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ON A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS WITH FIXED SECOND COEFFICIENT

Abstract. Using of Salagean operator, we define a new subclass of uniformly convex functions with negative coefficients and with fixed second coefficient. The main objective of this paper is to obtain coefficient estimates, distortion bounds, closure theorems and extreme points for functions belonging of this new class. The results are generalized to families with fixed finitely many coefficients.

1. Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$, let ST and CV the subclasses of S that are, respectively, starlike and convex. Goodman ([8] and [9]) introduced and defined the following subclasses of CV and ST.

A function $f(z)$ is uniformly convex (uniformly starlike) in U if $f(z)$ is in $CV(ST)$ and has the property that for every circular arc γ contained in U, with center ζ also in U, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\zeta)$. The class of uniformly convex functions is denoted by UCV and the class of uniformly starlike functions by UST (for details see [8]). It is well known from ([15] and [18]) that

$$f(z) \in UCV \Leftrightarrow \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{1.2}$$

Later on, Ronning [19] introduced a new class S_p of starlike functions related

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to UCV defined as

$$f(z) \in S_p \Leftrightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U. \tag{1.3}$$

Note that

$$f(z) \in UCV \Leftrightarrow zf'(z) \in S_p. \tag{1.4}$$

Also in [18], Ronning generalized the classes UCV and S_p by introducing a parameter α in the following way.

A function $f(z)$ of the form (1.1) is in $S_p(\alpha)$ if it satisfies the analytic characterization:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, -1 \leq \alpha < 1;\ z \in U, \tag{1.5}$$

and $f(z) \in UCV(\alpha)$, the class of uniformly convex functions of order α, if and only if $zf'(z) \in S_p(\alpha)$.

By $\beta - UCV$, $\beta \geq 0$, we denote the class of all β-uniformly convex functions introduced by Kanas and Wisniowska [13], it is known [13] that $f(z) \in \beta - UCV$ if and only if it satisfies the following condition:

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, z \in U; \beta \geq 0. \tag{1.6}$$

We consider the class $\beta - S_p$, $\beta \geq 0$, of β-starlike functions (see [14]) which are associated with β-uniformly convex functions by the relation:

$$f(z) \in \beta - UCV \Leftrightarrow zf'(z) \in \beta - S_p. \tag{1.7}$$

Thus, the class $\beta - S_p$, is the subclass of S, consisting of functions that satisfy the analytic condition:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, z \in U; \beta \geq 0. \tag{1.8}$$

For a function $f(z) \in S$, we define

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1, 2, \dots\}). \tag{1.9}$$

The differential operator D^n was introduced by Salagean [21].

For $\beta \geq 0$, $-1 \leq \alpha < 1$, $n \in N_0 = N \cup \{0\}$ and $m \in N$, we let $S(n, m, \alpha, \beta)$ denote the subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion:

$$\operatorname{Re}\left\{\frac{D^{n+m}f(z)}{D^n f(z)}-\alpha\right\}>\beta\left|\frac{D^{n+m}f(z)}{D^n f(z)}-1\right|,\qquad z\in U. \tag{1.10}$$

We denote by T the subclass of S consisting of functions of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k\qquad (a_k\geq 0). \tag{1.11}$$

Further, we define the class $TS(n,m,\alpha,\beta)$ by

$$TS(n,m,\alpha,\beta)=S(n,m,\alpha,\beta)\cap T. \tag{1.12}$$

We note that the operator D^{n+m} was studied by Sekine [22], Aouf et al. ([4] and [5]), Hossen et al. [11] and Aouf [2]. Also we note that $TS(n,1,\alpha,\beta)=T(n,\alpha,\beta)$ (Rosy and Murugusundaramoorthy [20]) and $TS(n,1,0,\beta)=S(n,1,0,\beta)$ (Kanas and Yaguchi [12]).

2. The Class $TS(n,m,\alpha,\beta)$

In this section we obtain necessary and sufficient conditions for functions $f(z)$ in the classes $TS(n,m,\alpha,\beta)$.

THEOREM 1. *A function $f(z)$ of the form (1.1) is in $S(n,m,\alpha,\beta)$ if*

$$\sum_{k=2}^{\infty}\delta(k,n,m,\alpha,\beta)\,|a_k|\leq 1-\alpha, \tag{2.1}$$

where $\delta(k,n,m,\alpha,\beta)=k^n[k^m(1+\beta)-(\alpha+\beta)]$, $-1\leq\alpha<1$, $\beta\geq 0$, $n\in N_0$ and $m\in N$.

Proof. It suffices to show that

$$\beta\left|\frac{D^{n+m}f(z)}{D^n f(z)}-1\right|-\operatorname{Re}\left\{\frac{D^{n+m}f(z)}{D^n f(z)}-1\right\}\leq 1-\alpha.$$

We have

$$\beta\left|\frac{D^{n+m}f(z)}{D^n f(z)}-1\right|-\operatorname{Re}\left\{\frac{D^{n+m}f(z)}{D^n f(z)}-1\right\}\leq(1+\beta)\left|\frac{D^{n+m}f(z)}{D^n f(z)}-1\right|$$

$$\leq\frac{(1+\beta)\sum\limits_{k=2}^{\infty}k^n(k^m-1)\,|a_k|}{1-\sum\limits_{k=2}^{\infty}k^n\,|a_k|}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}\delta(k,n,m,\alpha,\beta)\,|a_k|\leq 1-\alpha,$$

and hence the proof is complete.

THEOREM 2. *A necessary and sufficient condition for $f(z)$ of the form (1.11) to be in the class $TS(n,m,\alpha,\beta)$ is that*

$$\sum_{k=2}^{\infty}\delta(k,n,m,\alpha,\beta)a_k \le 1-\alpha. \tag{2.2}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f(z)\in TS(n,m,\alpha,\beta)$ and z is real, then

$$\frac{1-\sum_{k=2}^{\infty}k^{n+m}a_kz^{k-1}}{1-\sum_{k=2}^{\infty}k^na_kz^{k-1}}-\alpha \ge \beta\left|\frac{\sum_{k=2}^{\infty}k^n(k^m-1)a_kz^{k-1}}{1-\sum_{k=2}^{\infty}k^na_kz^{k-1}}\right|.$$

Letting $z\to 1^-$ along the real axis, we obtain the desired inequality (2.2).

COROLLARY 1. *Let the function $f(z)$ defined by (1.11) be in the class $TS(n,m,\alpha,\beta)$. Then*

$$a_k \le \frac{1-\alpha}{\delta(k,n,m,\alpha,\beta)} \qquad (k\ge 2). \tag{2.3}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\alpha}{\delta(k,n,m,\alpha,\beta)}z^k \qquad (k\ge 2). \tag{2.4}$$

Setting $k=2$ in (2.3), we have

$$a_2 \le \frac{1-\alpha}{\delta(2,n,m,\alpha,\beta)}. \tag{2.5}$$

Let $TS_c(n,m,\alpha,\beta)$ denote the class of functions $f(z)$ in $TS(n,m,\alpha,\beta)$ of the form

$$f(z)=z-\frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}z^2-\sum_{k=3}^{\infty}a_kz^k \qquad (a_k\ge 0), \tag{2.6}$$

where $0\le c\le 1$.

We note that:

(i) $TS_c(n,1,\alpha,0)=T_c(n,\alpha)\,(0\le\alpha<1, n\in N_0, 0\le c\le 1)$ (Aouf and Darwish [3]);

(ii) For $0\le\alpha<1, n\in N_0, m\in N, 0\le c\le 1$, we have

$$TS_c(n,m,\alpha,0)=\Bigg\{f(z)\in TS(n,m,\alpha):\mathrm{Re}\left\{\frac{D^{n+m}f(z)}{D^nf(z)}\right\}>\alpha,\quad f(z)=z-\frac{c(1-\alpha)}{2^n(2^m-\alpha)}z^2-\sum_{k=3}^{\infty}a_kz^k \quad (a_k\ge 0)\Bigg\}. \tag{2.7}$$

3. Coefficient estimates

THEOREM 3. *Let the function $f(z)$ be defined by (2.6). Then $f(z) \in TS_c(n,m,\alpha,\beta)$ if and only if*

$$\sum_{k=3}^{\infty} \delta(k,n,m,\alpha,\beta) a_k \leq (1-c)(1-\alpha). \tag{3.1}$$

The result is sharp for the function

$$f(z) = z - \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)} z^2 - \frac{(1-c)(1-\alpha)}{\delta(k,n,m,\alpha,\beta)} z^k \qquad (k \geq 3). \tag{3.2}$$

Proof. Putting

$$a_2 = \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}, \; 0 \leq c \leq 1, \tag{3.3}$$

in (2.2) and simplifying we get the result.

COROLLARY 2. *Let the function $f(z)$ defined by (2.6) be in the class $TS_c(n,m,\alpha,\beta)$. Then*

$$a_k \leq \frac{(1-c)(1-\alpha)}{\delta(k,n,m,\alpha,\beta)} \qquad (k \geq 3). \tag{3.4}$$

The result is sharp for the function $f(z)$ given by (3.2).

4. Extreme points

Employing the technique used earlier by Silverman and Silvia [23], Owa ([16] and [17]), Ganigi [7], Ahuja and Silverman [1], Aouf and Darwish [3], Aouf, Hossen and Srivastava [6] and Hossen[10] with the aid of Theorem 3, we can prove the following:

THEOREM 4. *Let*

$$f_2(z) = z - \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)} z^2 \tag{4.1}$$

and

$$f_k(z) = z - \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)} z^2 - \frac{(1-c)(1-\alpha)}{\delta(k,n,m,\alpha,\beta)} z^k \tag{4.2}$$

for $k = 3,4,\ldots$. Then $f(z)$ is in the class $TS_c(n,m,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z), \tag{4.3}$$

where $\lambda_k \geq 0$ and $\sum_{k=2}^{\infty} \lambda_k = 1$.

Corollary 3. *The extreme points of the class* $TS_c(n,m,\alpha,\beta)$ *are the functions* $f_k(z)\,(k \geq 2)$ *given by Theorem 4.*

5. Growth and distortion theorems for the class $TS_c(n,m,\alpha,\beta)$

Lemmas 1, 2 and 3 below will be required in our investigation of the growth and distortion properties of the general class $TS_c(n,m,\alpha,\beta)$.

Lemma 1. *Let the function* $f_3(z)$ *be defined by*

$$f_3(z) = z - \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}z^2 - \frac{(1-c)(1-\alpha)}{\delta(3,n,m,\alpha,\beta)}z^3. \tag{5.1}$$

Then for $0 \leq r < 1$ *and* $0 \leq c \leq 1$,

$$\left|f_3(re^{i\theta})\right| \geq r - \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}r^2 - \frac{(1-c)(1-\alpha)}{\delta(3,n,m,\alpha,\beta)}r^3 \tag{5.2}$$

with equality for $\theta = 0$. *For either* $0 \leq c < c_0$ *and* $0 \leq r \leq r_0$ *or* $c_0 \leq c \leq 1$,

$$\left|f_3(re^{i\theta})\right| \leq r + \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}r^2 - \frac{(1-c)(1-\alpha)}{\delta(3,n,m,\alpha,\beta)}r^3 \tag{5.3}$$

with equality for $\theta = \pi$. *Further, for* $0 \leq c < c_0$ *and* $r_0 \leq r < 1$,

$$\begin{aligned}\left|f_3(re^{i\theta})\right| \leq r\Bigg\{&(1 + \frac{c^2(1-\alpha)\delta(3,n,m,\alpha,\beta)}{4.\,2^n(1-c)\delta(2,n,m,\alpha,\beta)}) \\ &+ \left(\frac{c^2(1-\alpha)^2}{2(\delta(2,n,m,\alpha,\beta))^2} + \frac{2(1-c)(1-\alpha)}{\delta(3,n,m,\alpha,\beta)}\right)r^2 \\ &+ \left(\frac{(1-c)^2(1-\alpha)^2}{(\delta(3,n,m,\alpha,\beta))^2} + \frac{c^2(1-c)(1-\alpha)^3}{4\,(\delta(2,n,m,\alpha,\beta))^2\delta(3,n,m,\alpha,\beta)}\right)r^4\Bigg\}^{\frac{1}{2}}\end{aligned} \tag{5.4}$$

with equality for

$$\theta = \cos^{-1}\left(\frac{c(1-c)(1-\alpha)r^2 - c\delta(3,n,m,\alpha,\beta)}{4(1-c)\delta(2,n,m,\alpha,\beta)r}\right), \tag{5.5}$$

where

$$c_0 = \frac{[(1-\alpha) - 4\delta(2,n,m,\alpha,\beta) - \delta(3,n,m,\alpha,\beta)] + c_0^*}{2(1-\alpha)} \tag{5.6}$$

and

$$r_0 = \frac{-2(1-c)\delta(2,n,m,\alpha,\beta) + r_0^*}{c(1-c)(1-\alpha)} \tag{5.7}$$

where

$$(5.8)\quad c_0^* = \big\{\{(1-\alpha) - 4\delta(2,n,m,\alpha,\beta) - \delta(3,n,m,\alpha,\beta)\}^2 + 16(1-\alpha)\delta(2,n,m,\alpha,\beta)\big\}^{\frac{1}{2}}$$

and

$$(5.9)\quad r_0^* = \sqrt{4(1-c)^2(\delta(2,n,m,\alpha,\beta))^2 + c^2(1-c)(1-\alpha)\delta(3,n,m,\alpha,\beta)}.$$

Proof. We employ the same technique as it was used by Silverman and Silvia [23]. Since

$$(5.10)\quad \frac{\partial\left|f_3(re^{i\theta})\right|^2}{\partial\theta} = 2(1-\alpha)r^3\sin\theta\left\{\frac{c}{\delta(2,n,m,\alpha,\beta)} + \frac{4(1-c)(1-\alpha)}{\delta(3,n,m,\alpha,\beta)}r\cos\theta - \frac{c(1-c)(1-\alpha)}{\delta(2,n,m,\alpha,\beta)\delta(3,n,m,\alpha,\beta)}r^2\right\}$$

we can see that

$$(5.11)\quad \frac{\partial\left|f_3(re^{i\theta})\right|^2}{\partial\theta} = 0$$

for $\theta_1 = 0$, $\theta_2 = \pi$, and

$$(5.12)\quad \theta_3 = \cos^{-1}\left(\frac{c(1-c)(1-\alpha)r^2 - c\delta(3,n,m,\alpha,\beta)}{4(1-c)\delta(2,n,m,\alpha,\beta)r}\right).$$

Since θ_3 is a valid root only when $-1 \le \cos\theta_3 \le 1$, hence we have a third root if and only if $r_0 \le r < 1$ and $0 \le c < c_0$. Thus the results of the theorem follows from comparing the extremal values $\left|f_3(re^{i\theta_k})\right|$ $(k = 1,2,3)$ on the appropriate intervals.

LEMMA 2. *Let the functions* $f_k(z)\,(k \ge 4)$ *be defined by (4.2). Then*

$$(5.13)\quad \left|f_k(re^{i\theta})\right| \le |f_4(-r)| \qquad (k \ge 4).$$

Proof. Since $f_k(z) = z - \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}z^2 - \frac{(1-c)(1-\alpha)}{\delta(k,n,m,\alpha,\beta)}z^k$ and $\frac{(1-c)(1-\alpha)r^k}{\delta(k,n,m,\alpha,\beta)}$ is a decreasing function of k, we have

$$\left|f_k(re^{i\theta})\right| \le r + \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}r^2 + \frac{(1-c)(1-\alpha)}{\delta(4,n,m,\alpha,\beta)}r^4 = -f_4(-r),$$

which proves (5.13).

THEOREM 5. *Let the function* $f(z)$ *defined by (2.6) be in the class* $TS_c(n,m,\alpha,\beta)$. *Then, for* $0 \le r < 1$,

$$(5.14)\quad \left|f(re^{i\theta})\right| \ge r - \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}r^2 - \frac{(1-c)(1-\alpha)}{\delta(3,n,m,\alpha,\beta)}r^3$$

with equality for $f_3(z)$ *at* $z = r$, *and*

$$\left|f(re^{i\theta})\right| \le \max\{\max_\theta \left|f_3(re^{i\theta})\right|, -f_4(-r)\}, \tag{5.15}$$

where $\max_\theta \left|f_3(re^{i\theta})\right|$ *is given by Lemma* 1.

The proof of Theorem 5 is obtained by comparing the bounds given by Lemma 1 and Lemma 2.

REMARK 1. Putting (i) $c = 1$ and (ii) $c = 1$ and $\beta = 0$ in Theorem 5, we obtain the following results.

COROLLARY 4. *Let the function* $f(z)$ *defined by (1.11) be in the class* $TS(n, m, \alpha, \beta)$. *Then for* $|z| = r < 1$, *we have*

$$r - \frac{(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}r^2 \le |f(z)| \le r + \frac{(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}r^2 . \tag{5.16}$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}z^2. \tag{5.17}$$

COROLLARY 5. *Let the function* $f(z)$ *defined by (1.11) be in the class* $TS(n, m, \alpha, 0) = TS(n, m, \alpha)$. *Then for* $|z| = r < 1$, *we have*

$$r - \frac{1-\alpha}{\delta(2,n,m,\alpha,0)}r^2 \le |f(z)| \le r + \frac{1-\alpha}{\delta(2,n,m,\alpha,0)}r^2. \tag{5.18}$$

The result is sharp.

LEMMA 3. *Let the function* $f_3(z)$ *be defined by* (5.1). *Then, for* $0 \le r < 1$ *and* $0 \le c \le 1$,

$$\left|f_3'(re^{i\theta})\right| \ge 1 - \frac{c(1-\alpha)}{\delta(2,n-1,m,\alpha,\beta)}r - \frac{(1-c)(1-\alpha)}{\delta(3,n-1,m,\alpha,\beta)}r^2 \tag{5.19}$$

with equality for $\theta = 0$. *For either* $0 \le c < c_1$ *and* $r_1 \le r < 1$,

$$\left|f_3'(re^{i\theta})\right| \le 1 + \frac{c(1-\alpha)}{\delta(2,n-1,m,\alpha,\beta)}r - \frac{(1-c)(1-\alpha)}{\delta(3,n-1,m,\alpha,\beta)}r^2 \tag{5.20}$$

with equality for $\theta = \pi$. *Furthermore, for* $0 \le c < c_1$ *and* $r_1 \le r < 1$,

$$\begin{aligned}\left|f_3'(re^{i\theta})\right| \le \Bigg\{ \Bigg(1 + \frac{c^2(1-\alpha)\delta(3,n-1,m,\alpha,\beta)}{(1-c)(\delta(2,n,m,\alpha,\beta))^2} \Bigg) \\ + \left(\frac{2c^2(1-\alpha)^2}{(\delta(2,n,m,\alpha,\beta))^2} + \frac{6(1-c)(1-\alpha)}{\delta(3,n,m,\alpha,\beta)} \right) r^2 \\ + \left(\frac{(1-c)^2(1-\alpha)^2}{(\delta(3,n-1,m,\alpha,\beta))^2} + \frac{3c^2(1-c)(1-\alpha)^3}{2^n\delta(2,n,m,\alpha,\beta)\delta(3,n,m,\alpha,\beta)} \right) r^4 \Bigg\}^{\frac{1}{2}}\end{aligned} \tag{5.21}$$

with equality for

$$\theta = \cos^{-1}\left(\frac{3c(1-c)(1-\alpha)r^2 - c\delta(3,n,m,\alpha,\beta)}{6(1-c)\delta(2,n,m,\alpha,\beta)r}\right) \tag{5.22}$$

where

$$c_1 = \frac{\{3(1-\alpha) - 6\delta(2,n,m,\alpha,\beta) - \delta(3,n,m,\alpha,\beta)\} + c_1^*}{6(1-\alpha)} \tag{5.23}$$

and

$$r_1 = \frac{-6(1-c)\delta(2,n,m,\alpha,\beta) + r_1^*}{6c(1-c)(1-\alpha)} \tag{5.24}$$

where

$$c_1^* = \Big\{\{3(1-\alpha) - \delta(3,n,m,\alpha,\beta) - 6\delta(2,n,m;\alpha,\beta)\}^2 + 72(1-\alpha)\delta(2,n,m,\alpha,\beta)\Big\}^{\frac{1}{2}} \tag{5.25}$$

and

$$r_1^* = \sqrt{36(1-c)^2(\delta(2,n,m,\alpha,\beta))^2 + 12c^2(1-c)(1-\alpha)\delta(3,n,m,\alpha,\beta)}. \tag{5.26}$$

The proof of Lemma 3 is given in much the same way as Lemma 1.

THEOREM 6. *Let the function $f(z)$ defined by (2.6) be in the class $TS_c(n,m,\alpha,\beta)$. Then for $0 \le r < 1$,*

$$\left|f_3'(re^{i\theta})\right| \ge 1 - \frac{c(1-\alpha)}{\delta(2,n-1,m,\alpha,\beta)}r - \frac{(1-c)(1-\alpha)}{\delta(3,n-1,m,\alpha,\beta)}r^2 \tag{5.27}$$

with equality for $f_3'(z)$ at $z = r$, and

$$\left|f_3'(re^{i\theta})\right| \le \max\{\max_\theta \left|f_3'(re^{i\theta})\right|, f_4'(-r)\} \tag{5.28}$$

where $\max_\theta \left|f_3'(re^{i\theta})\right|$ is given by Lemma 3.

6. Radii of starlikeness and convexity

THEOREM 7. *Let the function $f(z)$ defined by (2.6) be in the class $TS_c(n,m,\alpha,\beta)$. Then $f(z)$ is starlike of order $\rho (0 \le \rho < 1)$ in the disc $|z| < r_1(n,m,\alpha,\beta,c,\rho)$, where $r_1(n,m,\alpha,\beta,c,\rho)$ is the largest value for which*

$$\frac{c(1-\alpha)(2-\rho)}{\delta(2,n,m,\alpha,\beta)}r + \frac{(1-c)(1-\alpha)(k-\rho)}{\delta(k,n,m,\alpha,\beta)}r^{k-1} \le 1-\rho \qquad (k \ge 3). \tag{6.1}$$

The result is sharp with the extremal function

$$f_k(z) = z - \frac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}z^2 - \frac{(1-c)(1-\alpha)}{\delta(k,n,m,\alpha,\beta)}z^k, \quad \textit{for some } k. \tag{6.2}$$

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho \quad (0 \leq \rho < 1) \text{ for } |z| < r_1(n,m,\alpha,\beta,c,\rho).$$

Note that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\dfrac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}r + \sum\limits_{k=3}^{\infty}(k-1)a_k r^{k-1}}{1 - \dfrac{c(1-\alpha)}{\delta(2,n,m,\alpha,\beta)}r - \sum\limits_{k=3}^{\infty} a_k r^{k-1}} \leq 1-\rho, \tag{6.3}$$

for $|z| \leq r$ if and only if

$$\frac{c(1-\alpha)(2-\rho)}{\delta(2,n,m,\alpha,\beta)}r + \sum_{k=3}^{\infty}(k-\rho)a_k r^{k-1} \leq 1-\rho. \tag{6.4}$$

Since $f(z)$ is in the class $TS_c(n,m,\alpha,\beta)$, from (2.1) we may take

$$a_k = \frac{(1-c)(1-\alpha)}{\delta(k,n,m,\alpha,\beta)}\lambda_k \qquad (k \geq 3), \tag{6.5}$$

where $\lambda_k \geq 0\,(k \geq 3)$ and

$$\sum_{k=3}^{\infty} \lambda_k \leq 1, \tag{6.6}$$

each fixed r, we choose the positive integer $k_0 = k_0(r)$ for which $\dfrac{(k_0-\rho)}{\delta(k_0,n,m,\alpha,\beta)}r^{k_0-1}$ is maximal. Then it follows that

$$\sum_{k=3}^{\infty}(k-\rho)a_k r^{k-1} \leq \frac{(1-c)(1-\alpha)(k_0-\rho)}{\delta(k_0,n,m,\alpha,\beta)}r^{k_0-1}. \tag{6.7}$$

Hence $f(z)$ is starlike of order ρ in $|z| \leq r_1(n,m,\alpha,\beta,c,\rho)$ provided that

$$\frac{c(1-\alpha)(2-\rho)}{\delta(2,n,m,\alpha,\beta)} + \frac{(1-c)(1-\alpha)(k_0-\rho)}{\delta(k_0,n,m,\alpha,\beta)}r^{k_0-1} \leq 1-\rho. \tag{6.8}$$

We find the value $r_1 = r_1(n,m,\alpha,\beta,c,\rho)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{c(1-\alpha)(2-\rho)}{\delta(2,n,m,\alpha,\beta)}r_0 + \frac{(1-c)(1-\alpha)(k_0-\rho)}{\delta(k_0,n,m,\alpha,\beta)}r_0^{k_0-1} = 1-\rho. \tag{6.9}$$

Then this value r_0 is the radius of starlikeness of order ρ for functions $f(z)$ belonging to the class $TS_c(n, m, \alpha, \beta)$.

In a similar manner, we can prove the following theorem concerning the radius of convexity of order ρ for functions in the class $TS_c(n, m, \alpha, \beta)$.

THEOREM 8. *Let the function $f(z)$ defined by (2.6) be in the class $TS_c(n, m, \alpha, \beta)$. Then $f(z)$ is convex of order $\rho(0 \leq \rho < 1)$ in the disc $|z| < r_2(n, m, \alpha, \beta, c, \rho)$, where $r_2(n, m, \alpha, \beta, c, \rho)$ is the largest value for which*

$$\frac{c(1-\alpha)(2-\rho)}{\delta(2, n-1, m, \alpha, \beta)} r + \frac{(1-c)(1-\alpha)(k-\rho)}{\delta(k, n-1, m, \alpha, \beta)} r^{k-1} \leq 1-\rho \quad (k \geq 3). \tag{6.10}$$

The result is sharp for the function $f(z)$ given by (6.2).

7. The Class $TS_{c_{k,N}}(n, m, \alpha, \beta)$

Instead of fixing just the second coefficient, we can fix finitely many coefficients. Let $TS_{c_{k,N}}(n, m, \alpha, \beta)$ denote the class of functions $f(z)$ in $TS_c(n, m, \alpha, \beta)$ of the form:

$$f(z) = z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)}{\delta(k, n, m, \alpha, \beta)} z^k - \sum_{k=N+1}^{\infty} a_k z^k, \tag{7.1}$$

where $0 \leq \sum_{k=2}^{N} c_k = c \leq 1$. Note that $TS_{c_{k,2}}(n, m, \alpha, \beta) = TS_c(n, m, \alpha, \beta)$.

THEOREM 9. *The extreme points of $TS_{c_{k,N}}(n, m, \alpha, \beta)$ are*

$$z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)}{\delta(k, n, m, \alpha, \beta)} z^k$$

and

$$z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)}{\delta(k, n, m, \alpha, \beta)} z^k - \sum_{k=N+1}^{\infty} \frac{(1-c)(1-\alpha)}{\delta(k, n, m, \alpha, \beta)} z^k \quad \text{for } k = N+1,\ N+2, \ldots$$

The details of the proof of Theorem 9 are omitted.

REMARK 2. The characterization of the extreme points for the general class $TS_{c_{k,N}}(n, m, \alpha, \beta)$ enables us to solve the standard extremal problems in the same manner as was done for the special case $TS_c(n, m, \alpha, \beta)$. The details involved may be left as an exercise for the interested reader.

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