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W. T. Sulaiman

NOTES ON INTEGRAL INEQUALITIES

Abstract. New results, generalizations and improvements concerning several integral inequalities are obtained.

1. Introduction

The following kinds of integral inequalities were presented first by F. Qi [10]. He proved the following results:

PROPOSITION 1.1. *Let $f(x)$ be differentiable on (a,b) and $f(a)=0$. If $0 \le f'(x) \le 1$, then*

$$\int_a^b (f(x))^3 dx \le \left(\int_a^b f(x)\,dx\right)^2. \tag{1.1}$$

If $f'(x) \ge 1$, the inequality (1.1) *reverses. The equality in* (1.1) *holds if $f(x)=0$ or $f(x)=x-a$.*

PROPOSITION 1.2. *Suppose $f(x)$ has continuous derivative of nth order on the interval $[a,b]$, $f^{(i)}(a) \ge 0$ and $f^{(n)}(x) \ge n!$, where $0 \le i \le n-1$. Then*

$$\int_a^b (f(x))^{n+2} dx \ge \left(\int_a^b f(x)\,dx\right)^{n+1}. \tag{1.2}$$

F. Qi as well-posed the following open problem:

Under what conditions does the inequality

$$\int_a^b (f(x))^t dx \ge \left(\int_a^b f(x)\,dx\right)^{t-1} \tag{1.3}$$

hold for $t > 1$?

Many mathematicians studied the complements, variants and continuations of Qi's integral inequality; see the references of this paper.

In [7], the authors proved the following result

THEOREM 1.3. *Assume that the condition: $f(x)$ is nonnegative continuous on $[0,1]$ satisfying*

$$\int_x^1 f(t)\,dt \geq \frac{1-x^2}{2}, \qquad \forall\, x \in [0,1]. \tag{1.4}$$

Then

$$\int_0^1 f^{\alpha+1}(x)\,dx \geq \int_0^1 x^{\alpha} f(x)\,dx \tag{1.5}$$

for every real number $\alpha > 0$.

They also posed the following open problem.

Let $f(x)$ be continuous function on $[0,1]$, satisfying

$$\int_x^1 f(t)\,dt \geq \int_x^1 t\,dt, \qquad \forall x \in [0,1]. \tag{1.6}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\alpha} f^{\beta}(x)\,dx, \tag{1.7}$$

hold for α and β?

For the above problem, L. Bougoffa [1], [2] found an answer. He in fact presented the following

THEOREM 1.4. *Let $f(x)$ be nonnegative function, continuous on $[a,b]$ and differentiable on (a,b), with $f'(x) \geq 1$ (see [2]), and let α and β be positive numbers. If*

$$\int_x^b f(t)\,dt \leq \int_x^b (t-a)\,dt, \qquad \forall x \in [a,b], \tag{1.8}$$

then

$$\int_a^b f^{\alpha+\beta}(x)\,dx \leq \int_a^b (x-a)^{\alpha} f^{\beta}(x)\,dx. \tag{1.9}$$

If (1.8) reverses, then (1.9) reverses.

2. Main results

We start with Theorem 1.4, and improving this result by dropping the condition (1.8). Indeed, we prove the following:

THEOREM 2.1. *Let $f(x)$ be nonnegative function, continuous on $[a,b]$ and differentiable on (a,b) with $f'(x) \geq 1$, and let α, β be positive numbers. Then (1.9) is satisfied.*

Proof. Set

$$F(t)=\int_a^t (x-a)^\alpha f^\beta(x)\,dx-\int_a^t f^{\alpha+\beta}(x)\,dx.$$

Then, we have

$$\begin{aligned}F'(t)&=(t-a)^\alpha f^\beta(t)-f^{\alpha+\beta}(t)=f^\beta(t)\left((t-a)^\alpha-f^\alpha(t)\right)=0\\&\qquad\qquad\qquad\qquad\qquad\qquad\text{if } f(t)=t-a,\\F''(t)&=\beta(t-a)f^{\beta-1}(t)f'(t)+\alpha(t-a)^{\alpha-1}f^\beta(t)\\&\quad-(\alpha+\beta)f^{\alpha+\beta}(t)f'(t),\\[F''(t)]_{f(t)=t-a}&=\alpha(t-a)^{\alpha+\beta-1}\left(f'(t)-1\right)\geq 0.\end{aligned}$$

This shows that F attains its minimum when $f(t)=t-a$ which is 0. That is $F(t)\geq 0$.

The following Lemma is needed for our aim

LEMMA 2.2. *Let f,g be two functions defined on $[a,b]$ such that f is nonnegat -ive, $g'(x)\geq 1$ with $g(a)=0$, and let $\gamma>0$. If*

$$\int_x^b f(t)\,dt\geq\int_x^b g(t)\,g'(t)\,dt=\frac{g^2(b)-g^2(x)}{2}\quad\forall\, x\in[a,b], \tag{2.1}$$

then

$$\int_a^b f(t)\,g^\gamma(t)\,dt\geq\frac{1}{\gamma+2}\,g^{\gamma+2}(b), \tag{2.2}$$

and

$$\int_a^b f^\alpha(x)\,g^\beta(x)\,dx\geq\frac{g^{\alpha+\beta+1}(b)}{\alpha+\beta+1},\qquad(\alpha\geq 1,\beta>0). \tag{2.3}$$

Proof. Since $g'(x)>0$, then g is increasing which gives $g(x)\geq g(a)=0$. By changing the order of integration, we have

$$\int_a^b\int_x^b f(t)\,g^\gamma(x)\,g'(x)\,dtdx=\int_a^b f(t)\,dt\int_a^t g^\gamma(x)\,g'(x)\,dx=\frac{1}{\gamma+1}\int_a^b f(t)\,g^{\gamma+1}(t)\,dt.$$

Also, we have

$$\begin{aligned}\int_a^b\int_x^b f(t)\,g^\gamma(x)\,g'(x)\,dt\,dx&=\int_a^b\Big(\int_x^b f(t)\,dt\Big)g^\gamma(x)\,g'(x)\,dx\\&\geq\frac{1}{2}\int_a^b\Big(g^2(b)-g^2(x)\Big)\,g^\gamma(x)\,g'(x)\,dx=\frac{1}{(\gamma+1)\,(\gamma+3)}\,g^{\gamma+3}(b),\end{aligned}$$

which together with the previous equality implies (2.2). Concerning (2.3), we apply AG inequality as follows

$$\frac{1}{\alpha} f^{\alpha}(x) + \frac{\alpha-1}{\alpha} g^{\alpha}(x) \geq f(x)\, g^{\alpha-1}(x),$$

and hence:

$$\frac{1}{\alpha} f^{\alpha}(x)\, g^{\beta}(x) + \frac{\alpha-1}{\alpha} g^{\alpha+\beta}(x) \geq f(x)\, g^{\alpha+\beta-1}(x),$$

$$\begin{aligned} f^{\alpha}(x)\, g^{\beta}(x) &\geq (1-\alpha)\, g^{\alpha+\beta}(x) + \alpha f(x)\, g^{\alpha+\beta-1}(x) \\ &\geq (1-\alpha)\, g^{\alpha+\beta}(x)\, g'(x) + \alpha f(x)\, g^{\alpha+\beta-1}(x). \end{aligned}$$

On integrating the above from a to b, and making use of (2.2), we obtain

$$\begin{aligned} \int_a^b f^{\alpha}(x)\, g^{\beta}(x)\, dx &\geq (1-\alpha) \int_a^b g^{\alpha+\beta}(x)\, g'(x) dx + \alpha \int_a^b f(x)\, g^{\alpha+\beta-1}(x)\, dx \\ &\geq (1-\alpha) \frac{g^{\alpha+\beta+1}(b)}{\alpha+\beta+1} + \alpha \frac{g^{\alpha+\beta+1}(b)}{\alpha+\beta+1} = \frac{g^{\alpha+\beta+1}(b)}{\alpha+\beta+1}. \end{aligned}$$

The above Lemma leads to the following

THEOREM 2.3. *If the functions f and g satisfy the conditions of Lemma (2.2) in addition to* (2.1), *then*

$$\int_a^b f^{\alpha+\beta}(x) dx \geq \int_a^b f^{\alpha}(x)\, g^{\beta}(x)\, dx, \tag{2.4}$$

for every real $\alpha \geq 1 and \beta > 0$.

Proof. Using the AG inequality, we have

$$\begin{aligned} \frac{\alpha}{\alpha+\beta} f^{\alpha+\beta}(x) &\geq f^{\alpha}(x)\, g^{\beta}(x) - \frac{\beta}{\alpha+\beta} g^{\alpha+\beta}(x) \\ &\geq f^{\alpha}(x)\, g^{\beta}(x) - \frac{\beta}{\alpha+\beta} g^{\alpha+\beta}(x)\, g'(x). \end{aligned}$$

Integrating both sides of the above inequality from a to b gives

$$\frac{\alpha}{\alpha+\beta} \int_a^b f^{\alpha+\beta}(x)\, dx \geq \int_a^b f^{\alpha}(x)\, g^{\beta}(x)\, dx - \frac{\beta}{\alpha+\beta} \int_a^b g^{\alpha+\beta}(x)\, g'(x)\, dx$$

$$= \frac{\alpha}{\alpha+\beta} \int_a^b f^{\alpha}(x) g^{\beta}(x)\, dx + \frac{\beta}{\alpha+\beta} \left(\int_a^b f^{\alpha}(x)\, g^{\beta}(x)\, dx - \int_a^b g^{\alpha+\beta}(x)\, g'(x)\, dx \right)$$

$$\geq \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx + \frac{\beta}{\alpha+\beta}\left(\frac{g^{\alpha+\beta+1}(b)}{\alpha+\beta+1} - \frac{g^{\alpha+\beta+1}(b)}{\alpha+\beta+1}\right)$$

$$= \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx.$$

The result follows.

REMARK 2.4. Theorem 2.3 covers the results of [3], [5], and [7] as these results are in fact special cases of this Theorem. As an example the result of [3] follows by putting $g(x) = x$, $a = 0$, $b = 1$.

Concerning F. Qi's result (Proposition 1.2), the authors in [5], [8] and [12] have all dealt with this result in order to get some improvement. Among all of these, the result of [12] which is the best, and states the following:

Let n be a positive integer. Suppose $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$, where $0 \leq i \leq n-1$, and

$$f^{(n)}(x) \geq \frac{n!}{(n+1)^{n-1}},$$

then

$$\int_a^b f^{n+2}(x)\,dx \geq \left(\int_a^b f(x)\,dx\right)^{n+1}. \tag{2.5}$$

Now we are in a position to give the following new results.

THEOREM 2.5. *Suppose f is positive and has continuous 2nd derivative on the interval $[a,b]$ such that $f(a) = 0$, $f'(a) = 0$, and let $\gamma > \alpha > 0$, $\beta > 1$, $\beta(\alpha+1) > (\gamma+1)$. If*

$$\frac{f(t)\,f''(t)}{(f'(t))^2} \geq \beta(\alpha+1) - (\gamma+1), \tag{2.6}$$

then

$$\int_a^b f^{\gamma}(x)\,dx \geq \left(\int_a^b f^{\alpha}(x)\,dx\right)^{\beta}. \tag{2.7}$$

If (2.6) *reverses, then* (2.7) *reverses as well.*

Proof. By the hypothesis, $f''(t) > 0$, then $f'(t)$ is increasing and hence $f'(t) > f'(a) = 0$. Set

$$F(t) = \int_a^t f^{\gamma}(x)\,dx - \left(\int_a^t f^{\alpha}(x)\,dx\right)^{\beta}.$$

We have

$$F'(t) = f^\gamma(t) - \beta\, f^\alpha(t)\Big(\int_a^t f^\alpha(x)\,dx\Big)^{\beta-1} = 0,$$

if

$$\beta\,\Big(\int_a^t f^\alpha(x)\,dx\Big)^{\beta-1} = f^{\gamma-\alpha}(t). \tag{2.8}$$

$$F''(t) = \gamma\, f^{\gamma-1}(t)\, f'(t) - \beta(\beta-1)\, f^{2\alpha}(t)\,\Big(\int_a^t f^\alpha(x)\,dx\Big)^{\beta-2}$$
$$- \alpha\beta\, f^{2\alpha-1}(t) f'(t)\,\Big(\int_a^t f(x)\,dx\Big)^{\beta-1}.$$

Now, for t satisfying (2.8), we have

$$F''(t) = (\gamma-\alpha)\, f^{\gamma-1}(t)\, f'(t) - (\beta-1)\, f^{\alpha+\gamma}(t)\,\Big(\int_a^t f^\alpha(x)\,dx\Big)^{-1}$$
$$= \frac{(\gamma-\alpha)\, f^{\gamma-1}(t)\, f'(t)}{\int_a^t f^\alpha(x)\,dx}\Big(\int_a^t f^\alpha(x)\,dx - \frac{(\beta-1)\, f^{\alpha+1}(t)}{(\gamma-\alpha)\, f'(t)}\Big).$$

If we are setting $G(t)$ for the quantity in the brackets above, we have

$$G'(t) = f^\alpha(t) - \frac{\beta-1}{\gamma-\alpha}\Big((\alpha+1)\, f^\alpha(t) - \frac{f^{\alpha+1}(t)\, f''(t)}{(f'(t))^2}\Big)$$
$$= f^\alpha(t)\Big(1 - \frac{\beta-1}{\gamma-\alpha}\Big(\alpha+1-\frac{f(t)\, f''(t)}{(f'(t))^2}\Big)\Big) \geq 0, \quad \text{by the hypothesis.}$$

Therefore G is nondecreasing. But $G(a) = 0$, then $G(t) \geq 0$. This implies that $F''(t) \geq 0$. That is F attains its minimum when $t = a$. But $F(a) \geq 0$, then $F(t) \geq 0$. This completes the proof.

REMARK 2.6. In Theorem 2.5, if f is convex, the theorem is valid as $f''(t) \geq 0$. But if f is concave, that is $f''(t) \leq 0$, and nondecreasing, then the theorem is valid if $\gamma + 1 > \beta(\alpha+1)$.

LEMMA 2.7. *Let f be positive and has continuous 2nd derivative on the interval $[a, b]$ such that $f(a) > 0$, $f'(a) > 0$. If*

$$\frac{f(t)f''(t)}{(f'(t))^2} > k > 1, \quad t \in [a,b],$$

then:

$$f(t) \leq \Big((k-1)\frac{f'(a)}{f(a)}(t-a) + f^{1-k}(a)\Big)^{1/(1-k)}.$$

Proof. By the hypothesis $f''(t) > 0$, then $f'(t)$ is increasing and hence positive. We have

$$\frac{f''(t)}{f'(t)} > k\frac{f'(t)}{f(t)},$$

which implies:

$$\begin{aligned}
&\int_a^t \frac{f''(x)}{f'(x)}\,dx > k\int_a^t \frac{f'(x)}{f(x)}\,dx,\\
&\ln f'(t) - \ln f'(a) > k(\ln f(t) - \ln f(a))\\
&\ln f'(t) - k\ln f(t) > \ln f'(a) - k\ln f(a)\\
&\ln\left(\frac{f'(t)}{f^k(t)}\right) > \ln\left(\frac{f'(a)}{f^k(a)}\right)
\end{aligned}$$

and hence:

$$\begin{aligned}
&\int_a^t f^{-k}(x)\,f'(x)\,dx > \left(\frac{f'(a)}{f^k(a)}\right)\int_a^t dx,\\
&f(t) \le \left((k-1)\left(\frac{f'(a)}{f(a)}\right)(t-a) + f^{1-k}(a)\right)^{1/(1-k)}.
\end{aligned}$$

The following is a good generalization for the result of [12], and hence for all similar results before, as well as the proof is via a very simple method.

THEOREM 2.8. *Let n be a positive integer. Suppose $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$, where $0 \le i \le n-1$. Let α, β, γ be positive numbers such that $\alpha\beta > \gamma$. If*

$$(f^{(n)}(x))^{\gamma-\alpha\beta} \ge \frac{(n\gamma+1)n!}{(n\alpha+1)^{\beta}}(b-a)^{\beta(n\alpha+1)-(n\gamma+1)}, \tag{2.9}$$

then (2.7) is satisfied. In particular for $\gamma = n+2$, $\beta = n+1$, $\alpha = 1$, we obtain (2.5).

Proof. Taylor's expansion applied to f with Lagrange remainder states that

$$\begin{aligned}
f(x) &= f(a) + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f(\xi)}{n!}(x-a)^n \qquad \text{for some } \xi \in (a,x)\\
&= \frac{f(\xi)}{n!}(x-a)^n.
\end{aligned}$$

On substituting for $f(x)$ above in (2.7), we obtain:

$$\int_a^b \left(\frac{f^{(n)}(\xi)}{n!}(x-a)^n\right)^{\gamma} dx \ge \left(\int_a^b \left(\frac{f^{(n)}(\xi)}{n!}(x-a)^n\right)^{\alpha} dx\right)^{\beta}.$$

Simplifying gives:

$$(f^{(n)}(\xi))^{\gamma-\alpha\beta} \geq \frac{(n\gamma+1)\,n!}{(n\alpha+1)^{\beta}}\,(b-a)^{\beta(n\alpha+1)-(n\gamma+1)}.$$

Therefore in order to have (2.7) satisfied, we get

$$(f^{(n)}(x))^{\gamma-\alpha\beta} \geq \frac{(n\gamma+1)\,n!}{(n\alpha+1)^{\beta}}\,(b-a)^{\beta(n\alpha+1)-(n\gamma+1)}.$$

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DEPARTMENT OF MATHEMATICS
COLLEGE OF COMPUTER SCIENCE AND MATHEMATICS
MOSUL UNIVERSITY
MOSUL, IRAQ

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