DEMONSTRATIO MATHEMATICA
Vol. XLI No 4 2008

Vinod K. Bhardwaj, Indu Bala

STRONG INVARIANT A-SUMMABILITY WITH RESPECT TO A SEQUENCE OF MODULUS FUNCTIONS IN A SEMINORMED SPACE

Abstract. The object of this paper is to introduce some new strongly invariant A-summable sequence spaces defined by a sequence of modulus functions $\mathcal{F} = (f_k)$ in a seminormed space, when $A = (a_{nk})$ is a non-negative regular matrix. Various algebraic and topological properties of these spaces, and some inclusion relations between these spaces have been discussed. Finally, we study some relations between A-invariant statistical convergence and strong invariant A-summability with respect to a sequence of modulus functions in a seminormed space.

1. Introduction and preliminaries

By w we shall denote the space of all scalar sequences. ℓ_∞, c and c_0 denote the spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. A sequence $x \in \ell_\infty$ is said to be almost convergent if all Banach limits of x coincide (see Banach [2]). Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [12] proved that

$$\hat{c} = \{x \in \ell_\infty : \lim_{m\to\infty} t_{mn}(x) \text{ exists, uniformly in } n\},$$

where $t_{mn}(x) = (m+1)^{-1}\sum_{k=0}^{m} x_{k+n}$. The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox [15] and also independently by Freedman et al. [7] as follows.

$$[\hat{c}] = \{x \in \ell_\infty : \lim_{m\to\infty} t_{mn}(|x - le|) = 0 \text{ uniformly in } n, \text{ for some } l\},$$

where $e = (1, 1, 1, \ldots)$.

Schaefer [26] defined the σ-convergence as follows.

2000 *Mathematics Subject Classification*: 40A05, 40C05, 46A45.
Key words and phrases: sequence space, invariant mean, modulus function, A-statistical convergence.

Let σ be an one-to-one mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(ii) $\phi(e) = 1$, and
(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

Let V_σ denote the set of bounded sequences which have unique σ-mean. It is known (see [26]) that

$$V_\sigma = \{x \in \ell_\infty : \lim_{k\to\infty} t_{km}(x) = l \,\text{uniformly in}\; m\},$$

$l = \sigma\text{-}\lim x$, where $t_{km}(x) = \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \dots + x_{\sigma^k(m)}}{k+1}$.

Here $\sigma^k(m)$ denotes the k^{th} iterate of the mapping σ at m.

In case σ is the translation mapping $n \to n+1$, a σ-mean reduces to the unique Banach limit and V_σ reduces to $\hat{c}$. A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits; that is to say, if and only if for all $n \geq 0$, $j \geq 1$, $\sigma^j(n) \neq n$ (see Mursaleen [18]).

Just as the concept of almost convergence led naturally to the concept of strong almost convergence, σ-convergence leads naturally to the concept of strong σ-convergence. A sequence $x = (x_k)$ is said to be strongly σ-convergent if there exists a number l such that

$$(|x_k - l|) \in V_\sigma \tag{1}$$

with the limit zero, (see Mursaleen [19]). We write $[V_\sigma]$ as the set of all strongly σ-convergent sequences. When (1) holds, we write $[V_\sigma]\text{-}\lim x = l$. Taking $\sigma(n) = n+1$, we obtain $[V_\sigma] = [\hat{c}]$ so that strong σ-convergence generalizes the concept of strong almost convergence. Note that $c \subset [V_\sigma] \subset V_\sigma \subset \ell_\infty$.

Let $A = (a_{nk})$ be an infinite matrix with real or complex numbers. A number sequence $x = (x_k)$ is called A-summable to a number l if the series $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ converge for all $n \in \mathbb{N}$ and $\lim_n A_n x = l$. A matrix method A is called regular if all convergent sequences $x = (x_k)$ are A-summable and $\lim_n A_n x = \lim_k x_k$. It is known (see [5], Theorem 4.1, II) that A is regular if and only if

(T_1) $\lim_n a_{nk} = 0\,(k \in \mathbb{N})$,
(T_2) $\lim_n \sum_k a_{nk} = 1$,
(T_3) $\sup_n \sum_k |a_{nk}| < \infty$.

We will denote the set of all non-negative regular matrices by $\mathcal{T}^+$. Cesàro method $C_1 = (c_{nk})$, where $c_{nk} = 1/n$ if $k \leq n$ and $c_{nk} = 0$ otherwise, is a well-known example of matrix method of summability. It is clear that $C_1 \in \mathcal{T}^+$. A similar summability method is defined as follows.

An increasing sequence of non-negative integers $\theta = (k_r)$ with $k_0 = 0$ is called a lacunary sequence if $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. A sequence $x = (x_k)$ is called lacunary convergent to l if $\lim_r h_r^{-1} \sum_{k \in I_r} x_k = l$. So, if $A_\theta = (a_{rk}^\theta)$ is the matrix, where $a_{rk}^\theta = 1/h_r$ if $k \in I_r$ and $a_{rk}^\theta = 0$ otherwise, then the A_θ-summability reduces to lacunary convergence. Clearly $A_\theta \in \mathcal{T}^+$.

A sequence $x = (x_k)$ is said to be strongly A-summable with index $p > 0$ to l if (see [13]) $\lim_n \sum_{k=1}^\infty a_{nk}|x_k - l|^p = 0$. The set of all strongly A-summable sequences is denoted by w_A^p.

Recall [16, 23] that $f : [0, \infty) \to [0, \infty)$ is called a modulus function if

(i) $f(t) = 0$ if and only if $t = 0$,
(ii) $f(t + u) \le f(t) + f(u)$ for all $t \ge 0, u \ge 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

Because of (ii), $|f(t) - f(u)| \le f(|t - u|)$ so that in view of (iv), f is continuous everywhere on $[0, \infty)$. A modulus may be unbounded (for example, $f(t) = t^p, 0 < p \le 1$) or bounded (for example, $f(t) = \frac{t}{1+t}$).

Ruckle [23], Maddox [16] and other authors used modulus function to construct new sequence spaces. In [10, 11, 22] some new sequence spaces are defined by means of a sequence of modulus functions $\mathcal{F} = (f_k)$.

The main object of this paper is to introduce and study the sequence spaces $w_0^p(A_\sigma, \mathcal{F}, q)$, $w^p(A_\sigma, \mathcal{F}, q)$ and $w_\infty^p(A_\sigma, \mathcal{F}, q)$ defined by means of sequence of modulus functions $\mathcal{F}$ in a seminormed space, when $A \in \mathcal{T}^+$. The definition of these sequence spaces is given in the following section. In §3, we propose to study various algebraic and topological properties of these spaces, and some inclusion relations between these spaces have been discussed. In §4, a new concept of A-invariant statistical convergence in a seminormed space is introduced. Some relations between A-invariant statistical convergence and strong invariant A-summability with respect to a sequence of modulus functions has been investigated.

2. Notation and definitions

Throughout the paper X denotes a seminormed space with seminorm q, $\mathcal{F} = (f_k)$ is a sequence of modulus functions and $A = (a_{nk})$ is a non-negative regular matrix. The symbol $w(X)$ denotes the space of all X-valued sequences. We define the following sequence spaces

$$w_0^p(A_\sigma, \mathcal{F}, q) = \{x \in w(X) : \lim_n \tau_{ni}(x) = 0 \,\text{uniformly in}\, i\},$$
$$w^p(A_\sigma, \mathcal{F}, q) = \{x \in w(X) : x - le \in w_0^p(A_\sigma, \mathcal{F}, q) \,\text{for some}\, l \in X\},$$
$$w_\infty^p(A_\sigma, \mathcal{F}, q) = \{x \in w(X) : \sup_{n,i} \tau_{ni}(x) < \infty\},$$

where $\tau_{ni}(x) = \sum_k a_{nk}[f_k(q(x_{\sigma^k(i)}))]^p$ and $p \geq 1$.

Some well-known spaces are obtained by specializing $X, q, A, \sigma, \mathcal{F}$ and p.

(i) If $X = \mathbb{C}, q(x) = |x|, f_k(t) = t$ for all k, $A = C_1, \sigma(i) = i + 1$ and $p = 1$, then $w_0^p(A_\sigma, \mathcal{F}, q) = [\hat{c}_0], w^p(A_\sigma, \mathcal{F}, q) = [\hat{c}]$ (Freedman et al. [7], Maddox [15]).

(ii) If $X = \mathbb{C}, q(x) = |x|, f_k(t) = t$ for all k, $A = C_1$ and $p = 1$, then $w^p(A_\sigma, \mathcal{F}, q) = [V_\sigma]$ (Mursaleen [19]).

(iii) If $X = \mathbb{C}, q(x) = |x|$, $f_k(t) = t$ for all k and $A = C_1$, then $w^p(A_\sigma, \mathcal{F}, q) = [V_\sigma]_p$ (Savas [25]).

(iv) If $X = \mathbb{C}, q(x) = |x|, \mathcal{F} = (f)$ and $p = 1$, then $w_0^p(A_\sigma, \mathcal{F}, q) = w_0(A_\sigma, f)$ and $w^p(A_\sigma, \mathcal{F}, q) = w(A_\sigma, f)$ (Nuray and Savas [20]).

(v) If $\mathcal{F} = (f)$ and $A = I$, the unit matrix, then $w_0^p(A_\sigma, \mathcal{F}, q) = c_0(f, \sigma, q)$, $w^p(A_\sigma, \mathcal{F}, q) = c(f, \sigma, q)$ and $w_\infty^p(A_\sigma, \mathcal{F}, q) = \ell_\infty(f, \sigma, q)$ (Altin and Isik [1]).

(vi) If $f_k(t) = t$ for all k, $A = A_\theta$ and $\sigma(i) = i + 1$, then $w_0^p(A_\sigma, \mathcal{F}, q) = (W, \theta, q)_0, w^p(A_\sigma, \mathcal{F}, q) = (W, \theta, q)$ and $w_\infty^p(A_\sigma, \mathcal{F}, q) = (W, \theta, q)_\infty$ (Colak et al. [3]).

We denote $w_0^p(A_\sigma, \mathcal{F}, q), w^p(A_\sigma, \mathcal{F}, q)$ and $w_\infty^p(A_\sigma, \mathcal{F}, q)$ by $w_0^p(A_\sigma, f, q)$, $w^p(A_\sigma, f, q)$ and $w_\infty^p(A_\sigma, f, q)$ when $\mathcal{F} = (f)$ and by $w_0^p(A_\sigma, q), w^p(A_\sigma, q)$ and $w_\infty^p(A_\sigma, q)$ when $f_k(t) = t$ for all k. If $x \in w^p(A_\sigma, \mathcal{F}, q)$, we say that x is strongly invariant A-summable to l with respect to the sequence of modulus functions $\mathcal{F}$ and a sequence $x \in w_\infty^p(A_\sigma, \mathcal{F}, q)$ is called strongly invariant A-bounded with respect to $\mathcal{F}$.

3. Linear topological structure of $w_0^p(A_\sigma, \mathcal{F}, q)$ space and inclusion theorems

In this section we examine some algebraic and topological properties of $w_0^p(A_\sigma, \mathcal{F}, q)$ space and investigate some inclusion relations between these spaces.

THEOREM 3.1. *$w_0^p(A_\sigma, \mathcal{F}, q), w^p(A_\sigma, \mathcal{F}, q)$ and $w_\infty^p(A_\sigma, \mathcal{F}, q)$ are linear spaces over the complex field $\mathbb{C}$.*

The proof is a routine verification by using standard techniques and hence is omitted.

THEOREM 3.2. *For a non-negative regular matrix $A = (a_{nk})$, $w_0^p(A_\sigma, \mathcal{F}, q)$ is a topological linear space with paranorm defined by*

$$g(x) = \sup_{n,i} \Big(\sum_k a_{nk} [f_k(q(x_{\sigma^k(i)}))]^p \Big)^{1/p}.$$

REMARK 3.3. From the properties of modulus function and seminorm it is clear that g is not a total paranorm.

LEMMA 3.4 ([21]). *Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$, we have $f(x) \le 2f(1)\delta^{-1}x$.*

THEOREM 3.5. *Let $A \in \mathcal{T}^+$ and $\mathcal{F} = (f_k)$ be a sequence of modulus functions such that $\lim_{t\to 0^+} \sup_k f_k(t) = 0$. Then $w_0^p(A_\sigma, q) \subseteq w_0^p(A_\sigma, \mathcal{F}, q)$.*

Proof. Let $x \in w_0^p(A_\sigma, q)$ and put $M = \sup_k f_k(1)$. Then

$$\sigma_n(i) = \sum_k a_{nk} [q(x_{\sigma^k(i)})]^p \to 0 \text{ as } n \to \infty, \text{ uniformly in } i.$$

Since $\lim_{t\to 0^+} \sup_k f_k(t) = 0$, for every $\epsilon > 0$ there is a number δ $(0 < \delta < 1)$ such that $f_k(t) < \epsilon$ $(k \in \mathbb{N})$ for $t \le \delta$. We can write

$$\sum_k a_{nk} [f_k(q(x_{\sigma^k(i)}))]^p = \sum_{k,\, q(x_{\sigma^k(i)}) \le \delta} a_{nk} [f_k(q(x_{\sigma^k(i)}))]^p + \sum_{k,\, q(x_{\sigma^k(i)}) > \delta} a_{nk} [f_k(q(x_{\sigma^k(i)}))]^p < \epsilon^p \sum_k a_{nk} + (2M\delta^{-1})^p \sigma_n(i)$$

by Lemma 3.4. Letting $n \to \infty$, it follows that $x \in w_0^p(A_\sigma, \mathcal{F}, q)$.

Our next result gives some sufficient conditions for $w_0^p(A_\sigma, q) = w_0^p(A_\sigma, \mathcal{F}, q)$.

THEOREM 3.6. *Let $A \in \mathcal{T}^+$ and $\mathcal{F} = (f_k)$ be a sequence of modulus functions such that $\lim_{t\to 0^+} \sup_k f_k(t) = 0$ and $\lim_{t\to\infty} \inf_k \frac{f_k(t)}{t} > 0$. Then $w_0^p(A_\sigma, q) = w_0^p(A_\sigma, \mathcal{F}, q)$.*

Proof. In view of Theorem 3.5, it is sufficient to show that $w_0^p(A_\sigma, \mathcal{F}, q) \subseteq w_0^p(A_\sigma, q)$.

Since $\lim_{t\to\infty} \inf_k \frac{f_k(t)}{t} > 0$, there exists a number $\beta > 0$ such that $f_k(t) > \beta t$ for $t > 0$ and $k \in \mathbb{N}$. For $x \in w_0^p(A_\sigma, \mathcal{F}, q)$, we have

$$\sum_k a_{nk} [q(x_{\sigma^k(i)})]^p < \beta^{-p} \sum_k a_{nk} [f_k(q(x_{\sigma^k(i)}))]^p$$

whence $x \in w_0^p(A_\sigma, q)$ and the proof is complete.

DEFINITION 3.7 ([27]). Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if there exists a constant L such that $q_2(x) \le Lq_1(x)$ for all $x \in X$. If each is stronger than the other, q_1 and q_2 are said to be equivalent.

THEOREM 3.8. *Let $A \in \mathcal{T}^+$, $\mathcal{F} = (f_k)$ be a sequence of modulus functions and q_1, q_2 be seminorms. Then*

(i) $w_0^p(A_\sigma, \mathcal{F}, q_1) \bigcap w_0^p(A_\sigma, \mathcal{F}, q_2) \subseteq w_0^p(A_\sigma, \mathcal{F}, q_1 + q_2)$,
(ii) *if q_1 is stronger than q_2, we have $w_0^p(A_\sigma, \mathcal{F}, q_1) \subseteq w_0^p(A_\sigma, \mathcal{F}, q_2)$,*
(iii) *if q_1 is equivalent to q_2, we have $w_0^p(A_\sigma, \mathcal{F}, q_1) = w_0^p(A_\sigma, \mathcal{F}, q_2)$.*

Proof. The proof of (i) is straightforward.

(ii) Let $x \in w_0^p(A_\sigma, \mathcal{F}, q_1)$. Then

$$\begin{aligned}\sum_k a_{nk}[f_k(q_2(x_{\sigma^k(i)}))]^p &\le \sum_k a_{nk}[f_k(Lq_1(x_{\sigma^k(i)}))]^p \\ &\le (1+[L])^p \sum_k a_{nk}[f_k(q_1(x_{\sigma^k(i)}))]^p \\ &\to 0 \text{ as } n \to \infty, \text{ uniformly in } i.\end{aligned}$$

Hence $x \in w_0^p(A_\sigma, \mathcal{F}, q_2)$.

4. Comparison with A-invariant statistical convergence

In this section we investigate some inclusion relations between A-invariant statistical convergence and strong invariant A-summability with respect to a sequence of modulus functions $\mathcal{F}$.

The idea of statistical convergence was introduced by Fast [6] and studied by various authors (e.g. [4], [9], [10], [17], [24]).

For $A \in \mathcal{T}^+$, Freedman and Sember [8] defined A-density as follows.

DEFINITION 4.1 ([8]). A set $K = \{k_i\} \subset \mathbb{N}$, with $k_i < k_{i+1}$ for all i (called an index set), is said to have A-density $\delta_A(K)$ equal to d if the characteristic sequence of K is A-summable to d, that is, $\lim_n \sum_{k \in K} a_{nk} = d$.

In particular case $A = C_1$, the A-density is called the asymptotic density.

Using A-density, we introduce the following definition.

DEFINITION 4.2. Let $A \in \mathcal{T}^+$. An X-valued sequence $x = (x_k)$ is said to be A-invariant statistically convergent to $l \in X$, briefly $st(A_\sigma, q)$-$\lim x = l$, if for each $\epsilon > 0$,

$$\lim_n \sum_{k \in L_{\epsilon,i}} a_{nk} = 0 \text{ uniformly in } i,$$

where $L_{\epsilon,i} = \{k : q(x_{\sigma^k(i)} - l) \ge \epsilon\}$.

We shall denote the set of all A-invariant statistically convergent sequences by $st(A_\sigma, q)$. If $X = \mathbb{C}, q(x) = |x|$, then this definition reduces to the definition introduced by Nuray and Savas [20].

THEOREM 4.3. *Let $A \in \mathcal{T}^+$. If $\mathcal{F} = (f_k)$ is a sequence of modulus functions which satisfies*

$$(M_1) \qquad \inf_k f_k(t) > 0 \ (t > 0),$$

then $w^p(A_\sigma, \mathcal{F}, q)\text{-}\lim x = l$ implies $st(A_\sigma, q)\text{-}\lim x = l$.

Proof. Let $\epsilon > 0$. If (M_1) holds then there exists a number $s > 0$ such that $f_k(\epsilon) \geq s$. If $w^p(A_\sigma, \mathcal{F}, q)\text{-}\lim x = l$ and $L_{\epsilon,i} = \{k : q(x_{\sigma^k(i)} - l) \geq \epsilon\}$, then

$$\sigma_n(i) = \sum_k a_{nk}[f_k(q(x_{\sigma^k(i)} - l))]^p \geq \sum_{k \in L_{\epsilon,i}} a_{nk}[f_k(\epsilon)]^p \geq s^p \sum_{k \in L_{\epsilon,i}} a_{nk}$$

whence $\sum_{k \in L_{\epsilon,i}} a_{nk} \leq s^{-p}\sigma_n(i) \to 0$ as $n \to \infty$, uniformly in i. Hence $st(A_\sigma, q)\text{-}\lim x = l$.

THEOREM 4.4. *Let $A \in \mathcal{T}^+$ and $\mathcal{F} = (f_k)$ be a sequence of modulus functions which satisfies*

$$(M_2) \qquad \lim_{t \to 0^+} \sup_k f_k(t) = 0,$$

$$(M_3) \qquad \sup_t \sup_k f_k(t) < \infty.$$

Then $st(A_\sigma, q)\text{-}\lim x = l$ implies $w^p(A_\sigma, \mathcal{F}, q)\text{-}\lim x = l$.

Proof. Let $st(A_\sigma, q)\text{-}\lim x = l$, $h(t) = \sup_k f_k(t)$, $h = \sup_t h(t)$ and choose $\epsilon > 0$. For every $i \in \mathbb{N}$, we split the sum $\sigma_n(i) = \sum_k a_{nk}[f_k(q(x_{\sigma^k(i)} - l))]^p$ into two sums $\sum_1$ and $\sum_2$ over $L_{\epsilon,i} = \{k : q(x_{\sigma^k(i)} - l) \geq \epsilon\}$ and $\{k : q(x_{\sigma^k(i)} - l) < \epsilon\}$, respectively. Then by (M_3),

$$\sum_1 \leq h^p \sum_{k \in L_{\epsilon,i}} a_{nk}$$

and by the increase of f_k, we have

$$\sum_2 \leq [h(\epsilon)]^p \sum_k a_{nk}.$$

Since $\lim_n \sum_{k \in L_{\epsilon,i}} a_{nk} = 0$ uniformly in i and using (T_2), we get $\lim_n \sigma_n(i) \leq [h(\epsilon)]^p$ uniformly in i. By (M_2) it follows that $\lim_n \sigma_n(i) = 0$ uniformly in i, that is, $w^p(A_\sigma, \mathcal{F}, q)\text{-}\lim x = l$.

From Theorems 4.3 and 4.4, we deduce the following result.

COROLLARY 4.5. *Let $A \in \mathcal{T}^+$ and $\mathcal{F} = (f_k)$ be a sequence of modulus functions which satisfies (M_1), (M_2) and (M_3). Then $st(A_\sigma, q) = w^p(A_\sigma, \mathcal{F}, q)$.*

In the case $f_k = f(k \in \mathbb{N})$, the conditions (M_1) and (M_2) hold. Thus we get

COROLLARY 4.6. *Let $A \in \mathcal{T}^+$ and f be a bounded modulus function, then $st(A_\sigma, q) = w^p(A_\sigma, f, q)$.*

The next theorem establishes the relation between A-invariant statistical convergence and strong invariant A-summability for bounded sequences.

THEOREM 4.7. *Let $A \in \mathcal{T}^+$. If $\mathcal{F} = (f_k)$ is a sequence of modulus functions which satisfies (M_2), then $\ell_\infty(q) \bigcap st(A_\sigma, q) \subseteq \ell_\infty(q) \bigcap w^p(A_\sigma, \mathcal{F}, q)$.*

Proof. Assume that (M_2) holds. Then $h(t) = \sup_k f_k(t) < \infty (t > 0)$. If $st(A_\sigma, q)\text{-}\lim x = l$ and $q(x_k) \leq M$, then

$$f_k(q(x_{\sigma^k(i)} - l)) \leq f_k(M + q(l)) \leq h(M + q(l)) < \infty$$

and $w^p(A_\sigma, \mathcal{F}, q)\text{-}\lim x = l$ follows from the proof of Theorem 4.4 with $h(M + q(l))$ instead of h.

Using also Theorem 4.3, we get

COROLLARY 4.8. *Let $A \in \mathcal{T}^+$ and $\mathcal{F} = (f_k)$ be a sequence of modulus functions which satisfies (M_1) and (M_2). Then*

$$\ell_\infty(q) \bigcap st(A_\sigma, q) = \ell_\infty(q) \bigcap w^p(A_\sigma, \mathcal{F}, q).$$

In the case $f_k = f(k \in \mathbb{N})$ from Corollary 4.8, we deduce

COROLLARY 4.9. *For any modulus function f and $A \in \mathcal{T}^+$,*

$$\ell_\infty(q) \bigcap st(A_\sigma, q) = \ell_\infty(q) \bigcap w^p(A_\sigma, f, q).$$

Acknowledgement. We wish to thank the referee for her/his several valuable suggestions that have improved the presentation of the paper.

References

[1] Y. Altin and M. Isik, *On some new seminormed sequence spaces defined by modulus functions*, Hokkaido Math. J. **35** (2006), 565–572.

[2] S. Banach, *Theorie des Operations Linéaries*, Warszawa, 1932.

[3] R. Colak, B. C. Tripathy and M. Et, *Lacunary strongly summable sequences and q-lacunary almost statistical convergence*, Vietnam J. Math. **34** (2) (2006), 129–138.

[4] J. S. Connor, *The statistical and strong p-Cesàro convergence of sequences*, Analysis **8** (1988), 47–63.

[5] R. G. Cooke, *Infinite Matrices and Sequence Spaces*, Macmillan and Co., London 1950.

[6] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[7] A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesàro type summability spaces*, Proc. London Math. Soc. **37** (3) (1978), 508–520.

[8] A. R. Freedman and J. J. Sember, *Densities and summability*, Pacific J. Math. **95** (1981), 293–305.
[9] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.
[10] E. Kolk, *The statistical convergence in Banach spaces*, Acta Comment. Univ. Tartu. **928** (1991), 41–52.
[11] E. Kolk, *On strong boundedness and summability with respect to a sequence of moduli*, Tartu Ül. Toimetised **960** (1993), 41–50.
[12] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.
[13] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford **18** (2) (1967), 345–355.
[14] I. J. Maddox, *A new type of convergence*, Math. Proc. Camb. Philos. Soc. **83** (1978), 61–64.
[15] I. J. Maddox, *On strong almost convergence*, Math. Proc. Camb. Philos. Soc. **85** (1979), 345–350.
[16] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Philos. Soc. **100** (1986), 161–166.
[17] I. J. Maddox, *Statistical convergence in a locally convex space*, Math. Proc. Camb. Philos. Soc. **104** (1988), 141–145.
[18] Mursaleen, *On some new invariant matrix methods of summability*, Quart. J. Math. Oxford **34** (1983), 77–86.
[19] Mursaleen, *Matrix transformation between some new sequence spaces*, Houston J. Math. **9** (1983), 505–509.
[20] F. Nuray and E. Savas, *Invariant statistical convergence and A-invariant statistical convergence*, Indian J. Pure Appl. Math. **25** (3) (1994), 267–274.
[21] S. Pehlivan and B. Fisher, *On some sequence spaces*, Indian J. Pure Appl. Math. **25** (10) (1994), 1067–1071.
[22] S. Pehliyan and B. Fisher, *Lacunary strong convergence with respect to a sequence of modulus functions*, Comment. Math. Univ. Carolinae **36** (1) (1995), 69–76.
[23] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. **25** (1973), 973–978.
[24] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30** (2) (1980), 139–150.
[25] E. Savas, *Strongly σ-convergent sequences*, Bull. Calcutta Math. Soc. **81** (4) (1989), 295–300.
[26] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. **36** (1972), 104–110.
[27] A. Wilansky, *Functional Analysis*, Blaisdell Publishing Company, New York, 1964.

DEPARTMENT OF MATHEMATICS
KURUKSHETRA UNIVERSITY
KURUKSHETRA-136 119, INDIA
e-mails: vinodk_ bhj @rediffmail.com, bansal_ indu @rediffmail.com

Received November 6, 2007; revised version March 21, 2008.