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ON THE CRITERIA FOR INITIAL OPERATORS POSSESSING (c)-PROPERTY AND GENERALIZED (c)-PROPERTY

Abstract. This paper gives the criteria for the system of initial operators to possess the $c(R)$-property and the generalized $c(R)$-property.

1. Introduction

In 1988, D. Przeworska-Rolewicz was the first who introduced and considered the general interpolation problems induced by a right invertible operator with initial operators (see [1]–[7]). In 1990, one of us gave the necessary and sufficient condition for the general interpolation problems to be well-posed (see [4]). Note that all results in mentioned works are based on the (c)-property of the initial operators. It is known that every initial operator F of the right invertible operator D possesses (c)-property if and only if $\dim\ker D = 1$ (see [1]). In particular, if D is the right invertible operator in the linear space X and $\dim\ker D \geq 2$, then there exists a class of initial operators of D which does not possess the (c)-property. In [5], the authors introduced so-called generalized (c)-property of the system of initial operators and studied some general interpolation problems. However, there is a lack of an acceptable general criterion for a system of initial operators to possess the (c)-property or the generalized (c)-property. If $\dim\ker D > 1$, there exist many initial operators that still possess the (c)-property, only. The initial operators of the right invertible operators play a key role for solving interpolation problems, and initial value problems (see [2], [5], [6], [7] and references therein).

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The main result of this paper is to give the necessary and sufficient condition for the system of initial operators to possess the (c)-property or the generalized (c)-property.

2. Initial operator

Let X be a linear space over a scalar field $\mathcal{K}$. In most application, one has $\mathcal{K} = \mathbb{R}$ or $\mathcal{K} = \mathbb{C}$. Denote by $L(X)$ the set of all linear operators having domain and image in X. Write $L_0(X) = \{A \in L(X) : \operatorname{dom} A = X\}$. Denote by $R(X)$ the set of all right invertible operators belonging to $L(X)$ (see [1]). For a given operator $D \in R(X)$, set $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$. In the sequel, we assume that $\dim \ker D > 0$, i.e. D is not invertible.

Definition 2.1 ([1]). The operator $F \in L_0(X)$ is said to be an initial operator of $D \in R(X)$, if the following conditions are satisfied

1. $\operatorname{Im} F = \ker D, \quad F^2 = F,$
2. There exists an $R \in \mathcal{R}_D$ such that $FR = 0$.

For every operator $D \in R(X)$, denote by $\mathcal{F}_D$ the set of all initial operators of D.

Definition 2.2 ([1]). Let $D \in R(X)$, $R \in \mathcal{R}_D$. We say that the operator $F \in \mathcal{F}_D$ possesses $c(R)$-property if for every $k \in \mathbb{N}$, there exists $c_k \in \mathcal{K}$ such that

$$FR^k z = c_k z \text{ for all } z \in \ker D,$$

where we admit $R^0 = I$.

Definition 2.3 ([5]). Let $D \in R(X), R \in \mathcal{R}_D$, and let $F_i \in \mathcal{F}_D, i = 1, 2, \ldots, n$. The system of the initial operators $\{F_i\}_{i=\overline{1,n}}$ is said to possess the generalized $c(R)$-property if there are nontrivial subspaces $Z_1, Z_2, \ldots, Z_p$ of $\ker D$ such that following conditions hold

1. $\ker D = \bigoplus_{\nu=1}^{p} Z_\nu.$
2. For any $i = 1, 2, \ldots, n, \ j \in \mathbb{N}$ there exists $c_{ij\nu} \in \mathcal{K}, \ \nu = 1, 2, \ldots, p$, such that
$$F_i R^j z = c_{ij\nu} z, \text{for all } z \in Z_\nu.$$

From Definition 2.3 it follows that if every initial operator $F_1, F_2, \ldots, F_n$ possesses the $c(R)$-property, then the system of initial operators $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property. The following example shows that there exist the initial operators possessing the generalized $c(R)$-property, but they do not possess $c(R)$-property.

EXAMPLE 2.1. Let

$$X = \mathcal{C}(\mathbb{R}), \quad D = d^2/dt^2, \quad R = \int_0^t \int_0^s .$$

Obviously, $\dim \ker D = 2$ and $e_1 = 1, e_2 = t$ are the basic vectors of $\ker D$. Consider the operators F_k given by the following

$$(F_k x)(t) = x(0) + tx'(0) + \frac{1}{2}(1+t)x''(k) + \frac{1}{2}(1-t)x''(-k), \quad k = 1, 2.$$

It is clearly that $F_k \in \mathcal{F}_D$, $\ker D = \operatorname{lin}\{e_1\} \oplus \operatorname{lin}\{e_2\}$, and

$$F_k R^j e_1 = \frac{k^{2j-2}}{(2j-2)} e_1,$$

$$F_k R^j e_2 = \frac{k^{2j-2}}{(2j-1)} e_2,$$

where $k = 1, 2$, $j \in \mathbb{N}$. Thus, two initial operators (F_1, F_2) possess the generalized $c(R)$-property, but they do not possess the $c(R)$-property.

3. The criterion for initial operators to possess the $c(R)$-property or the generalized $c(R)$-property

Let $D \in R(X)$ and let $1 < \dim \ker D = q < +\infty$. Denote by $E = \{e_m\}_{m=\overline{1,q}}$ the system of basic vectors of $\ker D$. Suppose that $F \in \mathcal{F}_D$ is the initial operator of D corresponding to an $R \in \mathcal{R}_D$. Note that $FR^i \in L_0(X)$ for every $i \in \mathbb{N}$. By the restriction the domain X to $\ker D$, one can consider FR^i the operator belonging to $L_0(\ker D)$. In the sequel, we write

$$T_i := FR^i|_{\ker D}, \quad i \in \mathbb{N}.$$

For every $i \in \mathbb{N}$, assume that $T_i e_j = \sum_{m=1}^{q} c_{ijm} e_m$, where $c_{ijm} \in \mathcal{K}$, $j = 1, 2, \ldots, q$, $i \in \mathbb{N}$. Put

$$C_i = [c_{ijm}]_{j,m=\overline{1,q}}, \tag{3.1}$$

i.e. C_i are the square matrices of order q. We say that the operators T_i are represented by the matrix C_i under the system of basic vectors $E = \{e_m\}_{m=\overline{1,q}}$, and write $T_i(E) = C_i(E)$.

THEOREM 3.1. *The operator $F \in \mathcal{F}_D$ possesses the $c(R)$-property if and only if $C_i = \alpha_i I$, where $\alpha_i \in \mathcal{K}$ and I is the identity matrix in* $\ker D$.
In another words, the operator $F \in \mathcal{F}_D$ possesses the $c(R)$-property if and only if there is a system of basic vectors of $\ker D$ *such that the matrices C_i are diagonal simultaneously.*

Proof. Suppose that $F \in \mathcal{F}_D$ possesses the $c(R)$-property. From Definition 2.2 it follows that for any $i \in \mathbb{N}$, there exists $c_i \in \mathcal{K}$ such that

$$T_i e_j = c_i e_j, \ \ j = 1, 2, \dots, q.$$

Then

$$\mathcal{C}_i = [\delta_{jm} c_i]_{j,m=\overline{1,q}},$$

where δ_{jm} is the Kronecker's symbol. Thus, $\mathcal{C}_i$ are the diagonal matrices.

Conversely, suppose that for every $i \in \mathbb{N}$, $\mathcal{C}_i$ defined by (3.1) are of the form $\mathcal{C}_i = \alpha_i I$. We have $T_i e_j = \mathcal{C}_i e_j = \alpha_i e_j$, $j = 1, 2, \dots, q$, for all $i \in \mathbb{N}$. It means that F possesses the $c(R)$-property. The proof is complete. □

Now we deal with the system of initial operators. Let $\{F_i\}_{i=\overline{1,n}}$ be the system of the initial operators of D corresponding to $R \in \mathcal{R}_D$. Write

$$T_{ij} := F_i R^j|_{\ker D}.$$

Suppose that

$$(3.2) \qquad T_{ij}(e_m) = \sum_{k=1}^{q} c_{ijmk} e_k, \ \ i = 1, 2, \dots, n, \quad m = 1, 2, \dots, q, \ \ i \in \mathbb{N}.$$

For every $i = 1, 2, \dots, n$ and $j \in \mathbb{N}$, we set

$$(3.3) \qquad \mathcal{C}_{ij} = [c_{ijmk}]_{m,k=\overline{1,q}},$$

i.e. $\mathcal{C}_{ij}$ are the square matrices of order q. We say that operators T_{ij} are represented by the matrices $\mathcal{C}_{ij}$ under the system of basic vectors $E = \{e_m\}_{m=\overline{1,q}}$. Set

$$(3.4) \qquad T_{ij}(E) = \mathcal{C}_{ij}(E).$$

THEOREM 3.2. *The system $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property if and only if there exists an invertible matrix S so that for every $i = 1, 2, \dots, n, j \in \mathbb{N}$, the operator $S^{-1}\mathcal{C}_{ij}S$ are the diagonal matrices.*
In another words, $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property if and only if there exists a system of basic vectors $E^* = \{e^*_m\}_{m=\overline{1,q}}$ of $\ker D$ such that all operators T_{ij} are represented by the diagonal matrices under system E^*.

Proof. *Necessity.* Suppose that $\ker D = \bigoplus_{\nu=1}^{p} Z_\nu$, and

$$(3.5) \qquad T_{ij} z = \gamma_{ij\nu} z \ \text{ for every } z \in Z_\nu, \ \ \nu = 1, 2, \dots, p.$$

In the subspaces Z_ν ($\nu = 1, 2, \dots, p$) we choose the system of basic vectors as follows

$$E^*_\nu = \{e^*_{\nu k}\}_{k=\overline{1,p_\nu}},$$

where $p_\nu = \dim Z_\nu$. Obviously, $E^* = \bigcup_{\nu=1}^{p} E_\nu^*$ is the system of basic vectors of $\ker D$. From (3.5) it follows

$$T_{ij} e_{\nu k}^* = \gamma_{ij\nu} e_{\nu k}^*, \ \ \nu = 1, 2, \dots, p, \ \ k = 1, 2, \dots, p_\nu.$$

We then write

$$T_{ij}(E^*) = \Gamma_{ij}(E^*). \tag{3.6}$$

Put

$$\Gamma_{ij\nu} = [\delta_{rs}\gamma_{ij\nu}]_{r,s=\overline{1,p_\nu}},$$

$$\Gamma_{ij} = [\delta_{\nu\mu}\Gamma_{ij\nu}]_{\nu\mu=\overline{1,p}}.$$

There exists a linear transform S domain E to image E^*, provided E and E^* are together the systems of basic vectors of $\ker D$. From (3.4) and (3.5) it follows

$$\Gamma_{ij} = S^{-1} C_{ij} S, \ \ i = 1, 2, \dots, n, \ \ j \in \mathbb{N}.$$

Sufficiency. Suppose that there exists an invertible matrix S so that for every $i = 1, 2, \dots, n$ and for $j \in \mathbb{N}$, $S^{-1} C_{ij} S = \Gamma_{ij}$ are the diagonal matrices. Suppose that

$$\Gamma_{ij} = \left[\gamma_{ijkm}\delta_{km}\right]_{k,m=\overline{1,p}}, \ \ i = 1, 2, \dots, n, \ \ j \in \mathbb{N}.$$

From the invertibility of S it implies that $E^* = S(E)$ is the system of basic vectors of $\ker D$. Hence, the operators $F_i R^j$ can be represented by the diagonal matrices Γ_{ij} under E^*, i.e.

$$\Gamma_{ij} = \left[\gamma_{ijkm}\delta_{km}\right]_{k,m=\overline{1,q}}, \ \ i = 1, 2, \dots, n, \ \ j \in \mathbb{N}.$$

For every $m = 1, 2, \dots, q$ we have

$$\ker D = \bigoplus_{m=1}^{q} \operatorname{lin}\{e_m^*\},$$

$$T_{ij} e_m^* = \Gamma_{ij} e_m^* = \gamma_{ij} e_m^*.$$

Thus, the system $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property corresponding to subspaces $Z_\nu = \operatorname{lin}\{e_m^*\}, m = 1, 2, \dots, q$. The theorem is proved. □

Note that if the system of initial operators $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property with respect to subspaces $Z_1, Z_2, \dots, Z_p$, then this system also possesses the generalized $c(R)$-property with respect to subspaces $\{Z_{\nu s}\}_{\nu=\overline{1,p}, s=\overline{1,p_\nu}}$, where $Z_{\nu s} \subset Z_\nu$, $p_\nu = \dim Z_\nu$. Hence, the following problem arises: *Let the system of the initial operators $\{F_i\}_{i=\overline{1,n}}$ be given. Find the minimal number p such that $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property with respect to p subspaces $Z_1, Z_2, \dots, Z_p$.*

Now we determine the minimal number p if possible. Suppose that the system of the initial operators $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property. By Theorem 3.2 we can assume that $E^* = \bigcup_{m=1}^{p} e_m^*$ is the system of basic vectors of $\ker D$ so that the operators T_{ij} are represented by the diagonal matrices Γ_{ij} under E^*. Suppose that

$$\Gamma_{ij} = [\delta_{mk}\gamma_{ijm}]_{m,h=\overline{1,q}}, \ \ i = 1,2,\ldots,n, \ \ i \in \mathbb{N}.$$

Then

$$T_{ij}e_m^* = \gamma_{ijm}e_m^*, \ \ i = 1,2,\ldots,n, \ \ j \in \mathbb{N}.$$

For every $m = 1,2,\ldots,q$, set

$$\Lambda_m = \{\gamma_{ijm}, i = 1,2,\ldots,n, j \in \mathbb{N}\}, \ \ m = 1,2,\ldots,q.$$

We say that Λ_k is equivalent to Λ_l if for all $i = 1,2,\ldots,n, j \in \mathbb{N}$, $\gamma_{ijk} = \gamma_{ijl}$. In this case, we write $\Lambda_k \equiv \Lambda_l$. We divide the set E^* into equivalent classes as follows

$$e_i^* \cong e_j^* \quad \text{if and only if} \quad \Lambda_i \equiv \Lambda_j.$$

Suppose that the system E^* is divided into p disjoint classes, i.e.

$$E^* = \bigcup_{\nu=1}^{p} E_\nu^*, \ \ E_\nu^* \bigcap_{\nu \mathbb{N} e \mu} E_\mu^* = \emptyset.$$

We set

$$Z_\nu^* = \operatorname{lin}\{E_\nu^*\}, \ \ \nu = 1,2,\ldots,p.$$

It is clearly that

$$\ker D = \bigoplus_{\nu=1}^{p} Z_\nu^*, \tag{3.7}$$

and

$$F_iR^jz = \Gamma_{ij}z = \gamma_{ij\nu}z \ \text{ for all } z \in Z_\nu^*, \ \ i = 1,2,\ldots,n, \ \ j \in \mathbb{N}.$$

Thus, the system of initial operators $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property corresponding to $Z_1^*, Z_2^*, \ldots, Z_p^*$. However, the decomposition (3.7) is *minimal* by mean that if $\{F_i\}_{i=\overline{1,n}}$ possesses the generalized $c(R)$-property respect to $Z_1', Z_2', \ldots, Z_q'$, then $q \geq p$.

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