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RATE OF APPROXIMATION FOR INTEGRATED SZASZ-MIRAKYAN OPERATORS

Abstract. Recently Jain et al. [3] proposed an integral modification of Szasz-Mirakyan operators $S_{n,\alpha}(f,x), \alpha > 0$ and studied some direct approximation theorems in simultaneous approximation. The present paper deals with the rate of approximation of such operators, for functions which have derivatives of bounded variation.

1. Introduction

To approximate integrable functions on the interval $[0,\infty)$, and for $\alpha > 0$, we proposed in [2] and [3] the integral modification of the Szasz-Mirakyan operators as

$$(1) \qquad \begin{aligned} S_{n,\alpha}(f,x) &= \int_0^\infty W_{n,\alpha}(x,t)f(t)dt \\ &= \sum_{v=0}^\infty s_{n,v}(x)\int_0^\infty b_{n,v,\alpha}(t)f(t)dt, \qquad x \in [0,\infty), \end{aligned}$$

where the kernel $W_{n,\alpha}(x,t)$ is defined as:

$$W_{n,\alpha}(x,t) = \sum_{v=0}^\infty s_{n,v}(x)b_{n,v,\alpha}(t),$$

and the Szasz and Beta basis functions are given by

$$s_{n,v}(x) = \exp(-nx)\frac{(nx)^v}{v!}, \qquad b_{n,v,\alpha}(t) = \alpha\frac{\Gamma(\frac{n}{\alpha}+v+1)}{\Gamma(v+1)\Gamma(\frac{n}{\alpha})}\frac{(\alpha t)^v}{(1+\alpha t)^{(\frac{n}{\alpha}+v+1)}}.$$

In case $\alpha = 1$, the above operators (1) reduce to the Szasz-Beta operators studied in [6]. Some direct results in simultaneous approximation on $S_{n,\alpha}(f,x)$ for iterative combinations and without combinations were discussed in [2] and [3] respectively. Very recently Gupta and Sinha [5] introduced similar type of operators, but they have considered the value of function at zero explicitly, the operators discussed in [5] for $\alpha > 0$, are defined as

$$V_{n,\alpha}(f,x)=(n-\alpha)\sum_{v=1}^{\infty}s_{n,v}(x)\int_0^{\infty}p_{n,v-1,\alpha}(t)f(t)dt+e^{-nx}f(0),\qquad x\in[0,\infty)$$

where $s_{n,v}(x)$ is as defined above and

$$p_{n,v,\alpha}(t)=\frac{\Gamma(\frac{n}{\alpha}+v)}{\Gamma(v+1)\Gamma(\frac{n}{\alpha})}\frac{(\alpha t)^v}{(1+\alpha t)^{\frac{n}{\alpha}+v}}.$$

The above two integral modifications $S_{n,\alpha}(f,x)$ and $V_{n,\alpha}(f,x)$ of Szasz-Mirakyan operators are very similar. The main difference between these two are that $V_{n,\alpha}(f,x)$ defined in [5] are discretely defined at $f(0)$ to preserve the constant functions, while the operators $S_{n,\alpha}(f,x)$ are the usual integral modification of the Szasz-Mirakyan operators having the weight function of generalized Baskakov operators. As the operators (1) are the generalization of the operators discussed in [6], this motivated us to study further on such operators.

We define $\beta_{n,\alpha}(x,t)=\int_0^t W_{n,\alpha}(x,s)ds$, then as a special case we have $\beta_{n,\alpha}(x,\infty)=\int_0^{\infty}W_{n,\alpha}(x,s)ds=1$. Let $DB_\gamma(0,\infty)$, $\gamma\geq 0$ be the class of absolutely continuous functions f defined on $(0,\infty)$ satisfying the growth condition $f(t)=O(t^\gamma)$, $t\to\infty$ and having a derivative f' on the interval $(0,\infty)$ coinciding a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that all functions $f\in BD_\gamma(0,\infty)$ posses for each $c>0$ a representation

$$f(x)=f(c)+\int_c^x\psi(t)dt,\quad x\geq c.$$

Another topic of interest is the rate of convergence for functions having derivatives of bounded variation. Such type of problems were discussed in [1] and [4], where the rate of convergence have been discussed for Bernstein and some other integral operators. In the present paper, we extend the study and obtain the rate of approximation for differential functions of bounded variation.

2. Auxiliary results

We shall use the following Lemmas to prove our main theorem.

LEMMA 1 ([2]). *Let the function $\mu_{n,m,\alpha}(x)$, $m\in\aleph^0$ and $\alpha>0$ be defined as*

$$\mu_{n,m,\alpha}(x)=\sum_{v=0}^{\infty}s_{n,v}(x)\int_0^{\infty}b_{n,v,\alpha}(t)(t-x)^m dt.$$

Then by easy computation, we have

$$\mu_{n,0,\alpha}(x) = 1, \qquad \mu_{n,1,\alpha}(x) = \frac{(1+\alpha x)}{n-\alpha},$$
$$\mu_{n,2,\alpha}(x) = \frac{(4\alpha x + 2\alpha^2 x^2 + 2) + nx(\alpha x + 2)}{(n-\alpha)(n-2\alpha)}.$$

Also for $n > \alpha(m+1)$, we have the recurrence relation:

$$[n - \alpha(m+1)]\mu_{n,m+1,\alpha}(x)$$
$$= x\mu^{(1)}_{n,m,\alpha}(x) + [(m+1)(1+2\alpha x) - \alpha x]\mu_{n,m,\alpha}(x)$$
$$+ mx(\alpha x + 2)\mu_{n,m-1,\alpha}(x).$$

Consequently for each $x \in [0,\infty)$, it follows from the recurrence relation that

$$\mu_{n,m,\alpha}(x) = O(n^{-[(m+1)/2]}).$$

REMARK 1. Particularly for any number $\lambda > 1$ and $x \in [0,\infty)$, using Lemma 1, for n sufficiently large, we have

$$S_{n,\alpha}((t-x)^2, x) \equiv \mu_{n,2,\alpha}(x) \le \frac{\lambda x(2+\alpha x)}{n}. \tag{2}$$

REMARK 2. In view of Remark 1, it can be easily verified by Holder's inequality that

$$S_{n,\alpha}(\mid t - x \mid, x) \le [\mu_{n,2,\alpha}(x)]^{1/2} \le \sqrt{\frac{\lambda x(2+\alpha x)}{n}}. \tag{3}$$

LEMMA 2. *Let $x \in [0,\infty)$, $\lambda > 1$, then for n sufficiently large, we have*

$$(i) \quad \beta_{n,\alpha}(x,y) = \int_0^y W_{n,\alpha}(x,t)dt \le \frac{\lambda x(2+\alpha x)}{n(x-y)^2}, \quad 0 \le y < x,$$

$$(ii) \quad 1 - \beta_{n,\alpha}(x,z) = \int_z^\infty W_{n,\alpha}(x,t)dt \le \frac{\lambda x(2+\alpha x)}{n(z-x)^2}, \quad x < z < \infty.$$

Proof. First we prove (i), using (2), we have

$$\int_0^y W_{n,\alpha}(x,t)dt \le \int_0^y \frac{(x-t)^2}{(x-y)^2} W_{n,\alpha}(x,t)dt$$
$$\le (x-y)^{-2}\mu_{n,2,\alpha}(x) \le \frac{\lambda x(2+\alpha x)}{x(x-y)^2}.$$

The proof of (ii) is similar, so we omit the details.

3. Main result

This section deals with the following main theorem.

THEOREM 1. *Let $f \in DB_\gamma(0,\infty)$, $\gamma > 0$ and $x \in (0,\infty)$. Then for $\lambda > 2$ and sufficiently large n, we have*

$$
\begin{aligned}
|S_{n,\alpha}(f,x)-f(x)| \le{}& \frac{\lambda(2+\alpha x)}{n}\left(\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x-x/v}^{x+x/v}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)\right)\\
&+\frac{\lambda(2+\alpha x)}{nx}(|f(2x)-f(x)-xf'(x^+)|+|f(x)|)\\
&+\sqrt{\frac{\lambda x(2+\alpha x)}{n}}\left(C2^\gamma O(n^{\frac{-\gamma}{2}})+|f'(x^+)|\right)\\
&+\frac{1}{2}\sqrt{\frac{\lambda x(2+\alpha x)}{n}}|f'(x^+)-f'(x^-)|\\
&+\frac{1+\alpha x}{2(n-\alpha)}|f'(x^+)+f'(x^-)|,
\end{aligned}
$$

where $\bigvee_a^b f(x)$ is the total variation of f_x on the interval $[a,b]$ and the auxiliary function f_x is defined as

$$
f_x(t)=\begin{cases} f(t)-f(x^-), & 0\le t<x;\\ 0, & t=x;\\ f(t)-f(x^+), & x<t<\infty;\end{cases}
$$

$f(x^-)$ and $f(x^+)$ represents the left and right hand limits at x.

Proof. It is easily observed from Lemma 1 that $\int_0^\infty W_{n,\alpha}(x,t)dt=1$, so we can write

$$
\begin{aligned}
S_{n,\alpha}(f,x)-f(x) &= \int_0^\infty W_{n,\alpha}(x,t)(f(t)-f(x))dt\\
&= \int_0^\infty \Big(\int_x^t W_{n,\alpha}(x,t)(f'(u)du)dt\Big).
\end{aligned}
$$

Also, we can write

$$
\begin{aligned}
f'(u)={}&\frac{[f'(x^+)+f'(x^-)]}{2}+(f')_x(u)+\frac{[f'(x^+)-f'(x^-)]}{2}sgn(u-x)\\
&+\left[f'(x)-\frac{[f'(x^+)+f'(x^-)]}{2}\right]\chi_x(u),
\end{aligned}
$$

where

$$
\chi_x(t)=\begin{cases}1, & x=u\\ 0, & x\neq u.\end{cases}
$$

Next, we have

$$
\int_0^\infty\left(\int_x^t f'(x)-\frac{[f'(x^+)+f'(x^-)]}{2}\chi_x(u)du\right)W_{n,\alpha}(x,t)dt=0,
$$

thus

$$S_{n,\alpha}(f,x)-f(x)=\int_0^\infty(\int_x^t W_n(x,t)(\frac{[f'(x^+)+f'(x^-)]}{2}+(f')_x(u))du)dt$$
$$+\int_0^\infty(\int_x^t W_n(x,t)\frac{[f'(x^+)-f'(x^-)]}{2}sgn(u-x)du)dt.$$

Also

$$\int_0^\infty\left(\int_x^t\frac{[f'(x^+)-f'(x^-)]}{2}sgn(u-x)du\right)W_{n,\alpha}(x,t)dt$$
$$=\frac{[f'(x^+)-f'(x^-)]}{2}S_{n,\alpha}(|t-x|,x)$$

and

$$\int_0^\infty\left(\int_x^t\frac{1}{2}[f'(x^+)+f'(x^-)]du\right)W_{n,\alpha}(x,t)dt=\frac{1}{2}[f'(x^+)+f'(x^-)]S_{n,\alpha}((t-x),x).$$

We can write

$$|S_{n,\alpha}(f,x)-f(x)|$$
$$\le|\int_x^\infty\left(\int_x^t(f')_x(u)du\right)W_{n,\alpha}(x,t)dt-\int_0^x\left(\int_x^t(f')_x(u)du\right)W_{n,\alpha}(x,t)dt|$$
$$+\frac{1}{2}|f'(x^+)-f'(x^-)|S_{n,\alpha}(|t-x|,x)$$
$$+\frac{1}{2}|f'(x^+)+f'(x^-)|S_{n,\alpha}((t-x),x)$$
$$\le|A_{n,\alpha}(f,x)+B_{n,\alpha}(f,x)|+\frac{1}{2}|f'(x^+)-f'(x^-)|S_{n,\alpha}(|t-x|,x)$$
$$+\frac{1}{2}|f'(x^+)+f'(x^-)|S_{n,\alpha}((t-x),x).$$

By applying Lemma 1 and Remark 2, we have

(4) $$|S_{n,\alpha}(f,x)-f(x)|$$
$$=|A_{n,\alpha}(f,x)+B_{n,\alpha}(f,x)|+\frac{1}{2}|f'(x^+)-f'(x^-)|S_{n,\alpha}(|t-x|,x)$$
$$+\frac{1}{2}|f'(x^+)+f'(x^-)|S_{n,\alpha}((t-x),x)$$
$$\le|A_{n,\alpha}(f,x)+B_{n,\alpha}(f,x)|+\frac{1}{2}|f'(x^+)-f'(x^-)|\sqrt{\frac{\lambda x(2+\alpha x)}{n}}$$
$$+\frac{1}{2}|f'(x^+)+f'(x^-)|\frac{(1+\alpha x)}{n-\alpha}.$$

In order to complete the proof of the theorem it is sufficient to estimate the terms $A_n(f,x)$ and $B_n(f,x)$. Using integration by parts, Lemma 2 and setting $y = x - x/\sqrt{n}$, we have

$$|B_{n,\alpha}(f,x)| = |\int_0^x \Big(\int_x^t (f')_x(u)du\Big) d_t(\beta_{n,\alpha}(x,t))|$$

$$= \Big|\int_0^x \beta_{n,\alpha}(x,t)(f')_x(t)dt\Big| \leq \Big(\int_0^y + \int_y^x\Big)|\bigvee_t^x((f')_x)||\beta_{n,\alpha}(x,t)|dt$$

$$\leq \frac{\lambda x(2+\alpha x)}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt + \int_y^x \bigvee_t^x((f')_x)dt$$

$$\leq \frac{\lambda x(2+\alpha x)}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt + \bigvee_{x-x/\sqrt{n}}^{x}((f')_x)\int_{x-x/\sqrt{n}}^{x} dt$$

$$\leq \frac{\lambda x(2+\alpha x)}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x).$$

Let $u = \frac{x}{x-t}$. Then we obtain

$$\frac{\lambda x(2+\alpha x)}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt = \frac{\lambda x(2+\alpha x)}{nx}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x}((f')_x)du$$

$$\leq \frac{\lambda(2+\alpha x)}{n}\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{v}}^{x}((f')_x).$$

Therefore

$$|B_{n,\alpha}(f,x)| \leq \frac{\lambda(2+\alpha x)}{n}\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{v}}^{x}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x). \tag{5}$$

Next, we have

$$|A_{n,\alpha}(f,x)| = \Big|\int_x^\infty \Big(\int_x^t (f')_x(u)du\Big)W_{n,\alpha}(x,t)dt\Big| \tag{6}$$

$$= \Big|\int_{2x}^\infty \Big(\int_x^t (f')_x(u)du\Big)W_{n,\alpha}(x,t)dt + \int_x^{2x}\Big(\int_x^t (f')_x(u)du\Big)dt(1-\beta_{n,\alpha}(x,t))\Big|dt$$

$$\leq \Big|\int_{2x}^\infty (f(t)-f(x))W_{n,\alpha}(x,t)dt\Big| + |f'(x^+)|\Big|\int_{2x}^\infty (t-x)W_{n,\alpha}(x,t)dt\Big|$$

$$+ \Big|\int_x^{2x}(f')_x(u)du\Big||(1-\beta_{n,\alpha}(x,2x)| + \int_x^{2x}|(f')_x(t)||1-\beta_n(x,t)|dt$$

$$\leq \frac{C}{x}\int_{2x}^{\infty} W_{n,\alpha}(x,t)t^{\gamma}|t-x|dt + \frac{|f(x)|}{x^2}\int_{2x}^{\infty} W_{n,\alpha}(x,t)(t-x)^2dt$$

$$+ |f'(x^+)|\int_{2x}^{\infty} W_{n,\alpha}(x,t)|(t-x)|dt + \frac{\lambda(2+\alpha x)}{nx}(|f(2x)-f(x)-xf'(x^+)|$$

$$+ \frac{\lambda(2+\alpha x)}{n}\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{v}}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}((f')_x).$$

Using Holder's inequality, and Lemma 1, we estimate the first two terms in the right hand side of (6) as follows:

$$\text{(7)}\qquad \frac{C}{x}\int_{2x}^{\infty} W_{n,\alpha}(x,t)t^{\gamma}|t-x|dt + \frac{|f(x)|}{x^2}\int_{2x}^{\infty} W_{n,\alpha}(x,t)(t-x)^2dt$$

$$\leq \frac{C}{x}\Big(\int_{2x}^{\infty} W_{n,\alpha}(x,t)t^{2\gamma}dt\Big)^{\frac{1}{2}}\Big(\int_{0}^{\infty} W_{n,\alpha}(x,t)(t-x)^2dt\Big)^{\frac{1}{2}}$$

$$+ \frac{|f(x)|}{x^2}\int_{2x}^{\infty} W_{n,\alpha}(x,t)(t-x)^2dt$$

$$\leq C2^{\gamma}O(n^{-\gamma/2})\frac{\sqrt{\lambda x(2+\alpha x)}}{\sqrt{n}} + |f(x)|\frac{\lambda(2+\alpha x)}{nx}.$$

Finally the third term of the right side of (6) is estimated as follows:

$$\text{(8)}\qquad |f'(x^+)|\int_{2x}^{\infty} W_{n,\alpha}(x,t)|t-x|dt \leq |f'(x^+)|\int_{0}^{\infty} W_{n,\alpha}(x,t)|t-x|dt$$

$$\leq |f'(x^+)|\Big(\int_{0}^{\infty} W_{n,\alpha}(x,t)(t-x)^2dt\Big)^{\frac{1}{2}}\Big(\int_{0}^{\infty} W_{n,\alpha}(x,t)dt\Big)^{\frac{1}{2}}$$

$$= |f'(x^+)|\frac{\sqrt{\lambda x(2+\alpha x)}}{\sqrt{n}}.$$

Combining the estimates (6)–(8), we get

$$\text{(9)}\qquad |A_{n,\alpha}(f,x)| \leq \frac{\lambda(2+\alpha x)}{nx}(|f(2x)-f(x)-xf'(x^+)|$$

$$+ \frac{\lambda(2+\alpha x)}{n}\sum_{v=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{v}}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}((f')_x) + |f'(x^+)|\frac{\sqrt{\lambda x(2+\alpha x)}}{\sqrt{n}}$$

$$+ C2^{\gamma}O(n^{-\gamma/2})\frac{\sqrt{\lambda x(2+\alpha x)}}{\sqrt{n}} + |f(x)|\frac{\lambda(2+\alpha x)}{nx}.$$

Finally combining (4), (5) and (9), we get the desired result. This completes the proof of Theorem 1.

REMARK 3. It may be noted that under the assumption of Theorem 1, the convergence rate of the operators $S_{n,\alpha}(f,x)$ to f is $O(\frac{1}{\sqrt{n}})$, which is the convergence rate of the classical Szasz-Mirakyan operators.

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