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ON TYPICALLY REAL FUNCTIONS WHICH OMIT TWO CONJUGATED VALUES

Abstract. In this paper we discuss the class $T_{\rho,\theta}$ consisting of typically real functions which do not admit values $w_0 = \rho e^{i\theta}$ and $\overline{w_0}$. We estimate the second and the third coefficients of a function $f \in T_{\rho,\theta}$ and we determine the Koebe domain for the class of typically real functions with fixed second coefficient.

Let T denote the class of analytic functions f in the unit disk $\Delta \equiv \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$ for which the condition

$$\operatorname{Im} z \cdot \operatorname{Im} f(z) \geq 0 \quad \text{for} \quad z \in \Delta$$

is satisfied. The class T is called the class of typically real functions. Rogosinski [5] gave the explicit relation between a function $f \in T$ and a probability measure μ defined on $[-1, 1]$. Namely,

$$f \in T \iff f(z) = \int_{-1}^{1} \frac{z}{1 - 2zt + z^2} d\mu(t).$$

In 1977 Goodman [1] defined the universal typically real function

$$G(z) = \frac{1}{\pi} \tan \frac{\pi z}{1 + z^2} \tag{1}$$

and determined the Koebe domain for T.

THEOREM A [1]. *The Koebe domain for the class T is symmetric with respect to the real axis and the boundary of this domain in the upper half plane is given by the polar equation $\rho(\theta)e^{i\theta}$, where*

$$\rho(\theta) = \begin{cases} \frac{\pi \sin \theta}{4\theta(\pi - \theta)}, & \theta \in (0, \pi) \\ \frac{1}{4}, & \theta = 0 \text{ or } \theta = \pi. \end{cases} \tag{2}$$

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In this paper we study a subclass of the class T, consisting of functions, which do not admit values $w_0 = \rho e^{i\theta}$ and $\overline{w_0}$. This class we denote by $T_{\rho,\theta}$. The class $T_{\rho,\theta}$ is defined as follows

$$T_{\rho,\theta} = \{f \in T : f \neq \rho e^{\pm i\theta}\}, \quad 0 \leq \theta \leq \pi, \quad \rho \geq \rho(\theta),$$

where $\rho(\theta)$ is given by (2).

In our paper we will use the class T_M which was investigated by Koczan [2]. The class T_M consists of typically real bounded functions, i.e

$$T_M = \{f \in T : |f(z)| < M, \ z \in \Delta\}, \ M > 1.$$

For $\theta \in (0, \pi)$ let

$$H_\theta(z) = 2\theta + \frac{az}{1 - 2tz + z^2}, \tag{3}$$

where

$$a = 8(\pi - \theta)\frac{\theta}{\pi},$$

and

$$t = 1 - \frac{2\theta}{\pi}. \tag{4}$$

Let

$$G_\theta(z) = \rho(\theta)\frac{e^{i\theta} - e^{-i\theta}e^{iH_\theta(z)}}{1 - e^{iH_\theta(z)}}. \tag{5}$$

Moreover, denote by

$$M = \frac{\rho}{\rho(\theta)}, \tag{6}$$

where $\rho(\theta)$ is given by (2).

THEOREM 1. *The class $T_{\rho,\theta}$ can be written as follows*

$$T_{\rho,\theta} = \left\{M \cdot \left(G_\theta \circ \frac{g}{M}\right) : \quad g \in T_M\right\}, \theta \in (0, \pi), \ \rho \geq \rho(\theta),$$

where M, $\rho(\theta)$ and G_θ are defined by (2), (5), (6), respectively.

Proof. In the proof of this theorem we use the method which was presented in [4].

Let $f \in T$ and $f(z) \neq \rho e^{\pm i\theta}$, $\theta \in (0, \pi)$, $\rho \geq \rho(\theta)$. We consider a function of the form

$$\frac{f(z) - \rho e^{i\theta}}{f(z) - \rho e^{-i\theta}}.$$

This function is analytic in Δ and omits 0. Therefore, the function

$$h(z) = \frac{1}{i}\log\frac{f(z) - \rho e^{i\theta}}{f(z) - \rho e^{-i\theta}}$$

is analytic in Δ, too. The branch of the logarithm is such that $h(0) = 2\theta$.

Hence

$$f(z)=\rho\frac{e^{i\theta}-e^{-i\theta}e^{ih(z)}}{1-e^{ih(z)}}.$$

Moreover, $h(z)\neq 2k\pi$, $k\in\mathbb{Z}$, because $1-e^{ih(z)}\neq 0$. From this fact and from the equality

$$\operatorname{Im} f(z)=\frac{\rho}{|1-e^{ih(z)}|^2}(1-e^{-2i\operatorname{Im} h(z)})\sin\theta,$$

we have $\operatorname{Im} z\operatorname{Im} h(z)\geq 0$, $z\in\Delta$.

Furthermore, $h(0)=2\theta$ and for real x there is $h(x)\neq 2k\pi$, $k\in\mathbb{Z}$. The function h is typically real, so we get $0<h(x)<2\pi$, $x\in(-1,1)$.

From these properties of h we conclude that h is subordinated to the univalent function H_θ, which is given by (3). For H_θ we have $H_\theta(0)=2\theta$, $H_\theta(\Delta)=\mathbb{C}\setminus\{p\in\mathbb{R}: p\in(-\infty,0)\cup(2\pi,\infty)\}$. Hence $h(z)=H_\theta(\omega(z))$ where $\omega(z)=H_\theta^{-1}(h(z))$. From this fact and $h,H_\theta\in T$ we get $\operatorname{Im} z\operatorname{Im}\omega(z)\geq 0$.

We have

$$\frac{2\sin\theta}{\rho}=h'(0)=H_\theta'(0)\cdot\omega'(0)=a\cdot\omega'(0),$$

therefore,

$$\omega'(0)=\frac{2\sin\theta}{\rho\cdot a}=\frac{\rho(\theta)}{\rho}.$$

Hence

$$\frac{\rho}{\rho(\theta)}\cdot\omega(z)\in T_{\frac{\rho}{\rho(\theta)}}.$$

Let

$$g(z)=\frac{\rho}{\rho(\theta)}\cdot\omega(z),\quad g\in T_M,$$

where M is given by (6). We have

$$f(z)=\rho\frac{e^{i\theta}-e^{-i\theta}e^{ih(z)}}{1-e^{ih(z)}}=\rho\frac{e^{i\theta}-e^{-i\theta}e^{iH_\theta\left(\frac{g(z)}{M}\right)}}{1-e^{iH_\theta\left(\frac{g(z)}{M}\right)}}=M\cdot G_\theta\left(\frac{g(z)}{M}\right),$$

where G_θ is given by (5). □

Observe that the function (5) we can extend onto limiting cases $\theta=0$ and $\theta=\pi$. When $\theta=0$ we have

$$H_0(z)\equiv 0\quad\text{and}\quad G_0(z)=\lim_{\theta\to 0}G_\theta(z)=\frac{z}{(1+z)^2}.$$

If $\theta=\pi$ then we get

$$H_\pi(z)\equiv 2\pi\quad\text{and}\quad G_\pi(z)=\lim_{\theta\to\pi}G_\theta(z)=\frac{z}{(1-z)^2}.$$

One can prove that the function (5) could be written in the form

$$G_\theta(z)=\frac{G\left(\frac{z+c}{1+cz}\right)-G(c)}{(1-c^2)G'(c)},\tag{7}$$

where G is given by (1) and $c=c(\theta)=\frac{2\theta-\pi}{2\sqrt{\pi\theta-\theta^2}+\pi}$.

LEMMA 1. *Let*

(8) $$\rho_1(\theta) = \frac{1}{2}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right), \theta \in \left[0, \frac{\pi}{2}\right]$$

and

(9) $$\rho^*(\theta) = \rho_1\left(\frac{\pi}{2} - \left|\frac{\pi}{2} - \theta\right|\right), \theta \in [0, \pi].$$

Then

(10) $$\rho(\theta) < \rho^*(\theta),$$

where $\rho(\theta)$ is given by (2) and $\rho^*(\theta)$ is described in (9), $\theta \in [0, \pi]$.

Proof. We shall prove the inequality (10). Observe that for $\theta = 0$ this inequality is obvious. Let $\theta \in (0, \frac{\pi}{2})$. Then (10) is equivalent to

$$(2\theta - \pi)\sin\theta + 2\theta(\pi - \theta)\cos\theta > 0.$$

Let us denote

$$l(\theta) = (2\theta - \pi)\sin\theta + 2\theta(\pi - \theta)\cos\theta,\ \theta \in \left(0, \frac{\pi}{2}\right).$$

Because

$$l''(\theta) = -3(\pi - 2\theta)\sin\theta - 2\theta(\pi - \theta)\cos\theta < 0,\ \theta \in (0, \frac{\pi}{2}),$$

so the function $l(\theta)$ attains its lowest value in 0 or in $\frac{\pi}{2}$. Hence

$$l(\theta) > \min\{\lim_{\theta\to 0} l(\theta), \lim_{\theta\to\frac{\pi}{2}} l(\theta)\} = 0,$$

and consequently (10) is true. Moreover,

$$\rho(\theta) = \rho(\pi - \theta) \le \rho_1(\pi - \theta),\ \theta \in \left[\frac{\pi}{2}, \pi\right].$$

Therefore,

$$\rho(\theta) < \rho^*(\theta),\quad \theta \in [0, \pi].$$ □

In the next theorem we estimate the second and third coefficients of a function from the class $T_{\rho,\theta}$.

THEOREM 2. *If $f \in T_{\rho,\theta}$ and $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ then*

(11) $$\frac{1}{\rho}\left(\frac{\sin\theta}{\theta} - \cos\theta\right) - 2 \le a_2 \le -\frac{1}{\rho}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right) + 2,\ \theta \in [0, \pi],$$

and

(12)
$$a_3 \ge \begin{cases} \frac{\sin^2\theta}{3\rho^2} - 1, & (\rho,\theta) \in A \\ 3 - \frac{4}{\rho}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right) + \frac{1}{\rho^2}\left(\frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{\sin 2\theta}{\pi-\theta} + \frac{2\cos^2\theta+1}{3}\right), & (\rho,\theta) \in B \\ 3 - \frac{4}{\rho}\left(\frac{\sin\theta}{\theta} - \cos\theta\right) + \frac{1}{\rho^2}\left(\frac{\sin^2\theta}{\theta^2} - \frac{\sin 2\theta}{\theta} + \frac{2\cos^2\theta+1}{3}\right), & (\rho,\theta) \in C \end{cases}$$

and

$$(13) \qquad a_3 \le 3 + \frac{4}{\rho}\left[\frac{\theta(\pi-\theta)\cos^2\theta}{\pi\sin\theta} - \frac{\sin\theta}{\pi} - \left(1 - \frac{2\theta}{\pi}\right)\cos\theta\right] + \frac{\sin^2\theta}{3\rho^2}.$$

The sets A, B, C are described as follows:
$A = \{(\rho,\theta), \rho \ge \rho^*(\theta), \theta \in [0,\pi]\}$,
$B = \left\{(\rho,\theta), \rho(\theta) \le \rho \le \rho_1(\theta), \theta \in [0, \frac{\pi}{2}]\right\}$,
$C = \left\{(\rho,\theta), \rho(\theta) \le \rho \le \rho_1(\pi-\theta), \theta \in [\frac{\pi}{2}, \pi]\right\}$,
where $\rho_1(\theta)$, $\rho^*(\theta)$ are defined by (8) and (9).

Proof. Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in T_{\rho,\theta}$, $\theta \in [0,\pi]$, $\rho \ge \rho(\theta)$, $\rho(\theta)$ is given by (2), and let $G_\theta(z) = z + A_2(\theta)z^2 + A_3(\theta)z^3 + \ldots$.. From Theorem 1 $f(z) = M\cdot G_\theta(\frac{g(z)}{M})$, where G_θ is given by (7) and $g(z) = z + c_2z + c_3z^3 + \cdots \in T_M$, M is defined by (6). Hence

$$(14) \qquad a_2 = c_2 + \frac{1}{2M}\cdot G_\theta''(0)$$

and

$$(15) \qquad a_3 = c_3 + c_2\frac{1}{M}\cdot G_\theta''(0) + \frac{1}{6M^2}G_\theta'''(0),$$

where M is defined by (6). We have

$$(16) \qquad G_\theta''(0) = \frac{G''(c)}{G'(c)}\cdot(1-c^2) - 2c$$

and

$$(17) \qquad G_\theta'''(0) = \frac{(1-c^2)^2G'''(c)}{G'(c)} - 6c\frac{(1-c^2)G''(c)}{G'(c)} + 6c^2.$$

From

$$(18) \qquad \frac{G''(c)}{G'(c)} = \frac{-2c(3-c^2)}{1-c^4} + \frac{2\pi(1-c^2)}{(1+c^2)^2}\tan\frac{\pi c}{1+c^2},$$

and

$$(19) \qquad c + \frac{1}{c} = \frac{2\pi}{2\theta-\pi}, \qquad \left(\frac{1-c^2}{1+c^2}\right)^2 = \frac{4}{\pi^2}\left(\pi\theta - \theta^2\right)$$

and (16) we get

$$(20) \qquad A_2(\theta) = \frac{G_\theta''(0)}{2} = \frac{2}{\pi}\left(\pi - 2\theta - 2\theta(\pi-\theta)\cot\theta\right).$$

Using (6) and (20) in the formula (14) we have

$$a_2 = c_2 + \frac{\sin\theta}{2\theta(\pi-\theta)\rho}\cdot(\pi - 2\theta - 2\theta(\pi-\theta)\cot\theta)$$

and, consequently,

$$a_2 = c_2 + \frac{(\pi - 2\theta)\sin\theta}{2\theta(\pi-\theta)\rho} - \frac{\cos\theta}{\rho}. \tag{21}$$

Taking advantage of the precise estimation of the second coefficient for functions in T_M (see [2])

$$-2\left(1-\frac{1}{M}\right) \le c_2 \le 2\left(1-\frac{1}{M}\right)$$

and equality (21) we have

$$\begin{aligned}-2\left(1-\frac{\pi\sin\theta}{4\theta(\pi-\theta)\rho}\right) + \frac{\sin\theta(\pi-2\theta)}{2\theta(\pi-\theta)\rho} - \frac{\cos\theta}{\rho} &\le a_2 \\ &\le 2\left(1-\frac{\pi\sin\theta}{4\theta(\pi-\theta)\rho}\right) + \frac{\sin\theta(\pi-2\theta)}{2\theta(\pi-\theta)\rho} - \frac{\cos\theta}{\rho}.\end{aligned}$$

The above inequality is equivalent to (11).

Now we are going to estimate the third coefficient of $f \in T_{\rho,\theta}$. Using

$$\begin{aligned}(1-c^2)^2\frac{G'''(c)}{G'(c)} &= \frac{-6(1-c^2)(c^4-6c^2+1)}{(1+c^2)^2} + 2\pi^2\frac{(1-c^2)^4}{(1+c^2)^4} \\ &+ \frac{12\pi c(1-c^2)^2(c^2-3)}{(1+c^2)^3}\tan\frac{\pi c}{1+c^2} + 6\frac{(1-c^2)^4}{(1+c^2)^4}\pi^2\tan^2\frac{\pi c}{1+c^2},\end{aligned}$$

(18) and (19) in the formula (17), we get

$$\begin{aligned}A_3(\theta) = \frac{G_\theta'''(0)}{6} &= -1 + 4\left(\frac{2\theta-\pi}{\pi}\right)^2 \\ &+ \frac{16(2\theta-\pi)\theta(\pi-\theta)}{\pi^2}\cot\theta + \left(\frac{4\theta(\pi-\theta)}{\pi}\cot\theta\right)^2 + \frac{16(\pi-\theta)^2\theta^2}{3\pi^2}.\end{aligned}$$

From the fact and equalities (15) and (20), we have

$$\begin{aligned}a_3 = c_3 + \frac{c_2}{\rho}\left(\frac{(\pi-2\theta)\sin\theta}{\theta(\pi-\theta)} - 2\cos\theta\right) &- \frac{\sin^2\theta}{4\rho^2\theta(\pi-\theta)} \\ &+ \frac{2\cos^2\theta+1}{3\rho^2} - \frac{(\pi-2\theta)\sin 2\theta}{2\rho^2(\pi-\theta)\theta}.\end{aligned}$$

Applying (2) and (4) we get

$$a_3 = c_3 + \frac{c_2}{\rho}\left(4t\rho(\theta) - 2\cos\theta\right) + (4t^2-1)\frac{\rho^2(\theta)}{\rho^2} - \frac{4t\rho(\theta)\cos\theta}{\rho^2} + \frac{1+2\cos^2\theta}{3\rho^2}.$$

From now on to the end of this proof we take M, t as in the formulae (6) and (4), respectively. In order to estimate the coefficient a_3 we shall use the region of values (c_2, c_3), where $g(z) = z + c_2 z^2 + c_3 z^3 + \cdots \in T_M$. This region is of the form [3]

$$A_{2,3}(T_M) = \left\{ (c_2, c_3) : \quad -2\left(1 - \frac{1}{M}\right) \leq c_2 \leq 2\left(1 - \frac{1}{M}\right), \right.$$
$$\left. c_2^2 - 1 + \frac{1}{M^2} \leq c_3 \leq 3 - \frac{4}{M} + \frac{1}{M^2} - \frac{c_2^2}{M-1} \right\}.$$

Let $c_2 = x$, $c_3 = y$. We consider a function

$$k(x,y) = y + \frac{x}{\rho}\left(4t\rho(\theta) - 2\cos\theta\right) + (4t^2 - 1)\frac{\rho^2(\theta)}{\rho^2} - \frac{4t\rho(\theta)\cos\theta}{\rho^2} + \frac{1 + 2\cos^2\theta}{3\rho^2}.$$

From the inequalities, given in the description of the set $A_{2,3}(T_M)$, we have

$$(22) \quad k(x,y) \leq 3 - \frac{4}{M} + \frac{1}{M^2} - \frac{x^2}{M-1} + \frac{2x}{\rho}(2t\rho(\theta) - \cos\theta) + (4t^2 - 1)\frac{\rho^2(\theta)}{\rho^2} - \frac{4t\rho(\theta)\cos\theta}{\rho^2} + \frac{1 + 2\cos^2\theta}{3\rho^2}$$

and

$$(23) \quad k(x,y) \geq x^2 - 1 + \frac{1}{M^2} + \frac{2x}{\rho}(2t\rho(\theta) - \cos\theta) + (4t^2 - 1)\frac{\rho^2(\theta)}{\rho^2} - \frac{4t\rho(\theta)\cos\theta}{\rho^2} + \frac{1 + 2\cos^2\theta}{3\rho^2}.$$

If we want to determine the upper estimate of a_3, we shall find the maximum of the function

$$K(x) = -\frac{x^2}{M-1} + \frac{2x}{\rho}(2t\rho(\theta) - \cos\theta) + 3 - \frac{4}{M} + \frac{1}{M^2} + (4t^2 - 1)\frac{\rho^2(\theta)}{\rho^2} - \frac{4t\rho(\theta)\cos\theta}{\rho^2} + \frac{1 + 2\cos^2\theta}{3\rho^2},$$

when $x \in [-2(1 - \frac{1}{M}), 2(1 - \frac{1}{M})]$. Observe that K is a quadratic function of the form

$$K(x) = -\frac{\rho(\theta)}{\rho - \rho(\theta)}x^2 + \frac{2x}{\rho}\left(2t\rho(\theta) - \cos\theta\right) + 3 - 4\frac{\rho(\theta)}{\rho} + 4t^2\frac{\rho^2(\theta)}{\rho^2} - \frac{4t\rho(\theta)\cos\theta}{\rho^2} + \frac{1 + 2\cos^2\theta}{3\rho^2},$$

which attains its local maximum in the point

$$x_w = \frac{(2t\rho(\theta) - \cos\theta)\,(\rho - \rho(\theta))}{\rho\rho(\theta)}.$$

It is easy to check that $x_w \in [-2(1-\frac{1}{M}), 2(1-\frac{1}{M})]$ for $\rho \geq \rho(\theta)$ and $\theta \in [0, \pi]$. The maximum of K is equal to

$$K_{max} = \frac{(\rho - \rho(\theta))(2t\rho(\theta) - \cos^2\theta)}{\rho^2\rho(\theta)} + 3 - 4\frac{\rho(\theta)}{\rho} + 4t^2\frac{\rho^2(\theta)}{\rho^2} - \frac{4t\rho(\theta)\cos\theta}{\rho^2} + \frac{1+2\cos^2\theta}{3\rho^2},$$

or equivalently

$$K_{max} = 3 + \frac{1}{\rho}\left(\frac{4\theta(\pi-\theta)\cos^2\theta}{\pi\sin\theta} + \frac{\sin^2\theta}{3\rho} - \frac{4\sin\theta}{\pi} - 4t\cos\theta\right).$$

From (22) we have $k(x,y) \leq K_{max}$ and consequently $a_3 \leq K_{max}$, $\rho \geq \rho(\theta)$ $\theta \in [0, \pi]$. Hence (13) is true.

Let us denote

$$\mathcal{K}(x) = x^2 - 1 + \frac{1}{M^2} + \frac{2x}{\rho}(2t\rho(\theta) - \cos\theta) + (4t^2-1)\frac{\rho^2(\theta)}{\rho^2} - \frac{4t\rho(\theta)\cos\theta}{\rho^2} + \frac{1+2\cos^2\theta}{3\rho^2}.$$

Now, we shall find the minimum of the function $\mathcal{K}$. The function $\mathcal{K}$ is quadratic function having its local minimum in the point

$$x_w = \frac{\cos\theta - 2t\rho(\theta)}{\rho}.$$

There are three possible cases: I. $x_w \in [-2(1-\frac{1}{M}), 2(1-\frac{1}{M})]$, II. $x_w > 2(1-\frac{1}{M})$, III. $x_w < -2(1-\frac{1}{M})$.

I. The case $x_w \in [-2(1-\frac{1}{M}), 2(1-\frac{1}{M})]$ holds only if the inequality

$$-2(\rho - \rho(\theta)) \leq \cos\theta - 2t\rho(\theta) \leq 2(\rho - \rho(\theta))$$

is satisfied. From the inequality

$$\cos\theta - 2t\rho(\theta) \leq 2(\rho - \rho(\theta))$$

we get

$$\rho \geq \frac{1}{2}\left[\cos\theta + \frac{\sin\theta}{\pi - \theta}\right].$$

Similarly, from the inequality

$$-2(\rho - \rho(\theta)) \leq \cos\theta - 2t\rho(\theta)$$

we have

$$\rho \geq \frac{1}{2}\left[\frac{\sin\theta}{\theta} - \cos\theta\right].$$

Hence $x_w \in [-2(1-\frac{1}{M}), 2(1-\frac{1}{M})]$ when $(\rho,\theta) \in A = \{(\rho,\theta) : \rho \geq \rho^*(\theta), \theta \in [0,\pi]\}$ and ρ^* given by (9).

For $(\rho,\theta) \in A$ the function $\mathcal{K}$ attains the minimum in the point x_w and it is equal to

$$\mathcal{K}_{min} = \frac{\sin^2\theta}{3\rho^2} - 1.$$

From (23) it follows that $\mathcal{K}_{min} \leq k(x,y)$ and consequently $a_3 \geq \mathcal{K}_{min}$ for $(\rho,\theta) \in A$.

II. Let now $x_w > 2(1-\frac{1}{M})$. This inequality, conditions (2) and (4), and Lemma 1 give us $\rho < \rho_1(\theta)$. Hence the minimum of $\mathcal{K}$ for $(\rho,\theta) \in B = \{(\rho,\theta) : \rho(\theta) \leq \rho < \rho_1(\theta), \theta \in [0,\frac{\pi}{2})\}$ is attained in the point $x = 2(1-\frac{1}{M})$. This minimum is equal to

$$\mathcal{K}_{min} = 3 - \frac{4}{\rho}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right) + \frac{1}{\rho^2}\left(\frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{\sin 2\theta}{\pi-\theta} + \frac{2\cos^2\theta+1}{3}\right).$$

Therefore, from (23) we conclude that $k(x,y) \leq \mathcal{K}_{min}$ and consequently $a_3 \geq \mathcal{K}_{min}$ for $(\rho,\theta) \in B$.

III. Consider the case $x_w < -2(1-\frac{1}{M})$. This condition is equivalent to $\rho < \rho_1(\pi-\theta)$. From this fact it follows that for $(\rho,\theta) \in C = \{(\rho,\theta) : \rho(\theta) \leq \rho < \rho_1(\pi-\theta), \theta \in (\frac{\pi}{2},\pi]\}$ the function $\mathcal{K}$ attains its minimum in the point $x = -2(1-\frac{1}{M})$. The minimum is equal to

$$\mathcal{K}_{min} = 3 - \frac{4}{\rho}\left(\frac{\sin\theta}{\theta} - \cos\theta\right) + \frac{1}{\rho^2}\left(\frac{\sin^2\theta}{\theta^2} - \frac{\sin 2\theta}{\theta} + \frac{2\cos^2\theta+1}{3}\right).$$

From (23) we get $k(x,y) \leq \mathcal{K}_{min}$ and consequently $a_3 \geq \mathcal{K}_{min}$ for $(\rho,\theta) \in C$. So we have proved (12). □

In the two following figures there are the regions of values (ρ, a_2) and (ρ, a_3) for $\theta = \frac{\pi}{3}$.

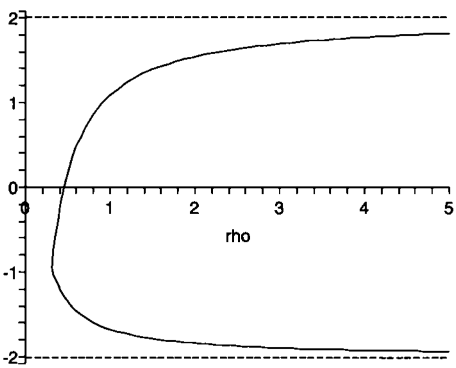

Fig. 1. The region of values of the point (ρ, a_2) (rightward of the curve)

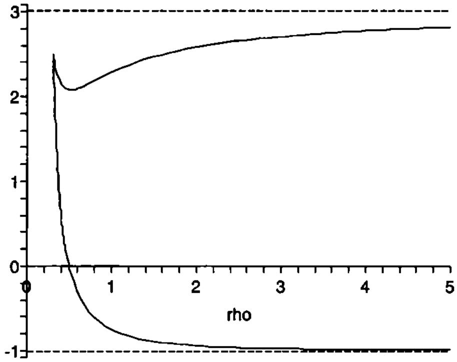

Fig. 2. The region of values of the point (ρ, a_3) (rightward of the curve)

Let b denote the fixed second coefficient of a function $f \in T_{\rho,\theta}$. Verifying ρ, θ, $\rho \in [0,\pi]$, $\rho > \rho(\theta)$, we obtain $b \in [-2,2]$. If $b = 2$, then the class $T_{\rho,\theta}$ consists of only one function $f(z) = \frac{z}{(1-z)^2}$. Analogously, if $b = -2$, then in this class there is only function $f(z) = \frac{z}{(1+z)^2}$. Therefore, we can restrict our research to the case $b \in (-2,2)$.

THEOREM 3. *The Koebe domain for the class of typically real functions with a fixed second coefficient b is bounded and symmetric with respect to the real axis. Its boundary in the upper half plane is given by the polar equation* $w = r(\theta)e^{i\theta}$, *where*

$$r(\theta) = \begin{cases} \frac{1}{2-b}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right), & \theta \in [0,\theta_0) \\ \frac{1}{2+b}\left(\frac{\sin\theta}{\theta} - \cos\theta\right), & \theta \in [\theta_0,\pi] \end{cases},$$

and θ_0 *is the only solution of the equation*

$$\frac{1}{2-b}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right) = \frac{1}{2+b}\left(\frac{\sin\theta}{\theta} - \cos\theta\right), \tag{24}$$

in $[0,\pi]$.

Proof. From (11) for fixed $a_2 = b$, we get

$$\begin{cases} \frac{1}{\rho}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right) \le 2-b \\ \frac{1}{\rho}\left(\frac{\sin\theta}{\theta} - \cos\theta\right) \le 2+b \end{cases}, \quad \theta \in [0,\pi]$$

and consequently

$$\begin{cases} \rho \ge \frac{1}{2-b}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right) \\ \rho \ge \frac{1}{2+b}\left(\frac{\sin\theta}{\theta} - \cos\theta\right) \end{cases}, \quad \theta \in [0,\pi].$$

Then

$$\rho \geq \max\{\frac{1}{2-b}\left(\frac{\sin\theta}{\pi-\theta}+\cos\theta\right), \frac{1}{2+b}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\}, \quad \theta\in[0,\pi].$$

Hence, the boundary of the Koebe domain is given by the following equation

$$r(\theta) = \max\{\frac{1}{2-b}\left(\frac{\sin\theta}{\pi-\theta}+\cos\theta\right), \frac{1}{2+b}\left(\frac{\sin\theta}{\theta}-\cos\theta\right)\}, \quad \theta\in[0,\pi].$$

Observe that the equation (24) can be written as

$$b\left(\frac{\sin\theta}{\pi-\theta}+\frac{\sin\theta}{\theta}\right) = 2\left(\frac{\sin\theta}{\theta}-\frac{\sin\theta}{\pi-\theta}-2\cos\theta\right).$$

The above condition is equivalent to

$$\frac{b\pi\sin\theta}{\theta(\pi-\theta)} = \frac{2(\pi-2\theta)\sin\theta}{\theta(\pi-\theta)} - 4\cos\theta.$$

Hence we have

$$b = \frac{2}{\pi}\left(\pi-2\theta-\theta(\pi-\theta)\cot\theta\right).$$

From (20) we get

$$b = A_2(\theta).$$

We are going to prove that $A_2(\theta)$ is increasing function for $\theta\in(0,\pi)$. We have

$$A_2'(\theta) = -\frac{4}{\pi}\left(1+(\pi-2\theta)\cot\theta-\theta(\pi-\theta)\frac{1}{\sin^2\theta}\right). \tag{25}$$

But for $\theta\in(0,\frac{\pi}{2}]$

$$\sin^2\theta+(\pi-2\theta)\sin\theta\cos\theta-\theta(\pi-\theta)\leq\theta^2+(\pi-2\theta)\theta-\theta(\pi-\theta)=0.$$

Hence $A_2'(\theta)>0$, $\theta\in(0,\frac{\pi}{2}]$ and $A_2(\theta)$ is the increasing function in this range. Moreover, $A_2(\pi-\theta)=-A_2(\theta)$. Then $A_2'(\theta)>0$, $\theta\in(\frac{\pi}{2},\pi)$. From these facts, we conclude that the function (20) is increasing in $(0,\pi)$. The monotonicity of (20) and the fact

$$\lim_{\theta\to 0}A_2(\theta)=-2 \qquad\text{and}\qquad \lim_{\theta\to\pi}A_2(\theta)=2$$

assure the existence of the only solution of the equation (24). This solution is denoted by θ_0. Finally, we get

$$r(\theta)=\begin{cases}\frac{1}{2-b}\left(\frac{\sin\theta}{\pi-\theta}+\cos\theta\right), & \theta\in[0,\theta_0)\\ \frac{1}{2+b}\left(\frac{\sin\theta}{\theta}-\cos\theta\right), & \theta\in[\theta_0,\pi]\end{cases}.$$

∎

From Theorem 3 we conclude

Corollary 1. *The Koebe domain for the class of typically real functions with a fixed second coefficient $b=0$ is bounded and symmetric with respect* *to both axes, whose boundary in the first quadrant of the complex plane is given by the polar equation $w = r(\theta)e^{i\theta}$, where*

$$r(\theta) = \frac{1}{2}\left(\frac{\sin\theta}{\pi-\theta} + \cos\theta\right).$$

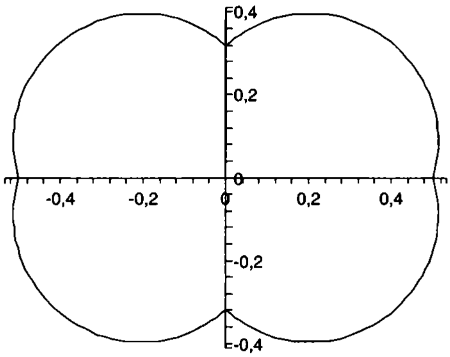

Fig. 3. The Koebe domain for the class of typically real function with fixed second coefficient $b = 0$

This set is the superdomain of the Koebe domain for the class of all typically real odd functions, which is still unknown.

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