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ON (m, n)-JORDAN DERIVATIONS AND COMMUTATIVITY OF PRIME RINGS

Abstract. The purpose of this paper is to prove the following result. Let $m \geq 1, n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with $char(R) \neq 2mn(m+n) \mid m-n \mid$. Suppose there exists a nonzero additive mapping $D : R \to R$ satisfying the relation $(m+n)D(x^2) = 2mD(x)x + 2nxD(x)$ for all $x \in R$ ((m,n)-Jordan derivation). If either $char(R) = 0$ or $char(R) > 3$ then D is a derivation and R is commutative.

This research is related to our earlier work [3] and [12]. Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. For $x, y \in R$ we write $[y,x]_1 = [y,x] = yx - xy$, and for $n \geq 1$, $[y,x]_n = \left[[y,x]_{n-1}, x\right]$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$, and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Obviously, any derivation is a Jordan derivation. The converse is in general not true. Herstein [9] has proved that any Jordan derivation on a prime ring with $char(R) \neq 2$ is a derivation. A brief proof of Herstein's result can be found in [1]. Cusack [7] has proved Herstenin's theorem for 2-torsion free semiprime rings (see [2] for an alternative proof). An additive mapping $D : R \to R$ is called a left derivation if $D(xy) = yD(x) + xD(y)$ holds for all pairs $x, y \in R$ and is called a left Jordan derivation (or Jordan left derivation) in case $D(x^2) = 2xD(x)$ is fulfilled for all $x \in R$. The concepts of left derivation and left Jordan derivation were introduced by Brešar and Vukman in [3]. One can easily prove (see [3]) that the existence of a nonzero

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left derivation on a prime ring forces the ring to be commutative. Moreover, we have the following result.

THEOREM 1. *Let R be a prime ring, and let $D: R \to R$ be a nonzero left Jordan derivation. If $char(R) \neq 2$ then D is a derivation and R is commutative.*

The result above has been first proved by Brešar and Vukman [3] under the additional assumption that $char(R) \neq 3$. Later on Deng [8] has proved that the assumption $char(R) \neq 3$ is superfluous. Theorem 1 is related to the theory of commuting and centralizing mappings. A mapping F, which maps a ring R into itself, is called centralizing on R in case $[F(x), x] \in Z(R)$ holds for all $x \in R$. In a special case when $[F(x), x] = 0$ is fulfilled for all $x \in R$, F is called commuting on R. A classical result of Posner (Posner's second theorem) [10] states that the existence of a nonzero centralizing derivation $D: R \to R$, where R is a prime ring, forces the ring to be commutative.

We proceed with the following definition.

DEFINITION 1. Let $m \geq 0$, $n \geq 0$ with $m + n \neq 0$ be some fixed integers. An additive mapping $D: R \to R$, where R is an arbitrary ring, is called a (m, n)-Jordan derivation in case

$$(1) \qquad (m+n)D(x^2) = 2mD(x)x + 2nxD(x)$$

holds for all $x \in R$.

Let us point out that the relation (1) appears naturally in the proof of Theorem 1 in [11]. Obviously, $(1, 1)$-Jordan derivation on a 2-torsion free ring is a Jordan derivation and $(1, 0)$-Jordan derivation is a left Jordan derivation.

We proceed with the following proposition.

PROPOSITION 1. *Let $m \geq 0$, $n \geq 0$ with $m + n \neq 0$ be some fixed integers, let R be a 2-torsion free ring, and let $D: R \to R$ be an (m, n)-Jordan derivation. In this case the relation*

$$(2) \qquad \begin{aligned}(m+n)^2D(xyx) = {} & m(n-m)D(x)xy + m(m-n)D(y)x^2 \\ & + n(n-m)x^2D(y) + n(m-n)yxD(x) \\ & + m(3m+n)D(x)yx + 4mnxD(y)x \\ & + n(3n+m)xyD(x), \qquad x, y \in R\end{aligned}$$

holds for all pairs $x, y \in R$.

Proof. The linearization of the relation (1) gives

$$(3) \quad (m+n)D(xy+yx) = 2mD(x)y + 2mD(y)x + 2nxD(y) + 2nyD(x), \qquad x, y \in R.$$

Putting in the relation (2) $(m+n)(xy+yx)$ for y, we obtain

$$
\begin{aligned}
&(m+n)^2D(x^2y+yx^2)+2(m+n)^2D(xyx)\\
&\quad=2m(m+n)D(x)(xy+yx)+2m(m+n)D(xy+yx)x\\
&\qquad+2n(m+n)xD(xy+yx)2n(m+n)(xy+yx)D(x), \qquad x,y\in R.
\end{aligned}
$$

Applying (2) and (1), we obtain

$$
\begin{aligned}
&4m^2D(x)xy+4mnxD(x)y+2m(m+n)D(y)x^2+2n(m+n)x^2D(y)\\
&\qquad+4mnyD(x)x+4n^2yxD(x)+2(m+n)^2D(xyx)\\
&\quad=2m(m+n)D(x)(xy+yx)+4m^2D(x)yx+4m^2D(y)x^2+4mnxD(y)x\\
&\qquad+4mnyD(x)x+4mnxD(x)y+4mnxD(y)x+4n^2x^2D(y)\\
&\qquad+4n^2xyD(x)+2n(m+n)(xy+yx)D(x), \qquad x,y\in R.
\end{aligned}
$$

After collecting terms we obtain

$$
\begin{aligned}
(m+n)^2D(xyx)&=m(n-m)D(x)xy+m(m-n)D(y)x^2\\
&\quad+n(n-m)x^2D(y)+n(m-n)yxD(x)\\
&\quad+m(3m+n)D(x)yx+4mnxD(y)x\\
&\quad+n(3n+m)xyD(x), \qquad x,y\in R
\end{aligned}
$$

which completes the proof.

It is our aim in this paper to prove the following result.

THEOREM 2. *Let $m\geq 1, n\geq 1$ be some fixed integers with $m\neq n$ and let R be a prime ring with $char(R)\neq 2mn(m+n)\mid m-n\mid$. Suppose $D:R\to R$ is a nonzero (m,n)-Jordan derivation. If $char(R)=0$ or $char(R)>3$, then D is a derivation and R is commutative.*

The proof of the above theorem depends heavily on the following result proved by Brešar [5] (see also [6]).

THEOREM 3. *Let R be a prime ring and $F:R\to R$ an additive mapping. Suppose that $[f(x),x]_n=0$, for all $x\in R$ and some fixed integer $n>1$. If either $char(R)=0$ or $char(R)>n$, then $[F(x),x]=0$, for all $x\in R$.*

Proof of Theorem 2. Putting in the relation (2) $y=x$ we obtain

$$
\begin{aligned}
(4)\qquad (m+n)^2D(x^3)&=m(3m+n)D(x)x^2\\
&\quad+4mnxD(x)x+n(3n+m)x^2D(x), \qquad x\in R.
\end{aligned}
$$

Now, putting $(m+n)^2x^3$ for y in the relation (3) and applying (4) we obtain

$$\begin{aligned}(m+n)^3D(x^4) &= m(m+n)^2D(x)x^3 + m(m+n)^2D(x^3)x \\ &\quad + n(m+n)^2xD(x^3) + n(m+n)^2x^3D(x) \\ &= m(m+n)^2D(x)x^3 + m((3m^2+mn)D(x)x^2 \\ &\quad + 4mnxD(x)x \\ &\quad + (3n^2+mn)x^2D(x))xnx((3m^2+mn)D(x)x^2 \\ &\quad + 4mnxD(x)x + (3n^2+mn)x^2D(x)) + n(m+n)^2x^3D(x) \\ &= (m(m+n)^2 + m(3m^2+mn))D(x)x^3 + (n(3m^2+mn) \\ &\quad + 4m^2n)xD(x)x^2 + (m(3n^2+mn) + 4mn^2)x^2D(x)x \\ &\quad + (n(m+n)^2 + n(3n^2+mn))x^3D(x), \qquad x \in R.\end{aligned}$$

We have therefore

$$\begin{aligned}(5)\quad (m+n)^3D(x^4) &= (4m^3+3m^2n+mn^2)D(x)x^3 \\ &\quad + (7m^2n+mn^2)xD(x)x^2 + (7mn^2+m^2n)x^2D(x)x \\ &\quad + (4n^3+3mn^2+m^2n)x^3D(x), \qquad x \in R.\end{aligned}$$

Putting $(m+n)x^2$ for x in (1) we obtain

$$\begin{aligned}(m+n)^3D(x^4) &= 2m(m+n)^2D(x^2)x^2 + 2n(m+n)^2x^2D(x^2) \\ &= 2m(m+n)(2mD(x)x + 2nxD(x))x^2 \\ &\quad + 2n(m+n)x^2(2mD(x)x + 2nxD(x)) \\ &= 4m^2(m+n)D(x)x^3 + 4mn(m+n)xD(x)x^2 \\ &\quad + 4mn(m+n)x^2D(x)x + 4n^2(m+n)x^3D(x), \qquad x \in R.\end{aligned}$$

We have therefore

$$\begin{aligned}(6)\qquad (m+n)^3D(x^4) &= 4m^2(m+n)D(x)x^3 + 4mn(m+n)xD(x)x^2 \\ &\quad + 4mn(m+n)x^2D(x)x \\ &\quad + 4n^2(m+n)x^3D(x), \qquad x \in R.\end{aligned}$$

By comparing (5) and (6) we obtain

$$\begin{aligned}&mn(n-m)D(x)x^3 + 3mn(m-n)xD(x)x^2 \\ &\qquad + 3mn(n-m)x^2D(x)x + mn(m-n)x^3D(x) = 0, \qquad x \in R,\end{aligned}$$

whence it follows $D(x)x^3 - 3xD(x)x^2 + 3x^2D(x)x - x^3D(x) = 0$, $x \in R$, which can be written in the form $[D(x), x]_3 = 0$, $x \in R$. According to Theorem 3 it follows that $[D(x), x] = 0$, for all $x \in R$, which makes it possible to replace in (1) $D(x)x$ with $xD(x)$. We have therefore $(m+n)D(x^2) = 2(m+n)xD(x)$, $x \in R$, which reduces to $D(x^2) = 2xD(x)$, $x \in R$. Applying again the fact that D is commuting on R, we arrive at $D(x^2) = D(x)x + xD(x)$, $x \in R$. In other words, D is a Jordan derivation, whence it follows that D is a derivation by Herstein's result. Since by Posner's second theorem the existence of commuting nonzero derivation on a prime ring forces the ring to be commutative, one can conclude that the proof of the theorem is complete.

In our forthcoming paper [12] we prove the following result. Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be a left Jordan derivation. In this case D is a derivation which maps R into $Z(R)$.

The result we have just mentioned above and Theorem 2 lead to the following conjecture.

CONJECTURE 1. *Let $m \geq 0$, $n \geq 0$ be fixed integers with $m+n \neq 0$, $m \neq n$ and let $D : R \to R$ be a (m,n)-Jordan derivation, where R is a semiprime ring with suitable torsion restrictions. In this case D is a derivation which maps R into $Z(R)$.*

An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation in case

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{7}$$

is fulfilled for all pairs $x, y \in R$. One can easily prove that any Jordan derivation which maps a 2-torsion free ring into itself, is a Jordan triple derivation. Brešar [4] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. According to all these observations and Proposition 1 we continue with the definition and the conjecture below.

DEFINITION 2. Let $m \geq 0$, $n \geq 0$ with $m+n \neq 0$ be some fixed integers. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called an (m,n)-Jordan triple derivation in case

$$\begin{aligned}(m+n)^2 D(xyx) = {} & m(n-m)D(x)xy + m(m-n)D(y)x^2 \\ & + n(n-m)x^2D(y) + n(m-n)yxD(x) \\ & + m(3m+n)D(x)yx + 4mnxD(y)x \\ & + n(3n+m)xyD(x)\end{aligned} \tag{8}$$

holds for all pairs $x, y \in R$.

According to Proposition 1 any (m,n)-Jordan derivation on arbitrary 2-torsion free ring is an (m,n)-Jordan triple derivation. We conclude with the following conjecture.

CONJECTURE 2. *Let $m \geq 0$, $n \geq 0$ be some fixed integers with $m + n \neq 0$, $m \neq n$, and let $D : R \to R$ be an (m,n)-Jordan triple derivation, where R is a semiprime ring with suitable torsion restrictions. In this case D is a derivation which maps R into $Z(R)$.*

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