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Adam Lecko, Millenia Lecko, Teruo Yaguchi

SUBCLASSES OF TYPICALLY-REAL FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

Abstract. For each $\lambda > -1$ let $\mathcal{T}_R(\lambda)$ be the class of all functions f analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $z \in \mathbb{D}$, having real coefficients and satisfying the condition

$$\operatorname{Re}\left\{\left(1-z^2\right)\frac{L_\lambda f(z)}{z}\right\} > 0, \qquad z \in \mathbb{D},$$

where L_λ denote the Ruscheweyh derivative. Some basic properties of functions in $\mathcal{T}_R(\lambda)$ are presented.

1. Introduction

1.1. For $0 < r \le 1$ let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, and let $\mathbb{D} = \mathbb{D}_1$. By $\mathcal{A}$ we denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad z \in \mathbb{D}, \tag{1.1}$$

which are analytic in $\mathbb{D}$. Let $\mathcal{P}$ denote the class of functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \qquad z \in \mathbb{D}, \tag{1.2}$$

which are analytic in $\mathbb{D}$ and have positive real part there. By $\mathcal{P}_R$ we denote the class of functions $p \in \mathcal{P}$ whose coefficients are real, i.e. $p_k = \overline{p_k}$ in (1.2) for all $k \in \mathbb{N}$.

1.2. For two analytic functions f and g of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

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the Hadamard product (the convolution) of f and g is defined as follows:

$$f * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k. \tag{1.3}$$

1.3. By L_λ, $\lambda \geq -1$, will be denoted the Ruscheweyh derivative (see [10]) over the class $\mathcal{A}$ defined as follows:

$$L_\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \qquad f \in \mathcal{A},\ z \in \mathbb{D}. \tag{1.4}$$

For $k \in \mathbb{N}$ and $\lambda \geq -1$ denote

$$B_k(\lambda) = \frac{(\lambda+1)(\lambda+2)\ldots(\lambda+k-1)}{(k-1)!}.$$

We see that

$$\frac{z}{(1-z)^{\lambda+1}} = z + \sum_{k=2}^{\infty} B_k(\lambda) z^k, \qquad z \in \mathbb{D},$$

and consequently by (1.3) and (1.4) we obtain

$$L_\lambda f(z) = z + \sum_{k=2}^{\infty} B_k(\lambda) a_k z^k, \qquad z \in \mathbb{D}, \tag{1.5}$$

where $f \in \mathcal{A}$ is of the form (1.1).

1.4. From the identity

$$\frac{z}{(1-z)^{\lambda+2}} = \frac{z}{(1-z)^{\lambda+1}} * \left(\frac{\lambda}{\lambda+1} \frac{z}{1-z} + \frac{1}{\lambda+1} \frac{z}{(1-z)^2} \right), \qquad z \in \mathbb{D},$$

it follows that

$$z\left(L_\lambda f\right)'(z) = (\lambda+1) L_{\lambda+1} f(z) - \lambda L_\lambda f(z), \qquad z \in \mathbb{D}, \tag{1.6}$$

for all $\lambda > -1$.

Let us notice that for every $\lambda > -1$ holds

$$\begin{aligned} L_1 L_\lambda f(z) &= z\left(L_\lambda f\right)'(z) = z\left(\frac{z}{(1-z)^{\lambda+1}} * f(z)\right)' \\ &= \frac{z}{(1-z)^{\lambda+1}} * \left(z f'(z)\right) = L_\lambda\left(z f'(z)\right) = L_\lambda L_1 f(z), \qquad z \in \mathbb{D}. \end{aligned} \tag{1.7}$$

For $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the formula (1.4) can be written as

$$L_n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}, \qquad f \in \mathcal{A},\ z \in \mathbb{D}. \tag{1.8}$$

Moreover then

$$B_k(n) = \binom{n+k-1}{n}, \qquad k \in \mathbb{N},\ n \in \mathbb{N}_0.$$

1.5. In this paper we examine the classes $\mathcal{T}_R(\lambda)$ defined below.

DEFINITION 1.1. Let $\lambda > -1$. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{T}_R(\lambda)$ if f has real coefficients and if

$$\operatorname{Re}\left\{\left(1-z^2\right)\frac{L_\lambda f(z)}{z}\right\} > 0, \qquad z \in \mathbb{D}. \tag{1.9}$$

Since $L_0 f = f$, so for $\lambda = 0$ the condition (1.9) defines well known class $\mathcal{T}_R(0)$ of functions called typically-real introduced by Rogosinski [9] (see also [2], vol. I, p. 185). For short let $\mathcal{T}_R = \mathcal{T}_R(0)$. If $\lambda = 1$, then $L_1 f(z) = zf'(z)$, $z \in \mathbb{D}$, and the condition (1.9) describes the class $\mathcal{T}_R(1)$ of functions f convex in the direction of the imaginary axis with real coefficients. The class $\mathcal{T}_R(1)$ will be denoted by $\mathcal{CVR}(i)$. It was introduced by Robertson [7] (see also [2], vol. I, pp. 205-206).

2. Integral formulas

Now we write the integral representation for functions in the class $\mathcal{T}_R(n)$, $n \in \mathbb{N}_0$.

From (1.8) and (1.9) we have

THEOREM 2.1. 1. *Let $\lambda > -1$. If $f \in \mathcal{T}_R(\lambda)$, then $L_\lambda f \in \mathcal{T}_R$.*

2. *Let $n \in \mathbb{N}$. If $g \in \mathcal{T}_R$, then the function $f \in \mathcal{A}$ being the solution of the differential equation $g(z) = L_n f(z)$, $z \in \mathbb{D}$, is in $\mathcal{T}_R(n)$. Moreover, if $n = 1$, then*

$$f(z) = \int_0^z \frac{g(u)}{u}\, du, \qquad z \in \mathbb{D},$$

and if $n \in \mathbb{N} \setminus \{1\}$, then

$$f(z) = \frac{n!}{z^{n-1}} \int_0^z \int_0^{u_1} \int_0^{u_2} \cdots \int_0^{u_{n-1}} \frac{g(u_n)}{u_n} du_n \cdots du_2 du_1, \qquad z \in \mathbb{D}.$$

Let $M(0, 2\pi)$ denote the set of all functions $m : [0, 2\pi] \longrightarrow \mathbb{R}$ which are nondecreasing in the interval $[0, 2\pi]$ and satisfy the condition $\int_0^{2\pi} dm(t) = 2\pi$.

From (1.9) we see that $f \in \mathcal{T}_R(\lambda)$, $\lambda > -1$, if and only if there exists a function $p \in \mathcal{P}_R$ such that

$$L_\lambda f(z) = \frac{zp(z)}{1-z^2}, \qquad z \in \mathbb{D}. \tag{2.1}$$

Using the integral representation for functions $p \in \mathcal{P}_R$ (see [2], vol. I, p. 186) we state from the above

THEOREM 2.2. *Let $\lambda > -1$. Then $f \in \mathcal{T}_R(\lambda)$ if and only if there exists a function $m \in M(0, 2\pi)$ such that*

$$L_\lambda f(z) = \frac{1}{2\pi} \int_0^{2\pi} J(z,t) dm(t), \tag{2.2}$$

where

$$J(z,t) = \frac{z}{1 - 2z\cos t + z^2}, \qquad z \in \mathbb{D},\ t \in [0, 2\pi]. \tag{2.3}$$

For $\lambda = 0$ and $\lambda = 1$ we get from the above the well known integral representations for functions in $\mathcal{T}_R$ and $\mathcal{CVR}(i)$, respectively.

COROLLARY 2.3 ([6]). *Every $f \in \mathcal{T}_R$ is of the form*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} J(z,t) dm(t), \qquad z \in \mathbb{D},$$

for some $m \in M(0, 2\pi)$.

COROLLARY 2.4 ([7]). *Every $f \in \mathcal{CVR}(i)$ is of the form*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{z} \frac{J(u,t)}{u} du dm(t) = \frac{1}{2\pi i} \int_0^{\pi} \frac{1}{\sin t} \log \frac{1 - ze^{-it}}{1 - ze^{it}} dm(t),$$

$z \in \mathbb{D}$, for some $m \in M(0, 2\pi)$.

Using (1.8) and (2.2) we can generalize the above integral formulas and we can obtain the integral representation for functions in $\mathcal{T}_R(n)$ for $n \geq 2$.

THEOREM 2.5. *Let $n \in \mathbb{N} \setminus \{1\}$. Then $f \in \mathcal{T}_R(n)$ if and only if there exists a function $m \in M(0, 2\pi)$ such that*

$$f(z) = \frac{n!}{2\pi z^{n-1}} \int_0^{2\pi} \int_0^{z} \int_0^{u_1} \cdots \int_0^{u_{n-1}} \frac{J(u_n, t)}{u_n} du_n \cdots du_1 dm(t), \qquad z \in \mathbb{D}.$$

3. Coefficient estimates

In this section estimates of the coefficients of functions in $\mathcal{T}_R(\lambda)$ are presented.

It is well known (see e.g. [2, p. 183]) that the function J defined by (2.3) is of the form

$$J(z,t) = z + \sum_{k=2}^{\infty} \frac{\sin kt}{\sin t} z^k, \qquad z \in \mathbb{D},\ t \in [0, 2\pi] \setminus \{0, \pi, 2\pi\}. \tag{3.1}$$

Since

$$\lim_{t \to 0} \frac{\sin kt}{\sin t} = \lim_{t \to 2\pi} \frac{\sin kt}{\sin t} = k, \qquad \lim_{t \to \pi} \frac{\sin kt}{\sin t} = (-1)^{k+1} k,$$

we see that J can be continuously extended to $t=0$, $t=\pi$ and $t=2\pi$. Using now Theorem 2.2 and (1.5) we have

THEOREM 3.1. *Let $\lambda > -1$. If $f \in \mathcal{T}_R(\lambda)$ and f is of the form* (1.1), *then*

$$a_k = \frac{1}{2\pi B_k(\lambda)} \int_0^{2\pi} \frac{\sin kt}{\sin t} dm(t), \qquad k = 2, 3, \dots, \quad m \in M(0, 2\pi).$$

Let now

$$m_k = \min_{t\in[0,2\pi]} \frac{\sin kt}{\sin t}, \qquad M_k = \max_{t\in[0,2\pi]} \frac{\sin kt}{\sin t},$$

where $k = 2, 3, \dots$. But $M_k = k$, $m_k = -k$ if k is even and $m_k > -k$ if k is odd. Hence and by Theorem 3.1 we get immediately coefficient bounds for functions in $\mathcal{T}_R(\lambda)$.

THEOREM 3.2. *Let $\lambda > -1$. If $f \in \mathcal{T}_R(\lambda)$ and f is of the form* (1.1), *then*

$$\frac{m_k}{B_k(\lambda)} \le a_k \le \frac{k}{B_k(\lambda)}, \qquad k = 2, 3, \dots \tag{3.2}$$

Especially, if k is even, then

$$\frac{-k}{B_k(\lambda)} \le a_k \le \frac{k}{B_k(\lambda)}.$$

Estimates (3.2) are sharp. The upper bound is achieved by the function f satisfying (2.1) with $p(z) = (1+z)/(1-z)$, $z \in \mathbb{D}$, i.e. for f such that

$$L_\lambda f(z) = \frac{z}{(1-z)^2}, \qquad z \in \mathbb{D}.$$

On account of (2.3) and (3.1), the lower bound is realized by the function f such that $L_\lambda f(z) = J(z, \theta)$, $z \in \mathbb{D}$, where $\theta \in [0, \pi]$ is choosen such that m_k is achieved for $t = \theta$.

In particular, from the above theorem we get the well known results.

COROLLARY 3.3. 1 ([9]). *If $f \in \mathcal{T}_R$, then*

$$m_k \le a_k \le k, \qquad k = 2, 3, \dots$$

2 ([7]). *If $f \in \mathcal{CVR}(i)$, then*

$$\frac{m_k}{k} \le a_k \le 1, \qquad k = 2, 3, \dots$$

4. Inclusion relations

In this section we will examine inclusion relations between classes $\mathcal{T}_R(\lambda)$ in view of parameter λ.

Let $\mathcal{S}^* \subset \mathcal{A}$ be the class of starlike functions, i.e. $f \in \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in \mathbb{D}.$$

By

$$K(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} kz^k, \qquad z \in \mathbb{D},$$

we denote the Koebe function. Clearly, $K \in \mathcal{S}^*$.

Define

$$h(z) = -\log(1-z) = z + \sum_{k=2}^{\infty} \frac{z^k}{k}, \qquad z \in \mathbb{D}.$$

The following theorem was proved by Robertson.

THEOREM 4.1 ([8]). *If $f, g \in \mathcal{T}_R$, then $f * g * h \in \mathcal{T}_R$.*

For each $c \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$ define

$$F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k, \qquad z \in \mathbb{D}, \tag{4.1}$$

where $f \in \mathcal{A}$ is of the form (1.1). The operator $F_n : \mathcal{A} \to \mathcal{A}$, $n \in \mathbb{N}_0$, was introduced by Bernardi [1]. In particular, the operator F_1 was examined by Libera [5]. The general case, when c is a complex number was studied by various authors. Lewandowski, Miller and Złotkiewicz [4] proved the following theorem.

THEOREM 4.2 ([4]). *If $f \in \mathcal{S}^*$, then $F_c(f) \in \mathcal{S}^*$ for every $c \in \mathbb{C}$ such that* $\operatorname{Re} c \geq 0$.

It is well known that typically-real functions and functions convex in the direction of the imaginary axis with the real coefficients are connected by the following Alexander type theorem.

THEOREM 4.3 ([7]). *$f \in \mathcal{CVR}(i)$ if and only if $g \in \mathcal{T}_R$, where $g(z) = zf'(z)$, $z \in \mathbb{D}$.*

LEMMA 4.4. *For every $f \in \mathcal{A}$ and $\lambda \geq 0$, holds $L_\lambda f = L_{\lambda+1} F_\lambda$.*

Proof. Fix $\lambda \geq 0$. From (4.1) by differentiating we have

$$z(F_\lambda(f))'(z) + \lambda F_\lambda(f)(z) = (\lambda+1)f(z).$$

Hence

$$L_\lambda\left(z(F_\lambda(f))'(z)\right) + \lambda L_\lambda F_\lambda(f)(z) = (\lambda+1)L_\lambda f(z).$$

Combining (1.6) and (1.7) with the above gives the assertion. □

If $f \in \mathcal{T}_R(\lambda)$ for $\lambda \geq 0$, then by Theorem 2.1 we see that $L_\lambda f \in \mathcal{T}_R$. By Lemma 4.4 we have $L_{\lambda+1}F_\lambda \in \mathcal{T}_R$. This shows that $F_\lambda \in \mathcal{T}_R(\lambda+1)$. Therefore we have

THEOREM 4.5. *Let $\lambda \geq 0$. Then $f \in \mathcal{T}_R(\lambda)$ if and only if $F_\lambda(f) \in \mathcal{T}_R(\lambda+1)$.*

THEOREM 4.6. *If $f \in \mathcal{T}_R$, then $F_c(f) \in \mathcal{T}_R$ for every $c \geq 0$.*

Proof. Fix $c \geq 0$. Since, as easy to verify,

$$F_c(K)(z) = \frac{c+1}{z^c}\int_0^z t^c(1-t)^{-2}dt = z + \sum_{k=2}^{\infty} k\frac{c+1}{c+k}z^k, \qquad z \in \mathbb{D},$$

by (4.1) we have

$$F_c(f) = f * F_c(K) * h.$$

By Theorem 4.2, $F_c(K) \in \mathcal{S}^*$. Moreover $F_c(K)$ has real coefficient so it is typically-real. Applying now Theorem 4.1 we deduce at once that $F_c(f)$ is typically-real also which ends the proof. □

Applying the last theorem we are able to prove the following theorem.

THEOREM 4.7. $\mathcal{T}_R(\lambda+1) \subsetneq \mathcal{T}_R(\lambda)$ *for every* $\lambda \geq 0$.

Proof. Fix $\lambda \geq 0$ and let $f \in \mathcal{T}_R(\lambda+1)$ be arbitrary. By Theorem 2.1, $L_{\lambda+1}f \in \mathcal{T}_R$. Hence, in view of (1.6) and Theorem 4.6 the function

$$L_\lambda f(z) = F_\lambda\left(L_{\lambda+1}f\right)(z) = \frac{\lambda+1}{z^\lambda}\int_0^z t^{\lambda-1}L_{\lambda+1}f(t)dt, \qquad z \in \mathbb{D},$$

is in $\mathcal{T}_R$. Thus again by Theorem 2.1 we see that $f \in \mathcal{T}_R(\lambda)$.

Now we prove that $\mathcal{T}_R(\lambda+1)$ is the proper subclass of $\mathcal{T}_R(\lambda)$. To this end, fix $\lambda \geq 0$. The function $p(z) = 1 - z^4$, $z \in \mathbb{D}$, is in $\mathcal{P}_R$. Setting p into (2.1) we have $L_\lambda f(z) = z + z^3$, $z \in \mathbb{D}$. Clearly, the function f is in $\mathcal{T}_R(\lambda)$. Using (1.6) we get from the above that $L_{\lambda+1}f(z) = z + (\lambda+3)z^3/(\lambda+1)$, $z \in \mathbb{D}$. But

$$\operatorname{Re}\left\{\left(1-z^2\right)\frac{L_{\lambda+1}f(z)}{z}\right\} = \operatorname{Re}\left\{\left(1-z^2\right)\left(1+\frac{\lambda+3}{\lambda+1}z^2\right)\right\} = 0$$

for $z_{1,2} = \pm i\sqrt{(\lambda+1)/(\lambda+3)} \in \mathbb{D}$ which implies that $f \notin \mathcal{T}_R(\lambda+1)$. □

From the last theorem it follows the following result.

COROLLARY 4.8. $\bigcup_{\lambda\geq 1}\mathcal{T}_R(\lambda) \subset \bigcup_{\lambda\in[0,1)}\mathcal{T}_R(\lambda)$.

COROLLARY 4.9. 1. $\mathcal{T}_R(n) \subsetneq \mathcal{T}_R$ *for every* $n \in \mathbb{N}$.

2. $\mathcal{T}_R(n) \subsetneq \mathcal{CVR}(i)$ *for every* $n \in \mathbb{N}$, $n \geq 2$.

Since the class $\mathcal{CVR}(i)$ is the set of univalent functions we have

COROLLARY 4.10. *For every $n \in \mathbb{N}$ the class $\mathcal{T}_R(n)$ is the set of univalent functions.*

For each $\lambda \in [0,1)$ denote

$$\mathcal{T}_R(\lambda,\infty) = \bigcap_{n=0}^{\infty} \mathcal{T}_R(\lambda+n).$$

By Theorem 4.7 we see that $\mathcal{T}_R(\lambda,\infty) \subsetneqq \mathcal{T}_R(\lambda+n)$ for every $n \in \mathbb{N}_0$. On the other hand the identity I, $I(z)=z$, $z \in \mathbb{D}$, belongs to $\mathcal{T}_R(\lambda,\infty)$ since $I \in \mathcal{T}_R(\lambda+n)$ for all $n \in \mathbb{N}_0$. Taking into account estimates of coefficients of functions in the class $\mathcal{T}_R(\lambda+n)$ we see that both sides of inequalities (3.2) tend to zero for every fixed $k = 2, 3, \cdots$ when n tends to infinity since then $B_k(\lambda+n)$ tends to infinity. Hence and again by Theorem 4.7 it may be concluded the following result.

THEOREM 4.11. *For each $\lambda \in [0,1)$ $\mathcal{T}_R(\lambda,\infty) = \bigcap_{n=0}^{\infty} \mathcal{T}_R(\lambda+n) = \{I\}$. Consequently,*

$$\mathcal{T}_R(\infty) = \bigcap_{\lambda \geq 0} \mathcal{T}_R(\lambda) = \{I\}.$$

5. Radius problem

Theorem 4.7 leads to the following radius problem.

DEFINITION 5.1. For each $\lambda \in [0,1]$ and $m \in \mathbb{N}$ by $R_{\lambda,m}(n)$, where $n \in \mathbb{N}_0$, we denote the largest radius of a disk $\mathbb{D}_{R_{\lambda,m}(n)}$ such that every function $f \in \mathcal{T}_R(\lambda+n)$ satisfy the condition

$$\operatorname{Re}\left\{ \left(1-z^2\right) \frac{L_{\lambda+n+m}f(z)}{z} \right\} > 0, \qquad z \in \mathbb{D}_{R_{\lambda,m}(n)}.$$

THEOREM 5.2. $R_{0,1}(0) = \sqrt{2}-1$.

Proof. Observe that $R_{0,1}(0)$ is the largest radius such that in the disk $\mathbb{D}_{R_{0,1}(0)}$ the condition

$$\operatorname{Re}\left\{ \left(1-z^2\right) f'(z) \right\} > 0, \tag{5.1}$$

is satisfied for every function $f \in \mathcal{T}_R$.

For each $z \in \mathbb{D}$ denote

$$\Lambda_z(f) = (1-z^2)f'(z), \qquad f \in \mathcal{A}.$$

Since for each $z \in \mathbb{D}$, Λ_z is a continuous linear function over the class $\mathcal{T}_R$ which is a convex compact subfamily of $\mathcal{A}$ with a standard topology, it suffices to prove (5.1) for the set $\{f_x : x \in [-1,1]\}$ of extreme points in $\mathcal{T}_R$, where for each $x \in [-1,1]$,

$$f_x(z) = \frac{z}{1-2xz+z^2}, \qquad z \in \mathbb{D}.$$

In this way by (5.1) we will find the largest radius $R_{0,1}(0)$ such that

$$\operatorname{Re} \Lambda_z(f_x) = \operatorname{Re}\left\{ \left(\frac{1-z^2}{1-2xz+z^2} \right)^2 \right\} > 0, \qquad z \in \mathbb{D}_{R_{0,1}(0)},$$

for all $x \in [-1,1]$. The above inequality can be rewritten as

$$|\operatorname{Arg} \Lambda_z(f_x)| = 2 \left| \operatorname{Arg}\left\{ \frac{1-z^2}{1-2xz+z^2} \right\} \right| < \frac{\pi}{2}. \tag{5.2}$$

Observe that for each $x \in [-1,1]$ and $r \in (0,1]$ holds

$$\left\{ \frac{1-z^2}{1-2xz+z^2} : z \in \mathbb{D}_r \right\} \subset \left\{ \frac{1+z}{1-z} : z \in \mathbb{D}_r \right\}.$$

Hence and from (5.2) it is enough to find the largest $r_0 \in (0,1]$ such that

$$\left| \operatorname{Arg}\left\{ \frac{1-z^2}{1-2xz+z^2} \right\} \right| < \frac{\pi}{4}, \qquad z \in \mathbb{D}_{r_0}.$$

From the above it is easy to see that r_0 is the unique solution in $(0,1]$ of the equation

$$\arctan \frac{2r}{1-r^2} = \frac{\pi}{4}.$$

Hence $r_0 = \sqrt{2}-1$. Consequently, $R_{0,1}(0) \geq \sqrt{2}-1$.

In order to prove that $R_{0,1}(0) = \sqrt{2}-1$ let us consider the Koebe function K. Clearly, $K \in \mathcal{T}_R$ and K is an extreme point in $\mathcal{T}_R$. We have

$$\operatorname{Re} \Lambda_z(K) = \operatorname{Re}\left\{ \left(\frac{1+z}{1-z} \right)^2 \right\} > 0, \qquad z \in \mathbb{D}_{r_0},$$

and $\operatorname{Re}\left\{ \left(1-z_0^2\right) K'(z_0) \right\} = 0$ at $z_0 = r_0 i$. □

REMARK 5.3. It is interesting that $R_{0,1}(0) = \sqrt{2}-1$ is equal to the radius of starlikeness and the radius of univalence in the class $\mathcal{T}_R$ (see [3]).

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A. Lecko, M. Lecko
DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF RZESZÓW
ul. W. Pola 2
35-959 RZESZÓW, POLAND

T. Yaguchi
DEPARTMENT OF MATHEMATICS
COLLEGE OF HUMANITIES & SCIENCES
NIHON UNIVERSITY
Skurajosui 3-25-40
SETAGAYA-KU, TOKYO 156, JAPAN

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