DEMONSTRATIO MATHEMATICA
Vol. XLI No 4 2008

Janusz Brzdęk

ON SOLUTIONS OF A GENERALIZATION OF THE REYNOLDS FUNCTIONAL EQUATION

Abstract. Let $(X, \cdot)$ be a group endowed with a topology and $F : \mathbb{C} \to X$. Under some assumptions on X and F, we describe the solutions $f : X \to \mathbb{C}$ of the functional equation

$$f(F(f(y)) \cdot x) = f(y)f(x),$$

that are continuous at a point or (universally, Baire, Christensen or Haar) measurable. We also show some consequences of those results.

Throughout the paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ stand, as usual, for the sets of positive integers, integers, reals and complex numbers, respectively.

In a very simple and natural way one may come across the problem of solving the following two functional equations

$$f(x + f(y)) = f(x) + f(y), \tag{1}$$

$$g(xg(y)) = g(x)g(y). \tag{2}$$

Namely let $\mathbb{K}$ be a field and $f : \mathbb{K} \to \mathbb{K}$, $g : \mathbb{K} \setminus \{0\} \to \mathbb{K} \setminus \{0\}$. Define binary operations $* : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $\circ : (\mathbb{K} \setminus \{0\}) \times (\mathbb{K} \setminus \{0\}) \to \mathbb{K} \setminus \{0\}$ by:

$$x * y = x + f(y), \qquad x \circ y = xg(y).$$

It is easy to check that the operation $*$ ($\circ$, respectively) is associative if and only if f (g, respectively) satisfies equation (1) ((2), respectively). Some further information concerning equations (1), (2) and the subsequent two similar functional equations

$$h(x + h(y)) = h(x)h(y), \tag{3}$$

$$j(xj(y)) = j(x) + j(y) \tag{4}$$

one can find in [1]–[5], [9]–[14], [17] and [18].

2000 *Mathematics Subject Classification*: 39B52.
Key words and phrases: functional equation, measurable solution, Reynolds operator, multiplicative symmetry, topological group.

All these four equations are connected with the problem of multiplicative symmetry originating in the operator theory (see e.g. [1]–[4]). Moreover, equation (2) arises in the averaging theory applied to the turbulent fluid motions (see e.g. [1, p. 330-1]). It is also one of conditions defining the Reynolds operator (see e.g. [15] and [18]); therefore we suggest to name it the Reynolds equation.

Z. Daróczy [9] has determined the solutions of (2) (in the class of functions $f : \mathbb{R} \to \mathbb{R}$) that are differentiable, the continuous solutions, and the solutions that are bounded; he has also proved that there exist nonmeasurable solutions of (2). Later the equation has been studied by N. Brillouët-Belluot [2]–[5], J. Dhombres [10]-[14], C.F.K. Jung, V. Boonyasombat, G. Barbançon and J.R. Jung [17], and Y. Matras [18]. The integrable solutions has been determined in [17]. For recent results, concerning the Hyers-Ulam stability of equations (1)–(4), we refer to [19] and [20].

Let $x_0 \in \mathbb{R}$ and $a : \mathbb{R} \to \mathbb{R}$ be an additive function (i.e. $a(x+y) = a(x) + a(y)$ for $x, y \in \mathbb{R}$) with $a(a(x)) = a(x)$ for $x \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by: $f(x) = a(x + x_0)$, $g(y) = \exp \circ f(\ln |y|)$ for $x, y \in \mathbb{R}$, $y \neq 0$. Then it is easy to check that the functions f, g are solutions of equations (1), (2), respectively. If a is continuous we obtain thus continuous solutions of the equations.

Note yet that if $f, j : \mathbb{R} \to \mathbb{R}$ are solutions of (1) and (4), respectively, then $\bar{f} = \exp \circ f$ and $\bar{j} = \exp \circ j$ satisfy the functional equations

$$\bar{f}(x + \ln \bar{f}(y)) = \bar{f}(x)\bar{f}(y), \tag{5}$$

$$\bar{j}(x \ln \bar{j}(y)) = \bar{j}(x)\bar{j}(y). \tag{6}$$

Let $(X, \cdot)$ be a group and $F : \mathbb{C} \to X$. Then each of the equations (2), (3), (5) and (6) is a particular case of the functional equation

$$f(F(f(y)) \cdot x) = f(y)f(x), \tag{7}$$

where the unknown function is $f : X \to \mathbb{C}$. On the other hand (7) is a particular case of the subsequent pexiderization of (2)

$$g(G(y)x) = h(y)h(x),$$

considered in [5] for continuous functions $g, h, G : \mathbb{K} \to \mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. (For yet another justification for equation (7) see Corollary 5). In this paper we deal with equation (7) in the situation where X is endowed with a topology. Under some additional assumptions on F and X we give descriptions of the solutions $f : X \to \mathbb{C}$ of (7) that are continuous at a point or (universally, Baire, Christensen or Haar) measurable. We generalize in this way in particular Theorem 5 and (to some extent) Theorem 7 in [17] (see Remark 4).

Let us start with the following very simple fact.

LEMMA 1. *Let $(X,\cdot)$ be a group and $F:\mathbb{C}\to X$. Suppose $f:X\to\mathbb{C}$ is a solution of equation (7) and $f(X)\neq\{0\}$. Then $f(X)$ is a multiplicative subgroup of $\mathbb{C}$ and*

$$f(F(f(y))^n\cdot x)=f(y)^n f(x)\qquad \text{for } x,y\in X, n\in\mathbb{Z}. \tag{8}$$

Proof. For every $x,y\in X$ we have

$$f(y)f(F(f(y))^{-1}\cdot x)=f(F(f(y))\cdot F(f(y))^{-1}\cdot x)=f(x), \tag{9}$$

which means that $f(y)\neq 0$ and $f(y)^{-1}f(x)\in f(X)$. Next, from (7) and (9), by induction we obtain (8). ∎

COROLLARY 1. *Let X and F be as in Lemma 1 and $f:X\to\mathbb{C}$ be a solution of equation (7). Suppose $f(X)\subset B(c,r):=\{a\in\mathbb{C}:|c-a|<r\}$ for some $c\in\mathbb{C}$ and $r\in\mathbb{R}$ with $0<r<|c|$. Then $f\equiv 1$.*

Proof. Lemma 1 implies $f(X)\subset B(c,r)$ is a multiplicative subgroup of $\mathbb{C}$. So $1\in B(c,r)$. Since $0\notin B(c,r)$, it is easily seen that the set $\{1\}$ is the only multiplicative subgroup of $\mathbb{C}$ that is contained in $B(c,r)$. Consequently $f(X)=\{1\}$. ∎

COROLLARY 2. *Let $(X,\cdot)$ be a group, $G:\mathbb{C}\to X$ and $g:X\to\mathbb{R}$ be a solution of*

$$g(G(g(y))\cdot x)=g(y)+g(x). \tag{10}$$

Suppose there is $c\in\mathbb{R}$ with $g(X)\subset(c,\infty)$ or $g(X)\subset(-\infty,c)$. Then $g\equiv 0$.

Proof. Let $f=\exp\circ g$, $F(x)=G\circ\ln|x|$ for $x\in\mathbb{C}\setminus\{0\}$ and $F(0)=0$. Then it is easy to check that (7) holds for every $x,y\in X$. Since, according to Lemma 1, $f(X)$ is a multiplicative subgroup of $\mathbb{R}$ and $f(X)\subset(e^c,\infty)$ or $f(X)\subset(0,e^c)$, $f\equiv 1$. ∎

In the next proposition we need the following three hypotheses.

(H) $(X,\cdot)$ is a group with the neutral element e, endowed with a topology such that the translations $X\ni x\to x\cdot y$ and $X\ni x\to y\cdot x$ are continuous for every $y\in X$.

(α) $F:\mathbb{C}\to X$ and $F(1)^k\in\lim_{z\to 0}F(z^{\operatorname{sign} l})^l$ with some $k,l\in\mathbb{Z}$, $l\neq 0$ (i.e. for every neighbourhood $V\subset X$ of $F(1)^k$ there exists $r\in\mathbb{R}$, $r>0$, such that $F(z^{\operatorname{sign} l})^l\in V$ for every $z\in B(0,r)$, $z\neq 0$).

(β) There exist $m,j\in\mathbb{Z}$, $j\neq 0$, such that $F(1)^m\in\lim_{z\to -1}F(z)^j$.

REMARK 1. Clearly (α) holds if F is continuous at 0 and $F(1)^k=F(0)^l$ for some $k,l\in\mathbb{Z}$, $l>0$, or if $F(1)^k=\lim_{z\to 0}F(\frac{1}{z})^l$ for some $k,l\in\mathbb{Z}$, $l<0$; (β) is valid if F is continuous at -1 and $F(1)^m=F(-1)^j$ for some $m,j\in\mathbb{Z}$, $j\neq 0$.

PROPOSITION 1. *Suppose that (H) and* (α) *hold,* $r \in \mathbb{R}$, $c \in \mathbb{C}$, $0 < r < |c|$, $f : X \to \mathbb{C}$ *is a solution of equation (7), the set* $f(X)$ *is not finite, and* $D := f^{-1}(B(c,r))$. *If* $f(X) \subset \mathbb{R}$ *or* (β) *holds, then* $e \notin \operatorname{int}(D \cdot D^{-1})$, *where* $D \cdot D^{-1} := \{x \cdot y^{-1} : x, y \in D\}$.

Proof. For the proof by contradiction suppose that there is a neighbourhood $U \subset X$ of e with $U \subset D \cdot D^{-1}$.

First assume that $|f(x)| \neq 1$ for some $x \in X$ and $l > 0$ ($l < 0$, respectively). Then, by Lemma 1, there is a sequence $\{x_n : n \in \mathbb{N}\} \subset X$ with $\lim_{n\to\infty} f(x_n) = 0$ ($\lim_{n\to\infty} |f(x_n)| = \infty$, respectively). Thus, on account of (α), there exists $m \in \mathbb{N}$ such that

$$|f(x_m)^l| < \frac{|c| - r}{2|c|}$$

and $F(f(x_m))^l \cdot F(1)^{-k} \in U$, which means that

$$|c - f(x_m)^l c| \geq |c| - |f(x_m)^l c| > |c| - \frac{|c| - r}{2} = r + \frac{|c| - r}{2} > r + |f(x_m)^l| r$$

and there are $y_1, y_2 \in D$ with $F(f(x_m))^l \cdot F(1)^{-k} \cdot y_1 = y_2$. Hence

$$B(c,r) \cap [f(x_m)^l B(c,r)] = B(c,r) \cap B(f(x_m)^l c, |f(x_m)^l| r) = \emptyset,$$

$$B(c,r) \ni f(y_2) = f(F(f(x_m))^l \cdot F(1)^{-k} \cdot y_1) = f(x_m)^l f(y_1) \in f(x_m)^l B(c,r).$$

This is a contradiction. Since the set $f(X)$ is not finite, that completes the proof in the case $f(X) \subset \mathbb{R}$.

It remains to consider the case $f(X) \subset S := \{a \in \mathbb{C} : |a| = 1\}$. Then, by Lemma 1, $f(X)$ is dense in S. Thus, according to (β), there exists $y_0 \in X$ with $F(f(y_0))^j \in U \cdot F(1)^m$ and

$$|f(y_0)^j + 1| < \frac{|c| - r}{|c|}.$$

Consequently $y_2 = F(f(y_0))^j \cdot F(1)^{-m} \cdot y_1$ for some $y_1, y_2 \in D$ and

$$|c - f(y_0)^j c| \geq 2|c| - |c + f(y_0)^j c| > 2|c| - (|c| - r) = |c| + r > 2r.$$

Further, in view of Lemma 1, $f(F(1)^{-m} \cdot y_1) = f(y_1)$. Since $|f(y_0)| = 1$,

$$B(c,r) \cap [f(y_0)^j B(c,r)] = B(c,r) \cap B(f(y_0)^j c, r) = \emptyset,$$

$$B(c,r) \ni f(y_2) = f(F(f(y_0))^j \cdot F(1)^{-m} \cdot y_1) = f(y_0)^j f(y_1) \in f(y_0)^j B(c,r).$$

This contradiction completes the proof. ∎

In the sequel, for every $k \in \mathbb{N}$, we write $U_k := \{z \in \mathbb{C} : z^k = 1\}$. Next we say a subset of a topological space is Baire measurable provided it has the property of Baire (see e.g. [21]).

Let us recall that, under suitable assumptions on the group $(X, \cdot)$ (see e.g. conditions (i)–(iv) of Theorem 1), a function $f : X \to \mathbb{C}$ is universally

(Baire, Christensen or Haar, respectively) measurable provided, for every open set $P \subset \mathbb{C}$, the set $f^{-1}(P)$ is universally (Baire, Christensen or Haar, respectively) measurable. Please note that in the subsequent Theorem 1 we assume less than a measurability of a function f; namely we only assume that $f^{-1}(P)$ is measurable for one particular open set P. The next two theorems are the main results of this paper.

THEOREM 1. *Let* (H) *and* (α) *be fulfilled and* $f : X \to \mathbb{C}$ *be a solution of* (7). *Assume that there exist* $c \in \mathbb{C}$, $r \in \mathbb{R}$, $0 < r < |c|$, *and a set* $P \subset B(c, r)$ *such that* $f^{-1}(\operatorname{int} P) \neq \emptyset$ *and one of the following four conditions is valid:*

(i) X *is abelian and metrizable with a complete metric and* $f^{-1}(P)$ *is universally measurable (see e.g. [7] or [8]).*
(ii) X *is a Baire space (see e.g. [21]) and* $f^{-1}(P)$ *is Baire measurable.*
(iii) X *is a Polish topological abelian group and* $f^{-1}(P)$ *is Christensen measurable (see e.g. [16]).*
(iv) X *is a locally compact topological group and* $f^{-1}(P)$ *is Haar measurable.*

Further suppose $f(X) \subset \mathbb{R}$ *or* (β) *holds. Then* $f(X) = U_k$ *for some* $k \in \mathbb{N}$. *Moreover* k *must be odd if additionally*

(γ) *there exist* $m, j \in \mathbb{Z}$ *with* $F(-1)^{2j-1} = F(1)^m$.

Proof. Write $P_0 := \operatorname{int} P$ and $T := f^{-1}(P)$. Since $T \neq \emptyset$, we have $f(X) \neq \{0\}$, whence, by Lemma 1, $f(X)$ is a multiplicative subgroup of $\mathbb{C}$ and consequently

$$f(X) \subset \bigcup_{b \in f(X)} b \cdot P_0 =: B_0.$$

Note that B_0, as a topological subspace of $\mathbb{C}$, has a countable basis of topology and therefore is a Lindelöf space. So there is a set $\{b_n : n \in \mathbb{N}\} \subset f(X)$ with

$$f(X) \subset \bigcup_{n \in \mathbb{N}} b_n \cdot P_0.$$

Take $x \in X$. There is $n \in \mathbb{N}$ with $f(x) \in b_n \cdot P_0$. Next, by Lemma 1,

$$f(F(b_n)^{-1} \cdot x) = b_n^{-1} f(x) \in b_n^{-1} b_n P_0 = P_0.$$

Hence $x \in F(b_n) \cdot T$.

In this way we have shown that

$$X = \bigcup_{n \in \mathbb{N}} F(b_n) \cdot T.$$

Further, for every $n \in \mathbb{N}$, the set $F(b_n) \cdot T$ is, respectively, universally, Baire, Christensen, or Haar measurable. Thus there is $m \in \mathbb{N}$ such that

$$e \in \operatorname{int} [F(b_m) \cdot T \cdot T^{-1} \cdot F(b_m)^{-1}] =: T_0,$$

(see e.g. Theorem 1 in [7], Proposition 1 in [6], Theorem 2 in [8] and Theorem in [22], respectively) and consequently

$$e = F(b_m)^{-1} \cdot F(b_m) \in F(b_m)^{-1} \cdot T_0 \cdot F(b_m) = \operatorname{int}[T \cdot T^{-1}].$$

(Here it seems that only the case of (iv) needs some comments. Note that then, for a compact neighbourhood $V \subset X$ of e, we have $V \subset \bigcup_{n\in\mathbb{N}} F(b_n)\cdot T$. Thus there is $m \in \mathbb{N}$ such that $V \cap [F(b_m) \cdot T]$ is of positive Haar measure and consequently we may apply the result in [22]).

Let $D := f^{-1}(B(c,r))$. Clearly $e \in \operatorname{int}(D\cdot D^{-1})$, because $T \subset D$. Hence, according to Proposition 1, $f(X)$ is finite, whence, by Lemma 1, $f(X) = U_k$ with some $k \in \mathbb{N}$.

Finally suppose (γ) holds and there is $z \in X$ with $f(z) = -1$. Then, according to Lemma 1, we have

$$-1 = f(z) = f(F(-1)^{-2j+1} \cdot F(-1)^{2j-1} \cdot z) = -f(F(1)^m \cdot z) = -f(z) = 1.$$

This is a contradiction. Consequently $-1 \notin f(X)$. ∎

REMARK 2. If the function f in Theorem 1 takes only real values and (γ) holds, then $f \equiv 1$, because $f(X) = U_k$ with some odd $k \in \mathbb{N}$.

COROLLARY 3. *Let $(X, \cdot)$ be a locally compact topological group, (α) be valid, μ be the Haar measure on X, $\mu(X) = \infty$, and $f : X \to \mathbb{C}$ be a solution of (7). Suppose $f(X) \subset \mathbb{R}$ or (β) holds. Then f is integrable if and only if $f \equiv 0$.*

Proof. Suppose $f \not\equiv 0$ is integrable. Then so is $|f|$. Since $\mu(X) = \infty$ and, by Theorem 1, $|f| \equiv 1$, this is a contradiction. ∎

THEOREM 2. *Let (H) and (α) be fulfilled and $f : X \to \mathbb{C}$ be a solution of (7), continuous at a point $x_0 \in X$. Suppose $f(X) \subset \mathbb{R}$ or (β) holds. Then $f \equiv 0$ or $f(X) = U_k$ for some $k \in \mathbb{N}$. Moreover k must be odd if additionally (γ) is valid.*

Proof. Suppose that $f(X) \neq \{0\}$. Then, by Lemma 1, $f(x_0) \neq 0$. Take $r \in \mathbb{R}$ with $0 < r < |f(x_0)|$. There is a neighbourhood $U \subset X$ of e with $f(U \cdot x_0) \subset B(f(x_0), r)$. Thus $e \in \operatorname{int}(D \cdot D^{-1})$ for $D := f^{-1}(B(f(x_0), r))$. Consequently, on account of Proposition 1, $f(X)$ is finite. We complete the proof in the same way as in the case of Theorem 1. ∎

COROLLARY 4. *Assume (H), (α) and (γ). Let $f : X \to \mathbb{R}$ be a solution of (7), continuous at a point. Then $f \equiv 1$ or $f \equiv 0$.*

Proof. Suppose $f(X) \neq \{0\}$. Then Theorem 2 implies $f(X) = U_k$ with some odd $k \in \mathbb{N}$. Since $f(X) \subset \mathbb{R}$, we must have $f(X) = U_1$. ∎

COROLLARY 5. *Assume that* (H), (α) *and* (γ) *are valid and define a binary operation* $\circ : (X \times \mathbb{C})^2 \to X \times \mathbb{C}$ *by:* $(y, w) \circ (x, z) = (F(w) \cdot x, wz)$. *Suppose*

$f : X \to \mathbb{R}$ is continuous at a point $x_0 \in X$. Then $gr\, f := \{(x, f(x)) : x \in X\}$ is a subgroupoid of the groupoid $(X \times \mathbb{C}, \circ)$ if and only if $f \equiv 0$ or $f \equiv 1$.

Proof. Assume $gr\, f$ is a subgroupoid of the groupoid $(X \times \mathbb{C}, \circ)$. Then

$$(F(f(y)) \cdot x, f(y)f(x)) = (y, f(y)) \circ (x, f(x)) \in gr\, f$$

for every $x, y \in X$. Hence f is a solution of (7) and consequently, by Corollary 4, $f \equiv 0$ or $f \equiv 1$.

The converse is trivial. ∎

REMARK 3. Replacing, in the proof above, Corollary 4 by Remark 2 we obtain an analogous result for measurable functions $f : X \to \mathbb{R}$.

COROLLARY 6. *Assume* (H) *and* (α). *Suppose X is connected and $f : X \to \mathbb{C}$ is a continuous solution of* (7). *If $f(X) \subset \mathbb{R}$ or* (β) *holds, then $f \equiv 1$ or $f \equiv 0$.*

Proof. Since $f(X)$ is connected, Theorem 2 implies $f(X) = \{0\}$ or $f(X) = U_1$. ∎

REMARK 4. In the case where one of the following two conditions is valid:
1° $(X, \cdot) = (\mathbb{C}, +)$ and $F(x) = x$ for $x \in \mathbb{C}$;
2° $(X, \cdot) = (\mathbb{R}, +)$, $f(X) \subset \mathbb{R}$, and $F(x) = \Re(x)$ (the real part of x) for $x \in \mathbb{C}$,

(7) takes the form

$$f(f(y) + x) = f(x)f(y) \tag{11}$$

and conditions $(\alpha), (\beta), (\gamma)$ hold with $m = -1$, $l = j = 1$ and $k = 0$. Thus Theorems 1 and 2, Remark 2, and Corollaries 3 and 4 generalize Theorem 5 and (to some extent) Theorem 7 in [17].

In view of the results that we have obtained so far, the solution $f : X \to \mathbb{C}$ of (7) with finite $f(X)$ seems to be quite significant. The subsequent theorem describes such solutions for commutative X. In the theorem we use the following notions: $\{d\} := \{d^n : n \in \mathbb{Z}\}$ for $d \in X$ and $r_k := \cos \frac{2\pi}{k} + i \sin \frac{2\pi}{k}$ for $k \in \mathbb{N}$. Moreover, for $k, n \in \mathbb{Z}$, we write $k|n$ provided $n = km$ with some $m \in \mathbb{Z}$.

THEOREM 3. *Let $(X, \cdot)$ be a commutative group and $F : \mathbb{C} \to X$. Then $f : X \to \mathbb{C}$ is a solution of equation* (7) *with the set $f(X) \neq \{0\}$ finite if and only if there exist $k \in \mathbb{N}$ and a selector $S \subset X$ of the factor group $X/\{F(r_k)\}$ (i.e. S has exactly one element in common with every element of $X/\{F(r_k)\}$) such that*

$$\text{if } \operatorname{card}\{F(r_k)\} \text{ is finite, then } k|\operatorname{card}\{F(r_k)\}, \tag{12}$$

$$F(r_k^m)\cdot S \subset \bigcup_{n\in\mathbb{Z}} F(r_k)^{m+nk}\cdot S \quad \textit{for } m\in\mathbb{Z}, \tag{13}$$

$$f(F(r_k)^m\cdot x) = r_k^m \quad \textit{for } x\in S, m\in\mathbb{Z}. \tag{14}$$

Proof. Suppose first that f has the form described in the statement. According to (12), formula (14) defining f is correct. Take $x, y \in X$. There exist $x_0, y_0 \in S$ and $l, m \in \mathbb{Z}$ such that $x = F(r_k)^l \cdot x_0$, $y = F(r_k)^m \cdot y_0$. Since, by (13), $F(r_k^m)\cdot x_0 = F(r_k)^{m+nk}\cdot z_0$ with some $z_0 \in S$, $n\in\mathbb{Z}$ and, by (14), $f(x) = r_k^l$, $f(y) = r_k^m$, we have $f(F(f(y))\cdot x) = f(F(r_k^m)\cdot F(r_k)^l\cdot x_0) = f(F(r_k)^{l+m+nk}\cdot z_0) = r_k^{l+m} = f(y)f(x)$.

Now assume $f : X \to \mathbb{C}$ satisfies (7). On account of Lemma 1, $f(X) = U_k$ with some $k\in\mathbb{N}$. Hence there exists $u\in X$ with $r_k = f(u)$. Let S_0 be a selector of the factor group $X/\{F(r_k)\}$. For every $x\in S_0$ there is $n(x)\in\mathbb{N}$ with $f(x) = r_k^{n(x)}$. Write $S = \{F(r_k)^{-n(x)}\cdot x : x\in S_0\}$. It is easily seen that $S\subset f^{-1}(\{1\})$ is a selector of $X/\{F(r_k)\}$ as well and, in view of Lemma 1,

$$\bigcup_{n\in\mathbb{Z}} F(r_k)^{nk}\cdot S = f^{-1}(\{1\})$$

(for every $x\in f^{-1}(\{1\})$, there are $l\in\mathbb{Z}$ and $y\in S$ with $x = F(r_k)^l\cdot y$ and $1 = f(x) = f(F(r_k)^l\cdot y) = r_k^l f(y) = r_k^l$, whence $k|l$).

Take $x\in S$ and $m\in\mathbb{Z}$. Then $f(z) = r_k^m$ for some $z\in X$ and, by (8),

$$f(F(r_k)^m\cdot x) = f(F(f(u))^m\cdot x) = f(u)^m f(x) = r_k^m,$$

$$f(F(r_k)^{-m}\cdot F(r_k^m)\cdot x) = f(u)^{-m} f(F(f(z))\cdot x) = r_k^{-m} f(z)f(x) = 1.$$

Thus we have shown (13) and (14). Since (14) implies (12), this completes the proof. ■

REMARK 5. Clearly (in Theorem 3), if $F(r_k^m) = F(r_k)^m$ for $m\in\mathbb{Z}$, then (13) holds for every $S\subset X$.

REMARK 6. Let $f : X\to\mathbb{C}$ have the form described in the statement of Theorem 3. Then it is easily seen that, for $m = 1,\ldots,k$,

$$\begin{aligned} f^{-1}(\{r_k^m\}) &= \bigcup_{n\in\mathbb{Z}} F(r_k)^{m+nk}\cdot S = F(r_k)^m\cdot \bigcup_{n\in\mathbb{Z}} F(r_k)^{nk}\cdot S \\ &= F(r_k)^m\cdot f^{-1}(\{1\}). \end{aligned}$$

Hence f is (universally, Baire, Christensen, or Haar, respectively) measurable if and only if the set

$$\widetilde{S} := \bigcup_{n\in\mathbb{Z}} F(r_k)^{nk}\cdot S$$

is (universally, Baire, Christensen, or Haar, respectively) measurable and f is continuous at a point $z \in X$ if and only if, for some $m \in \{1, \dots, k\}$,

$$z \in F(r_k)^m \cdot \operatorname{int} \widetilde{S}.$$

COROLLARY 7. *Assume (H). Let $G : \mathbb{R} \to X$ satisfy*

$$G(0)^k \in \lim_{x \to -\infty} G(x) \tag{15}$$

with some $k \in \mathbb{Z}$ (i.e. for every neighbourhood $V \subset X$ of $G(0)^k$ there is $b \in \mathbb{R}$ with $G(x) \in V$ for $x \in (-\infty, b)$) and $g : X \to \mathbb{R}$ be a solution of (10). Suppose g is continuous at a point $x_0 \in X$ or there exists a bounded set $P \subset \mathbb{R}$ such that $g^{-1}(\operatorname{int} P) \neq \emptyset$ and one of the conditions (i) – (iv) of Theorem 1 is valid, with $f^{-1}(P)$ replaced by $g^{-1}(P)$. Then $g \equiv 0$.

Proof. Let $f = \exp \circ g$ and define $F : \mathbb{C} \to X$ by

$$F(x) = \begin{cases} G(\ln(\Re(x))), & \text{if } \Re(x) > 0; \\ G(0)^k, & \text{if } \Re(x) \leq 0. \end{cases}$$

Then, on account of (15), F is continuous at 0 and $F(0) = G(0)^k = G(\ln 1)^k = F(1)^k$, which means (see Remark 1) that condition (α) is valid (with $l = 1$). Next, $f(y)f(x) = \exp(g(y) + g(x)) = \exp(g(G(g(y)) \cdot x)) = \exp(g(G(\ln(\exp(g(y)))) \cdot x)) = f(F(f(y)) \cdot x)$ for every $x, y \in X$. Thus, by Theorems 1 and 2, $f \equiv 1$, whence $g \equiv 0$. ∎

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia of Mathematics and its Applications, Vol. 31, Cambridge University Press, 1989.

[2] N. Brillouët-Belluot, *Multiplicative symmetry and related functional equations*, Aequationes Math. **51** (1996), 21–47.

[3] N. Brillouët-Belluot, *On Multiplicative Symmetry*, in: Advances in equations and inequalities, J. M. Rassias (ed.), Hadronic Press, 1999, 31–55.

[4] N. Brillouët-Belluot, *More about some functional equations of multiplicative symmetry*, Publicationes Math. Debrecen **58** (2001), 575–585.

[5] N. Brillouët-Belluot, *Pexider generalization of a functional equation of multiplicative symmetry*, Publicationes Math. Debrecen **64** (2004), 107–127.

[6] J. Brzdęk, *On almost additive functions*, Bull. Austral. Math. Soc. **54** (1996), 281–290.

[7] J. P. R. Christensen, *Borel structures in groups and semigroups*, Math. Scan. **28** (1971), 124-128.

[8] J. P. R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. **13** (1972), 255-260.

[9] Z. Daróczy, *Über die Funktionalgleichung: $\varphi(\varphi(x) \cdot y) = \varphi(x)\varphi(y)$*, Acta Univ. Debrecen Ser. Fiz. Chem. **8** (1962), 125–132.

[10] J. G. Dhombres, *Sur les opérateurs multiplicativement liés*, Mém. Soc. Math. France **27** (1971).
[11] J. G. Dhombres, *Quelques equations fonctionelles provenant de la théorie des moyennes*, C.R. Acad. Sci. Paris **273** (1971), 1–3.
[12] J. G. Dhombres, *Functional equations on semi-groups arising from the theory of means*, Nanta Math. **5** (3) (1972), 48–66.
[13] J. G. Dhombres, *Solution générale sur une groupe abélien de l'équation fonctionnelle:* $f(x * f(y)) = f(y * f(x))$, Aequationes Math. **15** (1977), 173–193.
[14] J. G. Dhombres, *Some Aspects of Functional Equations*, Chulalongkorn University, Department of Mathematics, Bangkok, 1979.
[15] M. L. Dubreil-Jacotin, *Propriétés algébriques des transformations de Reynolds*, C.R. Acad. Sci. Paris **236** (1953), 1950–1.
[16] P. Fisher and Z. Slodkowski, *Christensen zero sets and measurable convex functions*, Proc. Amer. Math. Soc. **79** (1980), 449–453.
[17] C. F. K. Jung, V. Boonyasombat, G. Barbançon and J. R. Jung, *On the functional equation* $f(x + f(y)) = f(x) \cdot f(y)$, Aequationes Math. **14** (1976), 41–48.
[18] Y. Matras, *Sur l'équation fonctionnelle:* $f(x \cdot f(y)) = f(x) \cdot f(y)$, Acad. Roy. Belg. Bull. Cl. Sci. (5) **55** (1969), 731–751.
[19] A. Najdecki, *On stability of a functional equation connected with the Reynolds operator*, J. Inequal. Appl. vol. 2007, Article ID 79816, 3 pages, 2007. doi:10.1155/2007/79816
[20] A. Najdecki, *On Stability of Some Generalizations of the Cauchy, d'Alembert and Quadratic Functional Equations*, Ph.D. Thesis, Pedagogical University in Cracow, 2006 (in Polish).
[21] J. C. Oxtoby, *Measure and Category*, Graduate Texts in Mathematics, Springer Verlag, 1971.
[22] K. Stromberg, *An elementary proof of Steinhaus's theorem*, Proc. Amer. Math. Soc. **36** (1972), 308.

DEPARTMENT OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
Podchorążych 2
30-084 KRAKÓW, POLAND
e-mail: jbrzdek@ap.krakow.pl

Received December 17, 2007; revised version March 1, 2008.