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ON PARTIAL SUMS OF CERTAIN ANALYTIC FUNCTIONS

Abstract. In the present paper we give some results concerning partial sums of certain analytic functions analogous to the results due to H. Silverman [J. Math. Anal. Appl. 209 (1997), 221-227]. All the results are sharp.

1. Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic (hence the series in (1.1) is convergent), and f is univalent in the open unit disc $U = \{z = |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of S that are, respectively, convex and starlike functions of order α, $0 \leq \alpha < 1$. For conveniece, we write $K(0) = K$ and $S^*(0) = S^*$(see, e.g. Srivastava an Owa [12]). Goodman ([2] and [3]) defined the following subclasses of K and S^*.

DEFINITION 1. A function f is uniformly convex (starlike) in U if f is in K (S^*) and has the property that for every circular arc γ contained in U, with center ζ also in U, the arc $f(\gamma)$ is convex (starlike with respect to $f(\zeta)$).

Goodman ([2] and [3]) then gave the following two-variable analytic characterizations of these classes, denoted, respectively, by UCV and UST.

THEOREM A. *A function f of the form* (1.1) *is in UCV if and only if*

$$\operatorname{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geqslant 0, \quad (z,\zeta) \in U \times U, \tag{1.2}$$

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and is in UST if and only if

$$\operatorname{Re}\left\{\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\right\} \geqslant 0, \quad (z,\zeta) \in U \times U. \tag{1.3}$$

Ma and Minda [6] and Ronning [7] independently found a more applicable one-variable characterization for UCV.

THEOREM B. *A function f of the form* (1.1) *is in UCV if and only if*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geqslant \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{1.4}$$

We note [3] that the classical Alexander's result, $f \in K \Leftrightarrow zf' \in S^*$ does not hold between the classes UCV and UST. Later on, Ronning [8] introduced a new class S_p of starlike functions related to UCV defined as

$$f(z) \in S_p \Leftrightarrow \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \left|\frac{zf'(z)}{f(z)}-1\right|, \quad z \in U. \tag{1.5}$$

Note that

$$f(z) \in UCV \Leftrightarrow zf'(z) \in S_p. \tag{1.6}$$

Also in [7], Ronning generalized the classes UCV and S_p by introducing a parameter α in the following way.

DEFINITION 2. A function f of the form (1.1) is in $S_p(\alpha)$, if it satisfies the analytic characterization:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\} \geqslant \left|\frac{zf'(z)}{f(z)}-1\right|, \quad \alpha \in R; z \in U, \tag{1.7}$$

and $f \in UCV(\alpha)$, the class of uniformly convex functions of order α, if and only if $zf' \in S_p$.

By $\beta - UCV$, $0 \leq \beta < \infty$, we denote the class of all β-uniformly convex functions introduced by Kanas and Wisniowska [4] . Recall that a function $f \in S$ is said to be β uniformly convex in U, if the image of every circular arc contained in U with center at ζ, where $|\zeta| \leq \beta$, is convex. Note that the class $1 - UCV$ coincides with the class UCV. Moreover, for $\beta = 0$ we get the class K. It is known that $f \in \beta - UCV$ if and only if it satisfies the following condition:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U, \quad 0 \leq \beta < \infty. \tag{1.8}$$

We consider the class $\beta - S_p, 0 \leq \beta < \infty$, of β-starlike functions (see[5]) which are associated with β-uniformly convex functions by the relation:

$$f(z) \in \beta - UCV \Leftrightarrow zf'(z) \in \beta - S_p. \tag{1.9}$$

Thus, the class $\beta - S_p, 0 \leq \beta < \infty$, is the subclass of S, consisting of functions that satisfy the analytic condition :

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U. \tag{1.10}$$

For a function f in S, we define

$$D^0 f(z) = f(z), \tag{1.11}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.12}$$

and

$$D^m f(z) = D(D^{m-1} f(z)) \quad (m \in N = \{1, 2, \dots\}). \tag{1.13}$$

The differential operator D^m was introduced by Salagean [10]. It is easy to see that

$$D^m f(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k, \quad m \in N_0 = N \cup \{0\}. \tag{1.14}$$

For $\beta \geqslant 0, -1 \leq \alpha < 1$ and $m \in N_0,$ we let $S^m(\alpha, \beta)$ denote the subclass of S consisting of functions f of the form (1.1) and satisfying

$$\text{Re}\left\{\frac{z(D^m f(z))'}{D^m f(z)} - \alpha\right\} > \beta\left|\frac{z(D^m f(z))'}{D^m f(z)} - 1\right|, \quad z \in U. \tag{1.15}$$

We note that $S^0(\alpha, 1) = S_p(\alpha)$ and $S^1(\alpha, 1) = UCV(\alpha)\,(-1 \leq \alpha < 1)$ (Bharati et al. [1]).

Also we note that

(*i*) $S^1(\alpha, \beta) = \beta - UCV(\alpha)$, the class of β-uniformly convex functions of order α,

$$= \left\{f(z) \in S : \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in U; \\ -1 \leq \alpha < 1, \beta \geqslant 0\right\}; \tag{1.16}$$

(*ii*) $S^0(\alpha, \beta) = \beta - S_p(\alpha)$, the class of β-starlike functions of order α,

$$= \left\{f(z) \in S : \text{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in U; \\ -1 \leq \alpha < 1, \beta \geqslant 0\right\}. \tag{1.17}$$

We denote by T the subclass of S consisting of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geqslant 0). \tag{1.18}$$

Further, we define the class $TS(m,\alpha,\beta)$ by

$$TS(m,\alpha,\beta) = S^m(\alpha,\beta) \cap T. \tag{1.19}$$

The class $TS(m,\alpha,\beta)$ was introduced and studied by Rosy and Murugusudaramoorthy [9]. The classes $T(0,\alpha,1) = S_pT(\alpha)$ and $T(1,\alpha,1) = UCV(\alpha)$ were studied by Bharati et al.[1] .

A sufficient condition for the function f of the form (1.1) to be in the class $S^m(\alpha,\beta)$ ($m \in N_0, -1 \le \alpha < 1$ and $\beta \geqslant 0$) is that

$$\sum_{k=2}^{\infty} k^m \left[k(1+\beta) - (\alpha+\beta)\right] |a_k| \le 1-\alpha. \tag{1.20}$$

For functions of the form (1.18), the sufficient condition (1.20) is also necessary (see [9]).

In this paper, applying the technique used by Silverman [11], we will investigate the ratio of a function of the form (1.1) to its sequence of partial sums $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ when the coefficients of f are sufficiently small to satisfy condition (1.20). More precisely, we will determine sharp lower bounds for

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\}, \ \operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\}, \ \operatorname{Re}\left\{\frac{f'(z)}{f'_n(z)}\right\} \text{ and } \operatorname{Re}\left\{\frac{f'_n(z)}{f'(z)}\right\}.$$

In the sequel, we will make use of the well-known result that $\operatorname{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0 \quad (z \in U)$ if and only if $w(z) = \sum_{k=1}^{\infty} c_k z^k$ satisfies the inequality $|w(z)| \le |z|$. Unless otherwise stated, we will assume that f is of the form (1.1) and its sequence of partial sums is denoted by $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$.

2. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $\beta \geqslant 0, 0 \le \alpha < 1$, and $m \in N_0$.

THEOREM 1. *If f of the form* (1.1) *satisfies the condition* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geqslant \frac{(n+1)^m\left[n(1+\beta)+1-\alpha\right] - (1-\alpha)}{(n+1)^m\left[n(1+\beta)+1-\alpha\right]} \quad (z \in U). \tag{2.1}$$

The result is sharp for every n, with the extremal function

$$f(z) = z + \frac{1-\alpha}{(n+1)^m\left[n(1+\beta)+1-\alpha\right]} z^{n+1}. \tag{2.2}$$

Proof. We may write

$$\frac{(n+1)^m\,[n(1+\beta)+1-\alpha]}{1-\alpha}\left\{\frac{f(z)}{f_n(z)}-\frac{(n+1)^m\,[n(1+\beta)+1-\alpha]-(1-\alpha)}{(n+1)^m\,[n(1+\beta)+1-\alpha]}\right\}$$

$$=\frac{1+\sum\limits_{k=2}^{n}a_kz^{k-1}+\frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_kz^{k-1}}{1+\sum\limits_{k=2}^{n}a_kz^{k-1}}$$

$$=\frac{1+A(z)}{1+B(z)}.$$

Set $\dfrac{1+A(z)}{1+B(z)}=\dfrac{1+w(z)}{1-w(z)}$, so that $w(z)=\dfrac{A(z)-B(z)}{2+A(z)+B(z)}$. Then

$$w(z)=\frac{\frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_kz^{k-1}}{2+2\sum\limits_{k=2}^{n}a_kz^{k-1}+\frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_kz^{k-1}}$$

and

$$|w(z)|\le\frac{\frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha}\sum\limits_{k=n+1}^{\infty}|a_k|}{2-2\sum\limits_{k=2}^{n}|a_k|-\frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha}\sum\limits_{k=n+1}^{\infty}|a_k|}.$$

Now $|w(z)|\le 1$ if

$$2\frac{(n+1)^m\,[n(1+\beta)+1-\alpha]}{1-\alpha}\sum_{k=n+1}^{\infty}|a_k|\le 2-2\sum_{k=2}^{n}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{n}|a_k|+\frac{(n+1)^m\,[n(1+\beta)+1-\alpha]}{1-\alpha}\sum_{k=n+1}^{\infty}|a_k|\le 1. \tag{2.3}$$

It is suffices to show that the left hand side of (2.3) is bounded above by $\sum\limits_{k=2}^{\infty}\dfrac{k^m\,[k(1+\beta)-(\alpha+\beta)]}{1-\alpha}|a_k|$, which is equivalent to

$$\sum_{k=2}^{n}\frac{k^m\,[k(1+\beta)-(\alpha+\beta)]-(1-\alpha)}{1-\alpha}|a_k|$$
$$+\sum_{k=n+1}^{\infty}\frac{k^m\,[k(1+\beta)-(\alpha+\beta)]-(n+1)^m\,[n(1+\beta)+1-\alpha]}{1-\alpha}|a_k|\geqslant 0.$$

To see that the function f given by (2.2) gives the sharp result, we observe for $z = re^{i\pi/n}$ that

$$\begin{aligned}\frac{f(z)}{f_n(z)} &= 1 + \frac{1-\alpha}{(n+1)^m\left[n(1+\beta)+1-\alpha\right]} z^n \\ &\to 1 - \frac{1-\alpha}{(n+1)^m\left[n(1+\beta)+1-\alpha\right]} \\ &= \frac{(n+1)^m\left[n(1+\beta)+1-\alpha\right]-(1-\alpha)}{(n+1)^m\left[n(1+\beta)+1-\alpha\right]}\end{aligned}$$

when r→ 1^-. Therefore we complete the proof of Theorem 1.

REMARK 1. Putting (*i*) $m = 0$, (*ii*) $m = 0$ and $\beta = 1$, (*iii*) $m = 1$ and (*iv*) $m = \beta = 1$ in Theorem 1, we have

COROLLARY 1. *If f of the form* (1.1) *satisfies the condition* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *(with $m = 0$), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geqslant \frac{n(1+\beta)}{n(1+\beta)+1-\alpha} \quad (z \in U). \tag{2.4}$$

The result is sharp for every n, with the extremal function

$$f(z) = z + \frac{(1-\alpha)}{n(1+\beta)+1-\alpha} z^{n+1}. \tag{2.5}$$

COROLLARY 2. *If f of the form* (1.1) *satisfies the condition* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *(with $m = 0$ and $\beta = 1$), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geqslant \frac{2n}{2n+1-\alpha} \quad (z \in U). \tag{2.6}$$

The result is sharp for every n, with the extremal function

$$f(z) = z + \frac{(1-\alpha)}{2n+1-\alpha} z^{n+1}. \tag{2.7}$$

COROLLARY 3. *If f of the form* (1.1) *satisfies the condition* (1.20) *(with $m = 1$), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geqslant \frac{n\left[(n+1)(1+\beta)+1-\alpha\right]}{(n+1)\left[n(1+\beta)+1-\alpha\right]} \quad (z \in U). \tag{2.8}$$

The result is sharp for every n, with the extremal function

$$f(z) = z + \frac{1-\alpha}{(n+1)\left[n(1+\beta)+1-\alpha\right]} z^{n+1}. \tag{2.9}$$

COROLLARY 4. *If f of the form* (1.1) *satisfies the condition* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *(with $m = \beta = 1$), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geqslant \frac{n(2n+3-\alpha)}{(n+1)(2n+1-\alpha)} \quad (z \in U). \tag{2.10}$$

The result is sharp for every n, with the extremal function

$$f(z) = z + \frac{1-\alpha}{(n+1)(2n+1-\alpha)} z^{n+1} \quad (z \in U). \tag{2.11}$$

We next determine bounds for $\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\}$.

THEOREM 2. *If $f(z)$ of the form* (1.1) *satisfies condition* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, *then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geqslant \frac{(n+1)^m [n(1+\beta)+1-\alpha]}{(n+1)^m [n(1+\beta)+1-\alpha]+1-\alpha} \quad (z \in U). \tag{2.12}$$

The result is sharp for every n, with the extremal function $f(z)$ given by (2.2).

Proof. We write

$$\frac{(n+1)^m [n(1+\beta)+1-\alpha]+1-\alpha}{1-\alpha}\left\{\frac{f_n(z)}{f(z)} - \frac{(n+1)^m [n(1+\beta)+1-\alpha]}{(n+1)^m [n(1+\beta)+1-\alpha]+1-\alpha}\right\}$$

$$= \frac{1 + \sum_{k=2}^{n} a_k z^{k-1} - \frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}} = \frac{1+w(z)}{1-w(z)},$$

where

$$w(z) = \frac{\frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} a_k z^{k-1} + \frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k-1}}.$$

Now

$$|w(z)| \leq \frac{\frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha} \sum_{k=n+1}^{\infty} |a_k|}{2 - 2\sum_{k=2}^{n} |a_k| - \frac{(n+1)^m[n(1+\beta)+1-\alpha]}{1-\alpha} \sum_{k=n+1}^{\infty} |a_k|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{k=2}^{n} |a_k| + \frac{(n+1)^m [n(1+\beta)+(1-\alpha)]}{1-\alpha} \sum_{k=n+1}^{\infty} |a_k| \leq 1. \tag{2.13}$$

The left hand side of (2.13) is bounded above by $\sum_{k=2}^{\infty} \frac{k^m[k(1+\beta)-(\beta+\alpha)]}{1-\alpha}|a_k|$ if

$$\sum_{k=2}^{n}\left\{\frac{k^m\left[k(1+\beta)-(\alpha+\beta)\right]-(1-\alpha)}{(1-\alpha)}\right\}|a_k|$$
$$+\sum_{k=n+1}^{\infty}\left\{\frac{k^m\left[k(1+\beta)-(\alpha+\beta)\right]-(n+1)^m\left[n(1+\beta)+1-\alpha\right]}{1-\alpha}\right\}|a_k| \geqslant 0,$$

and the proof is completed.

REMARK 2. Putting (*i*) $m = 0$, (*ii*) $m = 0$ and $\beta = 1$, (*iii*) $m = 1$ and (*iv*) $m = \beta = 1$ in Theorem 2, we obtain the following sharp results.

COROLLARY 5. *If $f(z)$ of the form* (1.1) *satisfies* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *(with $m = 0$), then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geqslant \frac{n(1+\beta)+1-\alpha}{n(1+\beta)+2-2\alpha} \quad (z \in U). \tag{2.14}$$

COROLLARY 6. *If $f(z)$ of the form* (1.1) *satisfies* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *(with $m = 0$ and $\beta = 1$), then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geqslant \frac{2n+1-\alpha}{2(n+1-\alpha)} \quad (z \in U). \tag{2.15}$$

COROLLARY 7. *If $f(z)$ of the form* (1.1) *satisfies* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *(with $m = 1$), then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geqslant \frac{(n+1)\left[n(1+\beta)+1-\alpha\right]}{(n+1)\left[n(1+\beta)+1-\alpha\right]+(1-\alpha)} \quad (z \in U). \tag{2.16}$$

COROLLARY 8. *If $f(z)$ of the form* (1.1) *satisfies* (1.20) *and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$ *(with $m = \beta = 1$), then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geqslant \frac{(n+1)\,(2n+1-\alpha)}{(n+1)(2n+1-\alpha)+(1-\alpha)} \quad (z \in U). \tag{2.17}$$

We next turn to ratios involving derivatives.

THEOREM 3. *If $f(z)$ of the form* (1.1) *satisfies* (1.20), *then for* $z \in U$,

$$(a)\ \operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geqslant \frac{n(1+\beta)+(1-\alpha)\left[1-(n+1)^{(1-m)}\right]}{n(1+\beta)+(1-\alpha)}, \tag{2.18}$$

and

$$(2.19) \qquad (b)\ \operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geqslant \frac{n(1+\beta)+(1-\alpha)}{n(1+\beta)+(1-\alpha)\left[1+(n+1)^{(1-m)}\right]}.$$

In both cases, the extremal function is given by (2.2).

Proof. We prove only (a), which is similar in spirit of the proof of Theorem 1. The proof of (b) follows the pattern of that in Theorem 2 . We write

$$\frac{(n+1)^{m-1}\left[n(1+\beta)+1-\alpha\right]}{1-\alpha}\left\{\frac{f'(z)}{f_n'(z)}-\frac{n(1+\beta)+(1-\alpha)\left[1-(n+1)^{(1-m)}\right]}{n(1+\beta)+(1-\alpha)}\right\} = \frac{1+w(z)}{1-w(z)},$$

where

$$w(z) = \frac{\frac{(n+1)^{m-1}[n(1+\beta)+1-\alpha]}{1-\alpha}\sum_{k=n+1}^{\infty} ka_k z^{k-1}}{2+2\sum_{k=2}^{n} ka_k z^{k-1}+\frac{(n+1)^{m-1}[n(1+\beta)+1-\alpha]}{1-\alpha}\sum_{k=n+1}^{\infty} ka_k z^{k-1}}.$$

Now $|w(z)| \leq 1$ if

$$(2.20) \qquad \sum_{k=2}^{n} k\,|a_k| + \frac{(n+1)^{m-1}\left[n(1+\beta)+1-\alpha\right]}{1-\alpha}\sum_{k=n+1}^{\infty} k\,|a_k| \leq 1.$$

Since the left hand side of (2.20) is bounded above by $\sum_{k=2}^{\infty}\frac{k^m[k(1+\beta)-(\alpha+\beta)]}{1-\alpha}|a_k|$, the proof is completed .

REMARK 3. Putting (i) $m=0$, (ii) $m=0$ and $\beta=1$, (iii) $m=1$ and (iv) $m=\beta=1$ in Theorem 3, we obtain the following sharp results.

COROLLARY 9. *If $f(z)$ of the form* (1.1) *satisfies* (1.20) *(with $m=0$), then*

$$(2.21) \qquad (a)\ \operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geqslant \frac{n(\alpha+\beta)}{n(1+\beta)+1-\alpha} \quad (z \in U),$$

and

$$(2.22) \qquad (b)\ \operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geqslant \frac{n(1+\beta)+1-\alpha}{n(1+\beta)+(n+2)(1-\alpha)} \quad (z \in U).$$

COROLLARY 10. *If $f(z)$ of the form* (1.1) *satisfies* (1.20) *(with $m=0$ and $\beta=1$), then*

$$(2.23) \qquad (a)\ \operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geqslant \frac{n(1+\alpha)}{2n+1-\alpha} \quad (z \in U),$$

and

$$\text{(2.24)} \qquad (b)\ \operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geqslant \frac{2n+1-\alpha}{2n+(n+2)(1-\alpha)} \quad (z \in U).$$

COROLLARY 11. *If $f(z)$ of the form* (1.1) *satisfies* (1.20) *(with $m = 1$), then*

$$\text{(2.25)} \qquad (a)\ \operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geqslant \frac{n(1+\beta)}{n(1+\beta)+1-\alpha} \quad (z \in U),$$

and

$$\text{(2.26)} \qquad (b)\ \operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geqslant \frac{n(1+\beta)+1-\alpha}{n(1+\beta)+2(1-\alpha)} \quad (z \in U).$$

COROLLARY 12. *If $f(z)$ of the form* (1.1) *satisfies* (1.20) *(with $m = \beta = 1$), then*

$$\text{(2.27)} \qquad (a)\ \operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geqslant \frac{2n}{2n+1-\alpha} \quad (z \in U),$$

and

$$\text{(2.28)} \qquad (b)\ \operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geqslant \frac{2n+1-\alpha}{2(n+1-\alpha)} \quad (z \in U).$$

REMARK 4. Putting $m = \beta = 0$ in the above results we get the results obtained by Silverman [11].

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