_{n-1}a_{n-1} + \gamma_{n-1}M' \\
&= \alpha_{n-1}L\|\lambda_{n-1}T^{n-1}z_{n-1} + \mu_{n-1}x_{n-1} + \nu_{n-1}v_{n-1} - x_{n-1}\| \\
&\quad + \alpha_{n-1}a_{n-1} + \gamma_{n-1}M' \\
&\leq \alpha_{n-1}\lambda_{n-1}L\|T^{n-1}z_{n-1} - x_{n-1}\| \\
&\quad + \alpha_{n-1}L\nu_{n-1}\|v_{n-1} - x_{n-1}\| + \alpha_{n-1}a_{n-1} + \gamma_{n-1}M' \\
&= \alpha_{n-1}\lambda_{n-1}L\|T^{n-1}z_{n-1} - T^{n-1}x_{n-1} + T^{n-1}x_{n-1} - x_{n-1}\| \\
&\quad + \alpha_{n-1}\nu_{n-1}L\|v_{n-1} - x_{n-1}\| + \alpha_{n-1}a_{n-1} + \gamma_{n-1}M' \\
&\leq \alpha_{n-1}\lambda_{n-1}L\|T^{n-1}z_{n-1} - T^{n-1}x_{n-1}\| \\
&\quad + \alpha_{n-1}\lambda_{n-1}L\|T^{n-1}x_{n-1} - x_{n-1}\| \\
&\quad + \alpha_{n-1}\nu_{n-1}L\|v_{n-1} - x_{n-1}\| + \alpha_{n-1}a_{n-1} + \gamma_{n-1}M' \\
&\leq \alpha_{n-1}\lambda_{n-1}L^2\|z_{n-1} - x_{n-1}\| + \alpha_{n-1}\lambda_{n-1}Lb_{n-1} + \alpha_{n-1}a_{n-1} \\
&\quad + \alpha_{n-1}\nu_{n-1}LM' + \gamma_{n-1}M' \\
&\leq \alpha_{n-1}\lambda_{n-1}\xi_{n-1}L^2\|R^{n-1}x_{n-1} - x_{n-1}\| + \alpha_{n-1}\lambda_{n-1}\zeta_{n-1}L^2\|w_{n-1} - x_{n-1}\| \\
&\quad + \alpha_{n-1}\lambda_{n-1}Lb_{n-1} + \alpha_{n-1}a_{n-1} + \alpha_{n-1}\nu_{n-1}LM' + \gamma_{n-1}M' \\
&\leq \alpha_{n-1}a_{n-1} + \alpha_{n-1}\lambda_{n-1}Lb_{n-1} + \alpha_{n-1}\lambda_{n-1}\xi_{n-1}L^2c_{n-1} \\
&\quad + \alpha_{n-1}\lambda_{n-1}\zeta_{n-1}L^2M' + \alpha_{n-1}\nu_{n-1}LM' + \gamma_{n-1}M'.
\end{aligned}
$$

Hence, we obtain

$$\|x_n - x_{n-1}\| \le a_{n-1} + Lb_{n-1} + L^2 c_{n-1} + \zeta_{n-1} L^2 M' + \nu_{n-1} L M' + \gamma_{n-1} M'. \tag{2.4}$$

Since $\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \nu_n = \lim_{n\to\infty} \zeta_n = 0$, then it follows from (2.5) that $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$ whenever $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$.

Using this by inequalities (2.2), (2.3) and (2.4) we conclude that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Rx_n\| = 0,$$

whenever

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = \lim_{n\to\infty} \|x_n - T^n x_n\| = \lim_{n\to\infty} \|x_n - R^n x_n\| = 0. \blacksquare$$

LEMMA 2.4. *Let C be a nonempty convex subset of a uniformly convex Banach space E. Let $S, T, R : C \to C$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings. Let $\{x_n\}$ be the iterative sequence defined in (1.8) with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \zeta_n < \infty$ and $0 < \delta \le \alpha_n, \lambda_n, \xi_n \le (1-\delta) < 1$, $n \ge 1$. If $F \ne \emptyset$, then*

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Rx_n\| = 0.$$

Proof. Let $p \in F$. Then, by Lemma 2.2, $\lim_{n\to\infty} \|x_n - p\|$ does exist. Suppose $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c \ge 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. Since $S, T, R : C \to C$ are asymptotically quasi-nonexpansive mappings, then there exists a sequence $\{k_n\}_{n=1}^{\infty}$ in [0,1] with $\lim_{n\to\infty} k_n = 0$ and we have

$$\begin{aligned}
\|y_n - p\| &= \|\lambda_n[T^n z_n - p + \nu_n(v_n - x_n)] + (1-\lambda_n)[x_n - p + \nu_n(v_n - x_n)]\| \\
&\le \lambda_n(1+k_n)\|z_n - p\| + (1-\lambda_n)\|x_n - p\| + \nu_n M' \\
&= \lambda_n(1+k_n)\|\xi_n[R^n x_n - p + \zeta_n(w_n - x_n)] + (1-\xi_n)[x_n - p \\
&\quad + \zeta_n(w_n - x_n)]\| + (1-\lambda_n)\|x_n - p\| + \nu_n M' \\
&\le \lambda_n(1+k_n)\xi_n\|R^n x_n - p\| + \lambda_n(1+k_n)(1-\xi_n)\|x_n - p\| \\
&\quad + (1-\lambda_n)\|x_n - p\| + \lambda_n(1+k_n)\zeta_n\|w_n - x_n\| + \nu_n M' \\
&\le [\lambda_n\xi_n(1+k_n)^2 + \lambda_n(1+k_n)(1-\xi_n) + (1-\lambda_n)]\|x_n - p\| \\
&\quad + [\lambda_n(1+k_n)\zeta_n + \nu_n]M'.
\end{aligned}$$

This implies

$$\begin{aligned}
\|y_n - p\| \le{} & [1 + \delta(1-\delta)k_n + (1-\delta)^2 k_n^2]\|x_n - p\| \\
& + +[(1-\delta)(1+k_n)\zeta_n + \nu_n]M'.
\end{aligned}$$

Since $\lim_{n\to\infty} k_n = 0$, $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \zeta_n < \infty$, then, taking lim sup on both sides of the above inequality, we get

$$\limsup_{n\to\infty} \|y_n - p\| \le c. \tag{2.5}$$

Since

$$\limsup_{n\to\infty} \|x_{n+1} - p\| = c,$$

then

$$\limsup_{n\to\infty} \|\alpha_n[S^n y_n - p + \gamma_n(u_n - x_n)] + (1-\alpha_n)[x_n - p + \gamma_n(u_n - x_n)]\| = c, \tag{2.6}$$

and since S is asymptotically nonexpansive, we have

$$\begin{aligned}\|S^n y_n - p + \gamma_n(u_n - x_n)\| &\le \|S^n y_n - p\| + \gamma_n\|u_n - x_n\| \\ &\le (1+k_n)\|y_n - p\| + \gamma_n M'.\end{aligned}$$

Hence, in view of (2.6), we obtain

$$\limsup_{n\to\infty} \|S^n y_n - p + \beta_n(u_n - x_n)\| \le c. \tag{2.7}$$

Furthermore, we have

$$\begin{aligned}\|x_n - p + \gamma_n(u_n - x_n)\| &\le \|x_n - p\| + \gamma_n\|u_n - x_n\| \\ &\le \|x_n - p\| + \gamma_n M',\end{aligned}$$

which implies

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(u_n - x_n)\| \le c. \tag{2.8}$$

Applying Lemma 1.1, in virtue of (2.7), (2.8) and (2.9), we obtain

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{2.9}$$

Now, we have

$$\begin{aligned}\|x_n - p\| &\le \|x_n - S^n y_n\| + \|S^n y_n - p\| \\ &\le \|x_n - S^n y_n\| + (1+k_n)\|y_n - p\|.\end{aligned}$$

Hence, taking lim inf and using (2.10), we get

$$c \le \liminf_{n\to\infty} \|y_n - p\|. \tag{2.10}$$

It follows from (2.6) and (2.11) that

$$\limsup_{n\to\infty} \|y_n - p\| \le c \le \liminf_{n\to\infty} \|y_n - p\|,$$

that is,

$$\lim_{n\to\infty} \|y_n - p\| = c,$$

which means that

$$\lim_{n\to\infty} \|\lambda_n[T^n z_n - p + \nu_n(v_n - x_n)] + (1-\lambda_n)[x_n - p + \nu_n(v_n - x_n)]\| = c. \tag{2.11}$$

Moreover, we have

$$\begin{aligned}\|T^n z_n - p + \nu_n(v_n - x_n)\| &\le \|T^n z_n - p\| + \nu_n\|v_n - x_n\| \\ &\le (1+k_n)\|z_n - p\| + \nu_n M' \\ &\le (1+k_n)\xi_n\|R^n x_n - p\| + (1-\xi_n)(1+k_n)\|x_n - p\| \\ &\quad + \zeta_n(1+k_n)\|w_n - x_n\| + \nu_n M' \\ &\le [(1+k_n)^2\xi_n + (1-\xi_n)(1+k_n)]\|x_n - p\| + [\zeta_n(1+k_n) + \nu_n]M',\end{aligned}$$

which yields

$$\|T^n z_n - p + \nu_n(v_n - x_n)\| \le [1 + k_n + (1-\delta)k_n + (1-\delta)k_n^2]\|x_n - p\| + [\zeta_n(1+k_n) + \nu_n]M'.$$

Taking lim sup on both sides of the above inequality, we get

$$\limsup_{n\to\infty} \|T^n z_n - p + \nu_n(v_n - x_n)\| \le c. \tag{2.12}$$

On the other hand, we have

$$\|x_n - p + \nu_n(v_n - p)\| \le \|x_n - p\| + \nu_n M',$$

which implies that

$$\limsup_{n\to\infty} \|x_n - p + \nu_n(v_n - p)\| \le c. \tag{2.13}$$

Applying Lemma 1.1, in view of (2.12), (2.13) and (2.14) we obtain that

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \tag{2.14}$$

Since

$$\begin{aligned}\|S^n x_n - x_n\| &\le \|S^n y_n - S^n x_n\| + \|S^n y_n - x_n\| \\ &\le (1+k_n)\|\lambda_n T^n z_n + (1-\lambda_n-\nu_n)x_n + \nu_n v_n - x_n\| \\ &\quad + \|S^n y_n - x_n\| \\ &\le (1+k_n)(1-\delta)\|T^n z_n - x_n\| + (1+k_n)\nu_n M' + \|S^n y_n - x_n\|.\end{aligned}$$

Thus, using (2.10) and (2.15), we obtain

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0. \tag{2.15}$$

Now observe that we have the following estimate

$$\begin{aligned}\|z_n - p\| &= \|\xi_n[R^n x_n - p + \zeta_n(w_n - x_n)] + (1-\xi_n)[x_n - p + \zeta_n(w_n - x_n)]\| \\ &\le \xi_n\|R^n x_n - p\| + (1-\xi_n)\|x_n - p\| + \zeta_n\|w_n - x_n\| \\ &\le [1 + (1-\delta)k_n]\|x_n - p\| + \zeta_n M'.\end{aligned}$$

Hence, taking lim sup on both sides, we obtain

$$\limsup_{n\to\infty} \|z_n - p\| \leq c. \tag{2.16}$$

Also,

$$\begin{aligned}\|x_n - p\| &\leq \|x_n - T^n z_n\| + \|T^n z_n - p\| \\ &\leq \|x_n - T^n z_n\| + (1 + k_n)\|z_n - p\|,\end{aligned}$$

which after using (2.15) and taking lim inf on both sides yields

$$c \leq \liminf_{n\to\infty} \|z_n - p\|. \tag{2.17}$$

It follows from (2.17) and (2.18) that

$$\lim_{n\to\infty} \|z_n - p\| = c. \tag{2.18}$$

Thus, from (2.19), we derive the equality

$$\lim_{n\to\infty} \|\xi_n[R^n x_n - p + \zeta_n(w_n - x_n)] + (1 - \xi_n)[x_n - p + \zeta_n(w_n - x_n)]\| = c. \tag{2.19}$$

Also, we have

$$\|R^n x_n - p + \zeta_n(w_n - x_n)\| \leq (1 + k_n)\|x_n - p\| + \zeta_n M'$$

i. e.,

$$\limsup_{n\to\infty} \|R^n x_n - p + \zeta_n(w_n - x_n)\| \leq c, \tag{2.20}$$

and

$$\limsup_{n\to\infty} \|x_n - p + \zeta_n(w_n - x_n)\| \leq c. \tag{2.21}$$

Using (2.20), (2.21) and (2.22) and applying Lemma 1.1, we get

$$\lim_{n\to\infty} \|R^n x_n - x_n\| = 0. \tag{2.22}$$

Finally, we have

$$\begin{aligned}\|T^n x_n - x_n\| &\leq \|T^n z_n - T^n x_n\| + \|T^n z_n - x_n\| \\ &\leq (1 + k_n)\xi_n\|R^n x_n - x_n\| + (1 + k_n)\zeta_n M' + \|T^n z_n - x_n\|.\end{aligned}$$

Hence, using (2.15) and (2.23), we obtain

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{2.23}$$

From Lemma 2.3 and using (2.16), (2.23) and (2.24) the desired result follows. ∎

Now, we are prepared to prove our strong convergence theorems.

THEOREM 2.5. *Let C be a nonempty closed convex subset of a real Banach space E. Let S, T and R be asymptotically quasi-nonexpansive mappings with a sequence $\{k_n\}_{n=1}^{\infty}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$. Let $F \neq \emptyset$ and*

$\{x_n\}$ be the sequence defined in Lemma 2.2. Then $\{x_n\}$ converges strongly to some common fixed point of S, T and R if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x, F) = \inf_{p\in F} \|x - p\|$.

Proof. In the proof of Lemma 2.2, we obtained that

$$\|x_{n+1} - p\| \leq (1 + k_n)^3\|x_n - p\| + [(1 + k_n)^2\zeta_n + (1 + k_n)\nu_n + \gamma_n]M,$$

which implies

$$d(x_{n+1}, F) \leq (1 + k_n)^3 d(x_n, F) + [(1 + k_n)^2\zeta_n + (1 + k_n)\nu_n + \gamma_n]M.$$

We can prove, by an argument similar to that in the proof of Lemma 2.2, that $\lim_{n\to\infty} d(x_n, F)$ exists for any $p \in F$. But $\liminf_{n\to\infty} d(x_n, F) = 0$, by hypothesis. Hence we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, for any $q \in F$, we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - q\| + \|x_n - q\|,$$

which implies that

$$\|x_{n+m} - x_n\| \leq d(x_{n+m}, F) + d(x_n, F). \tag{2.24}$$

Letting $n \to \infty$ on both sides of (3.1), we get

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0.$$

This means that $\{x_n\}$ is a Cauchy sequence. Since C is a closed subset of a Banach space E, then, $\{x_n\}$ converges to some $q \in C$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ then $d(q, F) = 0$.

Now we prove that $F = \{p : p \in F(S) \cap F(T) \cap F(R)\}$ is closed. So let $\{p_n\}$ be an arbitrary sequence of elements of F such that $p_n \to p$. We show that $p \in F$, i.e, show that $Sp = Tp = Rp = p$. For this purpose, consider the following estimate

$$\begin{aligned}\|Sp - p\| &\leq \|Sp - p_n\| + \|p_n - p\| \\ &\leq (1 + k_n)\|p - p_n\| + \|p_n - p\|,\end{aligned}$$

which , as $n \to \infty$, gives

$$\|Sp - p\| \leq 0.$$

Thus $Sp = p$. Similarly $Tp = p$ and $Rp = p$. By closedness of F and $d(q, F) = 0$ we have that $q \in F$, which completes the proof. ∎

COROLLARY 2.6. *If C is a nonempty closed convex subset of a real Banach space E, S, T and R are asymptotically nonexpansive mappings with a sequence $\{k_n\}_{n=1}^{\infty}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$, $F \neq \emptyset$ and $\{x_n\}$ is the sequence defined in Lemma 2.2. Then $\{x_n\}$ converges strongly to some common fixed point of S, T and R if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x, F) = \inf_{p\in F} \|x - p\|$.

REMARK 2.1. Theorem 2.5 generalizes Theorem 1 of [9] and extends Theorem 1 in [3] to the case of three asymptotically quasi-nonexpansive mappings using an easier proof than that in [3].

THEOREM 2.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S, T, R : C \to C$ be asymptotically quasi-nonexpansive mappings with a sequence $\{k_n\}_{n=1}^{\infty}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and satisfying condition (A''). Let $\{x_n\}$ be the iterative sequence defined in Lemma 2.4. If $F = F(S) \cap F(T) \cap F(R) \neq \emptyset$ then $\{x_n\}$ converges strongly to a common fixed point of S, T and R.*

Proof. By Lemma 2.4 and Condition (A''), we get that either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Rx_n\| = 0.$$

Hence

$$\lim_{n\to\infty} f(d(x_n, F)) = 0,$$

in any case. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, then we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Hence, by the conclusion of Theorem 2.5, we conclude that $\{x_n\}$ converges strongly to a common fixed point of S, T and R. ∎

COROLLARY 2.8. *If C is a nonempty closed convex subset of a uniformly convex Banach space E and $S, T, R : C \to C$ are asymptotically nonexpansive mappings with a sequence $\{k_n\}_{n=1}^{\infty}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and satisfying condition (A''), $F \neq \emptyset$ and $\{x_n\}$ is the iterative sequence defined in Lemma 2.4. Then $\{x_n\}$ converges strongly to a common fixed point of S, T and R .*

REMARK 2.2. Theorem 2 and Corollary 1 in [3] can be obtained from Theorem 3.3 and Corollary 3.4 as special cases when $\eta_n = 1$, $n \geq 1$. Since uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping is asymptotically nonexpansive mapping, then Theorem 2 in [6] is included in Theorem 3.3 with a weaker condition than compactness and moreover, for three mappings.

References

[1] Y. J. Cho, H. Zhou, G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl. **47** (2004), 707–717.

[2] H. Fukhar-ud-din, S. H. Khan, *Convergence of two-step iterative scheme with errors for two asymptotically nonexpansive mappings*, Int, J. Math. Sci. **37** (2004), 1956–1971.

[3] H. Fukhar-ud-din, S. H. Khan, *Convergence of iterates with errors for asymptotically quasi-nonexpansive mappings and applications*, J. Math. Anal. Appl. **328** (2007), 821–829.

[4] Z. Huang, *Mann and Ishikawa iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl. **37** (1999), 1–7.

[5] J. U. Jeong and S. H. Kim, *Weak and strong convergence of the Ishikawa iteration process with errors for two asymptotically nonexpansive mappings*, Appl. Math. Comp. **181** (2006), 1394–1401.

[6] S. H. Khan and W. Takahashi, *Approximating common fixed points of two asymptotically nonexpansive mappings*, Sci. Math. Jpn. **53** (2001), 143–148.

[7] S. H. Khan, H. Fukhar-ud-din, *Weak and strong convergence of a scheme with errors for two nonexpansive mappings*, Nonlinear Anal. **61** (2005), 1295–1301.

[8] L. S. Liu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1) (1995), 114-125.

[9] L. Qihou, *Iterative sequences for asymptotically quasi-nonexpansive mappings with errors member*, J. Math. Anal. Appl. **259** (2001), 18–24.

[10] R. A. Rashwan, A. A. Abdel Hakim, *Weak and Strong Convergence Theorems for Three-Step Iterations with Errors for Three Asymptotically Nonexpansive Mappings*, (submitted).

[11] J. Schu, *Weak and strong convergence of fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), 153–159.

[12] H. F. Senter and W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **44** (1974), 375–379.

[13] W. Takahashi and T. Tamura, *Convergence theorems for a pair of nonexpansive mappings,*, J. Convex Analysis **5** (1) (1998), 45–48.

[14] K. K. Tan, H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301–308.

[15] Y. G. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), 1–101.

[16] B. L. Xu, M. A. Noor, *Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **267** (2002), 444–453.

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