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ASYMPTOTIC BEHAVIOUR OF REAL TWO-DIMENSIONAL DIFFERENTIAL SYSTEM WITH A FINITE NUMBER OF CONSTANT DELAYS

Abstract. In this article stability and asymptotic properties of a real two-dimensional system $x'(t) = \mathbf{A}(t)x(t) + \sum_{j=1}^{n} \mathbf{B}_j(t)x(t-r_j) + \mathbf{h}(t, x(t), x(t-r_1), \dots, x(t-r_n))$ are studied, where $r_1 > 0, \dots, r_n > 0$ are constant delays, $\mathbf{A}$, $\mathbf{B}_1, \dots, \mathbf{B}_n$ are the matrix functions and $\mathbf{h}$ is the vector function. Generalization of results on stability of a two-dimensional differential system with one constant delay is obtained using the methods of complexification and Lyapunov-Krasovskii functional and some new corollaries and an example are presented.

1. Introduction

The investigation of the problem is based on the combination of the method of complexification and the method of Lyapunov-Krasovskii functional, which is to a great extent effective for two-dimensional systems. This combination was successfully used in [2] for two-dimensional system of ODE's and in [1] for system with one constant delay and led to interesting results.

This article is related to paper [3] where asymptotic properties of system with finite number of constant delays were studied. The aim is, under some special conditions, to improve the results presented in [3] and to illustrate the advancement with an example.

The subject of our study is the real two-dimensional system

$$(0)\quad x'(t) = \mathbf{A}(t)x(t) + \sum_{j=1}^{n} \mathbf{B}_j(t)x(t-r_j) + \mathbf{h}(t, x(t), x(t-r_1), \dots, x(t-r_n)),$$

where $\mathbf{A}(t) = \big(a_{ik}(t)\big)$, $\mathbf{B}_j(t) = \big(b_{jik}(t)\big)$ $(i, k = 1, 2)$ for $j \in \{1, \dots, n\}$ are real square matrices and

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$$\mathbf{h}(t,x,y_1,\dots,y_n) = \big(h_1(t,x,y_1,\dots,y_n), h_2(t,x,y_1,\dots,y_n)\big)$$

is a real vector function. We suppose that the functions a_{ik} are locally absolutely continuous on $[t_0,\infty)$, b_{jik} are locally Lebesgue integrable on $[t_0,\infty)$ and the function $\mathbf{h}$ satisfies Carathéodory conditions on

$$[t_0,\infty) \times \{[x_1,x_2] \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < R^2\} \times \{[y_{11},y_{12}] \in \mathbb{R}^2 \colon y_{11}^2 + y_{12}^2 < R^2\} \\ \times \dots \times \{[y_{n1},y_{n2}] \in \mathbb{R}^2 \colon y_{n1}^2 + y_{n2}^2 < R^2\},$$

where $0 < R \le \infty$ is a real constant.

The following notation will be used throughout the article:

$\mathbb{R}$	set of all real numbers
$\mathbb{R}_+$	set of all positive real numbers
$\mathbb{R}_+^0$	set of all nonnegative real numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{N}$	set of all positive integers
$\operatorname{Re} z$	real part of z
$\operatorname{Im} z$	imaginary part of z
$\overline{z}$	complex conjugate of z
$AC_{\text{loc}}(I,M)$	class of all locally absolutely continuous functions $I \to M$
$L_{\text{loc}}(I,M)$	class of all locally Lebesgue integrable functions $I \to M$
$K(I \times \Omega, M)$	class of all functions $I \times \Omega \to M$ satisfying Carathéodory conditions on $I \times \Omega$.

Introducing complex variables $z = x_1 + ix_2$, $w_1 = y_{11} + iy_{12}$, $\dots$, $w_n = y_{n1} + iy_{n2}$, we can rewrite the system (0) into an equivalent equation with complex-valued coefficients

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{j=1}^{n} \big[A_j(t)z(t-r_j) + B_j(t)\overline{z}(t-r_j)\big] \\ + g(t, z(t), z(t-r_1), \dots, z(t-r_n)),$$

where

$$a(t) = \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)),$$
$$b(t) = \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)),$$
$$A_j(t) = \frac{1}{2}(b_{j11}(t) + b_{j22}(t)) + \frac{i}{2}(b_{j21}(t) - b_{j12}(t)),$$

$$B_j(t) = \frac{1}{2}(b_{j11}(t) - b_{j22}(t)) + \frac{i}{2}(b_{j21}(t) + b_{j12}(t)),$$

$$g(t, z, w_1, \dots, w_n) = h_1(t, \frac{1}{2}(z + \overline{z}), \frac{1}{2i}(z - \overline{z}), \frac{1}{2}(w_1 + \overline{w_1}), \frac{1}{2i}(w_1 - \overline{w_1}), \dots, \frac{1}{2i}(w_n - \overline{w_n})) + ih_2(t, \frac{1}{2}(z + \overline{z}), \frac{1}{2i}(z - \overline{z}), \frac{1}{2}(w_1 + \overline{w_1}), \frac{1}{2i}(w_1 - \overline{w_1}), \dots, \frac{1}{2}(w_n + \overline{w_n}), \frac{1}{2i}(w_n - \overline{w_n})).$$

Conversely, the last equation can be written in the real form (0) as well, the relations are similar as in [2].

2. Results

We study the equation

$$(1) \quad z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{j=1}^{n} \big[A_j(t)z(t - r_j) + B_j(t)\overline{z}(t - r_j)\big] + g(t, z(t), z(t - r_1), \dots, z(t - r_n)),$$

where r_j are positive constants for $j = 1, \dots, n$, $A_j, B_j \in L_{\text{loc}}(J, \mathbb{C})$, $a, b \in AC_{\text{loc}}(J, \mathbb{C})$, $g \in K(J \times \Omega, \mathbb{C})$, where $J = [t_0, \infty)$, $\Omega = \{(z, w_1, \dots, w_n) \in \mathbb{C}^{n+1} \colon |z| < R, |w_j| < R,\ j = 1, \dots, n\}$, $R > 0$. Denote $r = \max\{r_j \colon j = 1, \dots, n\}$.

In this article we consider the case

$$(2') \qquad \liminf_{t\to\infty}\big(|\operatorname{Im} a(t)| - |b(t)|\big) > 0$$

and study the behavior of solutions of (1) under this assumption.

Obviously, this case is included in the case $\liminf_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0$ considered in [3], but in this special case we are able to derive more useful results as we will see later in an example. The idea is based upon the well known result that the condition $|a| > |b|$ in an autonomous equation $z' = az + b\overline{z}$ ensures that zero is a focus, a centre or a node while under the condition $|\operatorname{Im} a| > |b|$ zero can be just a focus or a centre. Details are contained in [2].

The inequality (2') is equivalent to the existence of $T \geq t_0 + r$ and $\mu > 0$ such that

$$(2) \qquad |\operatorname{Im} a(t)| > |b(t)| + \mu \quad \text{for } t \geq T - r.$$

Denote

$$(3) \quad \gamma(t) = \operatorname{Im} a(t) + \sqrt{(\operatorname{Im} a(t))^2 - |b(t)|^2}\,\operatorname{sgn}(\operatorname{Im} a(t)), \quad c(t) = -ib(t).$$

Since $|\gamma(t)| > |\operatorname{Im} a(t)|$ and $|c(t)| = |b(t)|$, the inequality

$$(4) \qquad |\gamma(t)| > |c(t)| + \mu$$

is true for all $t \geq T - r$. It is easy to verify that $\gamma, c \in AC_{\mathrm{loc}}([T-r,\infty),\mathbb{C})$.

For the purpose of this paper we denote

$$\vartheta(t) = \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) + |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2}. \tag{5}$$

In the text we will consider following conditions:

(i) The numbers $T \geq t_0 + r$ and $\mu > 0$ are such that (2) holds.

(ii) There are functions $\kappa_0, \kappa_1, \dots, \kappa_n, \lambda\colon [T,\infty) \to \mathbb{R}$ such that

$$|\gamma(t)g(t,z,w_1,\dots,w_n) + c(t)\overline{g}(t,z,w_1,\dots,w_n)| \leq \kappa_0(t)|\gamma(t)z(t) + c(t)\bar{z}(t)| + \sum_{j=1}^{n} \kappa_j(t)|\gamma(t-r_j)w_j + c(t-r_j)\overline{w_j}| + \lambda(t)$$

for $t \geq T$, $|z| < R$ and $|w_j| < R$ for $j = 1,\dots,n$, where $\kappa_0, \lambda \in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$.

(iii) $\beta \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}^0_+)$ is a function satisfying

$$\beta(t) \geq \psi(t) \quad \text{a. e. on } [T,\infty), \tag{6}$$

where ψ is defined for every $t \geq T$ by

$$\psi(t) = \max_{j=1,\dots,n} \left\{ \kappa_j(t) + (|A_j(t)| + |B_j(t)|)\frac{|\gamma(t)| + |c(t)|}{|\gamma(t-r_j)| - |c(t-r_j)|} \right\}. \tag{7}$$

(iv) The function $\Lambda \in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$ satisfies the inequalities $\beta'(t) \leq \Lambda(t)\beta(t)$, $\theta(t) \leq \Lambda(t)$ for almost all $t \in [T,\infty)$, where the function θ is defined by

$$\theta(t) = \operatorname{Re} a(t) + \vartheta(t) + \kappa_0(t) + n\beta(t). \tag{8}$$

Clearly, if A_j, B_j, κ_j are absolutely continuous on $[T,\infty)$ for $j = 1,\dots,n$ and $\psi(t) \geq 0$ on $[T,\infty)$, we may choose $\beta(t) = \psi(t)$.

Under the assumption (i), we can estimate

$$|\vartheta| \leq \frac{|\operatorname{Re}(\gamma\gamma' - \bar{c}c')| + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \leq \frac{(|\gamma'| + |c'|)(|\gamma| + |c|)}{\gamma^2 - |c|^2} = \frac{|\gamma'| + |c'|}{|\gamma| - |c|} \leq \frac{1}{\mu}(|\gamma'| + |c'|),$$

hence the functions ϑ and θ are locally Lebesgue integrable on $[T,\infty)$. Moreover, if $\beta \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$, then in (iv) we may choose

$$\Lambda(t) = \max\Big(\theta(t), \frac{\beta'(t)}{\beta(t)}\Big),$$

from which one can see that we slightly generalized the situation considered in [3].

Notice that the condition (ii) implies that the functions $\kappa_j(t)$ are non-negative on $[T,\infty)$ for $j = 0,\dots,n$, and due to this, $\psi(t) \ge 0$ on $[T,\infty)$. Finally, if $\lambda(t) \equiv 0$ in (ii), then equation (1) has the trivial solution $z(t) \equiv 0$.

In the proof of the main theorem we will need

LEMMA 1. *Let a_1, a_2, b_1, $b_2 \in \mathbb{C}$ and $|a_2| > |b_2|$. Then*

$$\operatorname{Re}\frac{a_1 z + b_1\overline{z}}{a_2 z + b_2\overline{z}} \le \frac{\operatorname{Re}(a_1\overline{a_2} - b_1\overline{b_2}) + |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2}$$

for $z \in \mathbb{C}$, $z \neq 0$.

For the proof see [3] or [2].

THEOREM 1. *Let the conditions* (i), (ii), (iii) *and* (iv) *hold and* $\lambda(t) \equiv 0$.

a) *If*

$$\limsup_{t\to\infty} \int^{t} \Lambda(s)ds < \infty, \tag{9}$$

then the trivial solution of (1) *is stable on* $[T,\infty)$*;*

b) *if*

$$\lim_{t\to\infty} \int^{t} \Lambda(s)ds = -\infty, \tag{10}$$

then the trivial solution of (1) *is asymptotically stable on* $[T,\infty)$.

Proof. The proof is similar to that of Theorem 1 from [3].

Choose arbitrary $t_1 \ge T$. Let $z(t)$ be any solution of (1) satisfying the condition $z(t) = z_0(t)$ for $t \in [t_1 - r, t_1]$, where $z_0(t)$ is a continuous complex-valued initial function defined on $t \in [t_1 - r, t_1]$. Consider Lyapunov function

$$V(t) = U(t) + \beta(t)\sum_{j=1}^{n} \int_{t-r_j}^{t} U(s)ds, \tag{11}$$

where $U(t) = |\gamma(t)z(t) + c(t)\overline{z}(t)|$.

To simplify the computations, denote $w_j(t) = z(t - r_j)$ and write the functions of variable t without brackets, for example, z instead of $z(t)$.

From (11) we get

$$V' = U' + \beta'\sum_{j=1}^{n} \int_{t-r_j}^{t} U(s)ds + n\beta|\gamma z + c\overline{z}| \tag{12}$$

$$-\beta\sum_{j=1}^{n} |\gamma(t - r_j)w_j + c(t - r_j)\overline{w_j}|$$

for almost all $t \ge t_1$ for which $z(t)$ is defined and $U'(t)$ exists.

Denote $\mathcal{K} = \{t \geq t_1 \colon z(t) \text{ exists}, U(t) \neq 0\}$ and $\mathcal{M} = \{t \geq t_1 \colon z(t) \text{ exists}, U(t) = 0\}$. It is clear that the derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$, and the existence of the derivative almost everywhere in the set $\mathcal{M}$ can be proved in the same way as in [3].

In particular, the derivative U' exists for almost all $t \geq t_1$ for which $z(t)$ is defined, thus (12) holds for almost all $t \geq t_1$ for which $z(t)$ is defined.

Now turn our attention to the set $\mathcal{K}$. For almost all $t \in \mathcal{K}$ it holds that $UU' = U\big(\sqrt{(\gamma z + c\overline{z})(\overline{\gamma z} + \overline{c}z)}\big)' = \operatorname{Re}\big[(\gamma\overline{z} + \overline{c}z)(\gamma' z + \gamma z' + c'\overline{z} + c\overline{z}')\big]$. As $z(t)$ is a solution of (1), we have

$$\begin{aligned} UU' = {} & \operatorname{Re}\Big\{(\gamma\overline{z} + \overline{c}z)\Big[\gamma' z + c'\overline{z} + \gamma\Big(az + b\overline{z} + \sum_{j=1}^{n}(A_j w_j + B_j\overline{w_j}) + g\Big) \\ & + c\Big(\overline{az} + \overline{b}z + \sum_{j=1}^{n}(\overline{A_j}\overline{w_j} + \overline{B_j}w_j) + \overline{g}\Big)\Big]\Big\} \\ = {} & \operatorname{Re}\Big\{(\gamma\overline{z} + \overline{c}z)\Big[\gamma' z + c'\overline{z} + (\gamma a + c\overline{b})z + (\gamma b + c\overline{a})\overline{z} \\ & + \gamma\Big(\sum_{j=1}^{n}(A_j w_j + B_j\overline{w_j}) + g\Big) + c\Big(\sum_{j=1}^{n}(\overline{A_j}\overline{w_j} + \overline{B_j}w_j) + \overline{g}\Big)\Big]\Big\} \end{aligned}$$

for almost all $t \in \mathcal{K}$. Short computation gives $(\gamma a + c\overline{b})c = (\gamma b + c\overline{a})\gamma$, and from this we get

$$\begin{aligned} UU' \leq {} & \operatorname{Re}\{(\gamma\overline{z} + \overline{c}z)(\gamma' z + c'\overline{z})\} + \operatorname{Re}\Big\{(\gamma\overline{z} + \overline{c}z)(\gamma a + c\overline{b})\Big(z + \frac{c}{\gamma}\overline{z}\Big)\Big\} \\ & + \operatorname{Re}\Big\{(\gamma\overline{z} + \overline{c}z)\Big(\gamma\sum_{j=1}^{n}(A_j w_j + B_j\overline{w_j}) + c\sum_{j=1}^{n}(\overline{A_j}\overline{w_j} + \overline{B_j}w_j)\Big)\Big\} \\ & + \operatorname{Re}\{(\gamma\overline{z} + \overline{c}z)(\gamma g + c\overline{g})\}. \end{aligned}$$

Consequently,

$$\begin{aligned} UU' \leq U^2 \operatorname{Re}\Big(a + \frac{c}{\gamma}\overline{b}\Big) + U(|\gamma| + |c|)\Big(\sum_{j=1}^{n}|A_j w_j + B_j\overline{w_j}|\Big) \\ + U|\gamma g + c\overline{g}| + U^2 \operatorname{Re}\frac{\gamma' z + c'\overline{z}}{\gamma z + c\overline{z}} \end{aligned}$$

for almost all $t \in \mathcal{K}$. Applying Lemma 1 to the last term, we obtain

$$\operatorname{Re}\frac{\gamma' z + c'\overline{z}}{\gamma z + c\overline{z}} \leq \vartheta.$$

Using this inequality together with (7), the assumption (ii) and the relation

$\operatorname{Re}(a + \frac{c}{\gamma}\bar{b}) = \operatorname{Re} a$, we obtain

$$\begin{aligned}
UU' &\le U^2(\operatorname{Re} a + \vartheta + \kappa_0) + U \sum_{j=1}^{n} \big(\kappa_j |\gamma(t-r_j)w_j + c(t-r_j)\overline{w_j}|\big) \\
&\quad + U(|\gamma| + |c|)\Big(\sum_{j=1}^{n} \frac{|A_j||w_j| + |B_j||\overline{w_j}|}{|\gamma(t-r_j)| - |c(t-r_j)|}(|\gamma(t-r_j)| - |c(t-r_j)|)\Big) \\
&\le U^2(\operatorname{Re} a + \vartheta + \kappa_0) + U\Big\{\sum_{j=1}^{n}\Big[\kappa_j + (|A_j| + |B_j|)\frac{|\gamma| + |c|}{|\gamma(t-r_j)| - |c(t-r_j)|}\Big] \\
&\quad \times |\gamma(t-r_j)w_j + c(t-r_j)\overline{w_j}|\Big\} \\
&\le U^2(\operatorname{Re} a + \vartheta + \kappa_0) + U\psi \sum_{j=1}^{n} |\gamma(t-r_j)w_j + c(t-r_j)\overline{w_j}|
\end{aligned}$$

for almost all $t \in \mathcal{K}$. Consequently,

$$U' \le U(\operatorname{Re} a + \vartheta + \kappa_0) + \psi \sum_{j=1}^{n} |\gamma(t-r_j)w_j + c(t-r_j)\overline{w_j}| \tag{13}$$

for almost all $t \in \mathcal{K}$.

Recalling that $U'(t) = 0$ for almost all $t \in \mathcal{M}$, we can see that the inequality (13) is valid for almost all $t \ge t_1$ for which $z(t)$ is defined.

From (12) and (13) we have

$$\begin{aligned}
V' &\le U(\operatorname{Re} a + \vartheta + \kappa_0 + n\beta) + (\psi - \beta)\sum_{j=1}^{n} |\gamma(t-r_j)w_j + c(t-r_j)\overline{w_j}| \\
&\quad + \beta' \sum_{j=1}^{n} \int_{t-r_j}^{t} |\gamma(s)z(s) + c(s)\bar{z}(s)|\,ds.
\end{aligned}$$

As $\beta(t)$ fulfills the condition (6), we obtain

$$V'(t) \le U(t)\theta(t) + \beta'(t)\sum_{j=1}^{n} \int_{t-r_j}^{t} |\gamma(s)z(s) + c(s)\bar{z}(s)|\,ds,$$

and from the assumption (iv) (which is more general than relation (7) in [3]) we get

$$V'(t) - \Lambda(t)V(t) \le 0 \tag{14}$$

for almost all $t \ge t_1$ for which the solution $z(t)$ exists.

The rest of the proof is same as in the proof of Theorem 1 in [3]. ∎

REMARK 1. Since

$$\vartheta = \frac{\operatorname{Re}(\gamma\gamma' - \bar{c}c') + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \le \frac{(|\gamma'| + |c'|)(|\gamma| + |c|)}{\gamma^2 - |c|^2} = \frac{|\gamma'| + |c'|}{|\gamma| - |c|},$$

it follows from (4) that we can replace the function ϑ in (8) by $\frac{1}{\mu}(|\gamma'| + |c'|)$.

The proofs of following two corollaries are identical to the proofs of corresponding corollaries in [3].

COROLLARY 1. *Let the assumptions* (i), (ii) *and* (iii) *be fulfilled and* $\lambda(t) \equiv 0$. *If for some* $K \in \mathbb{R}_+$ *and* $T_1 \ge T$ *the function* $\beta(t)$ *satisfies* $\beta(T_1) = K$, $\beta(t) \le K$ *for all* $t \ge T_1$ *and*

$$\lim_{t\to\infty} \int^t [\theta^*(s)]_+ ds < \infty,$$

where $\theta^*(t) = \theta(t) - n\beta(t) + nK$ *and* $[\theta^*(t)]_+ = \max\{\theta^*(t), 0\}$, *then the trivial solution of* (1) *is stable.*

COROLLARY 2. *Assume that the conditions* (i), (ii) *and* (iii) *are valid with* $\lambda(t) \equiv 0$. *If* $\beta(t)$ *is monotone and bounded on* $[T, \infty)$ *and if*

$$\lim_{t\to\infty} \int^t [\theta(s)]_+ ds < \infty,$$

where $[\theta(t)]_+ = \max\{\theta(t), 0\}$, *then the trivial solution of* (1) *is stable.*

We use following Corollary 3 to find an important example which shows, in connection with the article [3], that it is worth to consider the condition (2').

COROLLARY 3. *Let* $a(t) \equiv a \in \mathbb{C}$, $b(t) \equiv b \in \mathbb{C}$, $|\operatorname{Im} a| > |b|$. *Suppose that* $\rho_0, \rho_1, \dots, \rho_n \colon [T, \infty) \to \mathbb{R}$ *are such that*

$$|g(t, z, w_1, \dots, w_n)| \le \rho_0(t)|z| + \sum_{j=1}^{n} \rho_j(t)|w_j| \tag{15}$$

for $t \ge T$, $|z| < R$, $|w_j| < R$ *for* $j = 1, \dots, n$ *and* $\rho_0 \in L_{\text{loc}}([T, \infty), \mathbb{R})$.

Let $\beta \in AC_{\text{loc}}([T, \infty), \mathbb{R}_+)$ *satisfy*

$$\beta(t) \ge \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \max_j \big(\rho_j(t) + |A_j(t)| + |B_j(t)|\big) \quad a.e.\ on\ [T, \infty)$$

for $j = 1, \dots, n$. *If*

$$\limsup_{t\to\infty} \int^t \max\left(\operatorname{Re} a + \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \rho_0(s) + n\beta(s), \frac{\beta'(s)}{\beta(s)}\right) ds < \infty, \tag{16}$$

then the trivial solution of equation (1) *is stable. If*

$$(17)\quad \lim_{t\to\infty}\int^t \max\left(\operatorname{Re} a + \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \rho_0(s) + n\beta(s), \frac{\beta'(s)}{\beta(s)}\right) ds = -\infty,$$

then the trivial solution of (1) *is asymptotically stable.*

Proof. First part of the proof is identical to the first part of the proof of Corollary 3 in [3]. We continue with the idea that since

$$\frac{|\gamma| + |c|}{|\gamma| - |c|} = \frac{|\operatorname{Im} a| + \sqrt{|\operatorname{Im} a|^2 - |b|^2} + |b|}{|\operatorname{Im} a| + \sqrt{|\operatorname{Im} a|^2 - |b|^2} - |b|} = \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}},$$

in view of (8) we obtain

$$\psi(t) = \max_j \psi_j(t) = \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \max_j\{\rho_j(t) + |A_j(t)| + |B_j(t)|\},$$

$$\theta(t) = \operatorname{Re} a + \frac{|\gamma| + |c|}{|\gamma| - |c|}\rho_0(t) + n\beta(t) = \operatorname{Re} a + \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \rho_0(t) + n\beta(t).$$

Since $\beta(t)$ is positive on $[T,\infty)$, we may choose $\Lambda(t) = \max\bigl(\theta(t), \frac{\beta'(t)}{\beta(t)}\bigr)$ and the assertion follows from Theorem 1. ∎

Now we are able to give an example mentioned before Corollary 3.

EXAMPLE 1. Consider equation (1), where $a(t) \equiv -\sqrt{5} + 2i$, $b(t) \equiv 1$, $A_j(t) \equiv 0$, $B_j(t) \equiv 0$ for $j \in \{1,\dots,n\}$,

$$g(t, z, w_1, \dots, w_n) = \frac{2}{\sqrt{3}} \mathrm{e}^{it} z + \sum_{j=1}^{n} \frac{1}{2n}(\sqrt{15} - \sqrt{14}) \mathrm{e}^{-t} w_j.$$

Assume that $t_0 = 0$ and $R = \infty$, r_j may be arbitrary positive constant delays. Put $T = t_0 + r$. Then $\rho_0(t) \equiv \frac{2}{\sqrt{3}}$, $\rho_j(t) = \frac{1}{2n}(\sqrt{15} - \sqrt{14}) \mathrm{e}^{-t}$. We have

$$\begin{aligned}\max&\left(\frac{|a| - |b|}{|a|}\operatorname{Re} a + \left(\frac{|a| + |b|}{|a| - |b|}\right)^{\frac{1}{2}} \rho_0(t) + n\beta(t), \frac{\beta'(t)}{\beta(t)}\right)\\ &= \max\left(-\frac{2}{3}\sqrt{5} + \sqrt{2}\frac{2}{\sqrt{3}} + n\beta(t), \frac{\beta'(t)}{\beta(t)}\right) \geq \frac{2}{3}(\sqrt{6} - \sqrt{5}) > 0\end{aligned}$$

for

$$\beta(t) \geq \left(\frac{|a| + |b|}{|a| - |b|}\right)^{\frac{1}{2}} \max_j\bigl(\rho_j(t) + |A_j(t)| + |B_j(t)|\bigr) = \frac{1}{n\sqrt{2}}(\sqrt{15} - \sqrt{14}) \mathrm{e}^{-t},$$

where $j \in \{1,\dots,n\}$, hence we cannot apply Corollary 3 from the paper [3].

On the other hand, if we use

$$\beta(t)=\frac{\sqrt{3}}{2n}(\sqrt{15}-\sqrt{14})\,\mathrm{e}^{-t}\geq\left(\frac{|\operatorname{Im}a|+|b|}{|\operatorname{Im}a|-|b|}\right)^{\frac{1}{2}}\max_j\bigl(\rho_j(t)+|A_j(t)|+|B_j(t)|\bigr),$$

where $j\in\{1,\dots,n\}$, we have

$$\begin{aligned}&\max\left(\operatorname{Re}a+\left(\frac{|\operatorname{Im}a|+|b|}{|\operatorname{Im}a|-|b|}\right)^{\frac{1}{2}}\rho_0(t)+n\beta(t),\frac{\beta'(t)}{\beta(t)}\right)\\&=\max\Bigl(-\sqrt{5}+2+n\frac{\sqrt{3}}{2n}(\sqrt{15}-\sqrt{14})\,\mathrm{e}^{-t},-1\Bigr)\\&\leq-\sqrt{5}+2+\frac{\sqrt{3}}{2}(\sqrt{15}-\sqrt{14})<-\frac{12}{100}<0.\end{aligned}$$

Thus Corollary 3 guarantees the stability and also asymptotic stability of the trivial solution of the considered equation.

In the following corollary, we denote

$$H_1(t)=\sqrt{\frac{|\operatorname{Im}a|-|b|}{|\operatorname{Im}a|+|b|}}\operatorname{Re}a+\rho_0(t)+n\max_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\},$$

$$H_2(t)=\sqrt{\frac{|\operatorname{Im}a|-|b|}{|\operatorname{Im}a|+|b|}}\frac{\rho_i'(t)}{\max_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\}},$$

where, for every t, the index i in H_2 is such that $\rho_i(t)+|A_i|+|B_i|=\max_j\bigl\{\rho_j(t)+|A_j|+|B_j|\bigr\}$.

Corollary 4. *Let $a(t)\equiv a\in\mathbb{C}$, $b(t)\equiv b\in\mathbb{C}$, $|\operatorname{Im}a|>|b|$ and $A_j(t)\equiv A_j\in\mathbb{C}$, $B_j(t)\equiv B_j\in\mathbb{C}$ for all $j\in\{1,\dots,n\}$. Let there exist $\rho_0,\rho_1,\dots,\rho_n\colon[T,\infty)\to\mathbb{R}$, ρ_0 locally Lebesgue integrable and $\rho_1,\dots,\rho_n$ locally absolutely continuous, such that* (15) *holds for $t\geq T$, $|z|<R$, $|w_j|<R$, $j\in\{1,\dots,n\}$. Suppose $\max_j\{\rho_j(t)+|A_j|+|B_j|\}>0$ on $[T,\infty)$ for $j\in\{1,\dots,n\}$. If*

$$\limsup_{t\to\infty}\int^t\max\bigl(H_1(s),H_2(s)\bigr)ds<\infty,$$

then the trivial solution of equation (1) *is stable. If*

$$\lim_{t\to\infty}\int^t\max\bigl(H_1(s),H_2(s)\bigr)ds=-\infty,$$

then the trivial solution of (1) *is asymptotically stable.*

Proof. We can choose

$$\beta(t) = \left(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\right)^{\frac{1}{2}} \max_j\{\rho_j(t) + |A_j| + |B_j|\}$$

in Corollary 3 since this function is locally absolutely continuous on $[T, \infty)$ for $j \in \{1, \dots, n\}$. ∎

The proofs of following theorems and corollaries except for Corollary 5 are omitted since they are almost identical to the proofs of corresponding propositions in [3].

THEOREM 2. *Let the assumptions* (i), (ii), (iii) *and* (iv) *hold and*

$$V(t) = |\gamma(t)z(t) + c(t)\overline{z}(t)| + \beta(t)\sum_{j=1}^{n}\int_{t-r_j}^{t}|\gamma(s)z(s) + c(s)\overline{z}(s)|ds, \tag{18}$$

where $z(t)$ is any solution of (1) *defined on $[t_1, \infty)$, where $t_1 \geq T$. Then*

$$\mu|z(t)| \leq V(t) \leq V(s)\exp\Bigl(\int_s^t \Lambda(\tau)d\tau\Bigr) + \int_s^t \lambda(\tau)\exp\Bigl(\int_\tau^t \Lambda(\sigma)d\sigma\Bigr)d\tau \tag{19}$$

for $t \geq s \geq t_1$.

From Theorem 2 we obtain several consequences.

COROLLARY 5. *Let the conditions* (i), (ii), (iii) *and* (iv) *be fulfilled and*

$$\limsup_{t\to\infty}\int_s^t \lambda(\tau)\exp\Bigl(-\int_s^\tau \Lambda(\sigma)d\sigma\Bigr)d\tau < \infty$$

for some $s \geq T$.

If $z(t)$ is any solution of (1) *defined for $t \to \infty$, then*

$$z(t) = O\Bigl[\exp\Bigl(\int_s^t \Lambda(\tau)d\tau\Bigr)\Bigr].$$

Proof. From the assumptions and (19) we can see that there are $K > 0$ and $S \geq s$ such that for $t \geq S$ we have

$$V(t)\exp\Bigl(-\int_s^t \Lambda(\tau)d\tau\Bigr) - V(s) \leq \int_s^t \lambda(\tau)\exp\Bigl(-\int_s^\tau \Lambda(\sigma)d\sigma\Bigr)d\tau \leq K < \infty.$$

Then

$$\mu|z(t)| \leq V(t) \leq (K + V(s))\exp\Bigl(\int_s^t \Lambda(\tau)d\tau\Bigr). \ \blacksquare$$

COROLLARY 6. *Let the assumptions* (i), (ii), (iii) *and* (iv) *hold and let*

$$\limsup_{t\to\infty}\Lambda(t) < \infty \quad \textit{and} \quad \lambda(t) = O(\mathrm{e}^{\eta t}), \tag{20}$$

where $\eta > \limsup_{t\to\infty} \Lambda(t)$. *If* $z(t)$ *is any solution of* (1) *defined for* $t \to \infty$, *then* $z(t) = O(\mathrm{e}^{\eta t})$.

REMARK 2. If $\lambda(t) \equiv 0$, from Corollary 6 we obtain the following statement: there is an $\eta^* < \eta_0 < \eta$ such that $z(t) = o(\mathrm{e}^{\eta_0 t})$ holds for the solution $z(t)$ defined for $t \to \infty$.

Consider now a special case of equation (1) with $g(t, z, w_1, \dots, w_n) \equiv h(t)$:

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{j=1}^{n} \big(A_j(t)z(t-r_j) + B_j(t)\overline{z}(t-r_j)\big) + h(t), \tag{21}$$

where $h(t) \in L_{\mathrm{loc}}([t_0,\infty),\mathbb{C})$.

COROLLARY 7. *Let the assumption* (i) *be satisfied and suppose*

$$\limsup_{t\to\infty} \big(|\gamma(t)| + |c(t)|\big) < \infty. \tag{22}$$

Let $\tilde{\beta} \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$ *be such that*

$$\tilde{\beta}(t) \geq \max_j \Big\{ \big(|A_j(t)| + |B_j(t)|\big) \frac{|\gamma(t)| + |c(t)|}{|\gamma(t-r_j)| - |c(t-r_j)|} \Big\} \quad \textit{a.e. on} \quad [T,\infty). \tag{23}$$

If h *is bounded,*

$$\limsup_{t\to\infty} \big[\mathrm{Re}\, a(t) + \vartheta(t) + n\tilde{\beta}(t)\big] < 0 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} < 0, \tag{24}$$

then any solution of equation (21) *is bounded.*

If $h(t) = O(\mathrm{e}^{\eta t})$ *for any* $\eta > 0$,

$$\limsup_{t\to\infty} \big[\mathrm{Re}\, a(t) + \vartheta(t) + n\tilde{\beta}(t)\big] \leq 0 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \leq 0,$$

then any solution of (21) *satisfies* $z(t) = o(\mathrm{e}^{\eta t})$ *for any* $\eta > 0$.

REMARK 3. If $h(t) \equiv 0$ in Corollary 7, then, with respect to Corollary 6 and Remark 2, we gain the following assertion:

Suppose that assumptions (i) and (22) hold and for $\tilde{\beta}$ from Corollary 7 the inequality (23) is valid. If conditions (24) are satisfied, then there is $\eta_0 < 0$ such that $z(t) = o(\mathrm{e}^{\eta_0 t})$ for any solution $z(t)$ of

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{j=1}^{n} \big(A_j(t)z(t-r_j) + B_j(t)\overline{z}(t-r_j)\big)$$

defined for $t \to \infty$.

THEOREM 3. *Let the assumptions* (i), (ii), (iii) *and* (iv) *be satisfied. Let* $\Lambda(t) \le 0$ *a.e. on* $[T^*, \infty)$, *where* $T^* \in [T, \infty)$. *If*

$$\lim_{t\to\infty} \int^{t} \Lambda(s)ds = -\infty \quad \text{and} \quad \lambda(t) = o(\Lambda(t)), \tag{25}$$

then any solution $z(t)$ *of equation* (1) *defined for* $t \to \infty$ *satisfies* $\lim_{t\to\infty} z(t) = 0$.

COROLLARY 8. *Let the assumptions* (i) *and* (22) *be satisfied, and* $\tilde{\beta} \in AC_{\text{loc}}([T, \infty), \mathbb{R}_+)$ *satisfy* (23). *If the conditions* (24) *are fulfilled and* $h \in L_{\text{loc}}([t_0, \infty), \mathbb{C})$ *satisfies* $\lim_{t\to\infty} h(t) = 0$, *then* $\lim_{t\to\infty} z(t) = 0$ *for any solution* $z(t)$ *of equation* (21).

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