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TRANSFORMATIONS BETWEEN MENGER SYSTEMS

Abstract. To define transformations between based universal algebras we must introduce representations that depend on the bases, contrary to what was possible for general vector spaces and believed possible for universal algebras. In fact, a counterexample shows that by representation-free transformations alone one cannot even ascertain whether a universal algebra has any dimension or not.

A transformation notion, which can do, concerns basis dependent Menger systems. It enjoys a basic geometric property of universal algebras, the preservation of reference flocks, and generalizes the transformation groups of Linear Algebra into groupoids.

0. Preliminaries

0.0. Introduction. The necessity of a transformation notion, distinct from isomorphisms, was acknowledged in vector spaces since 1889 [13]. In Universal Algebra, on the contrary, no notion of a transformation appeared, just some isomorphism variants (equivalence between algebras [3] and Marczewski's weak or general [1] isomorphisms) did. Since till last year even simple definitions of vector spaces as universal algebras [11, 12] lacked, this made conceivable that their two fields are distinct.

After introducing some notions of "Universal Mathematics", this paper provides universal algebras with a candidate for a transformation notion together with a counterexample to the belief that isomorphism ideas suffice. Its continuation (to appear here under the title "Sameness between based universal algebras") will validate this candidate by proving its equivalence to other new notions. Other motivations are in [10].

0.1. Notation. We conform to [4], but for the following few differences. We denote the set-theoretical pair $\{\{a\},\{a,b\}\}$ by $\langle a,b\rangle$, yet we still simplify $f(\langle a,b\rangle)$ into $f(a,b)$ and $\langle\langle x,y\rangle,z\rangle$ into (x,y,x) as in [4]. PX denotes the set of subsets of set X and $\boldsymbol{i}_X$ its identity function.

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We consider functional composition as the restriction of relational composition, here denoted by $\cdot$, namely $f \cdot g$ is "the composition of g and f" and $(f \cdot g)(x) = f(g(x))$. Accordingly, we perform the restriction of a function f to some set S merely by functional composition: $f \cdot \boldsymbol{i}_S$.

As usual, we write $f: A \to B$ to say that f is a function with arguments in the whole set A and values in B, $f: A \rightarrowtail B$ or $f: A \twoheadrightarrow B$ to say that it also is one to one or onto B and $f: A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ to say it is a bijection onto B. We will forget that "function – domain" and "family – index" are pairwise synonymic and we avoid the notation $\{a_i\}_{i \in I}$ or $(a_i \mid i \in I)$. Within informal comments we will replace "function" with "indexing", to emphasize values. Also, we denote the set-theoretical power ${}^A B = \{f \mid f: A \to B\}$ as the arithmetic one B^A. (The latter will not occur here.)

0.2. Endomorphism representations. Let $\mathbb{E}_\alpha \subseteq A^A$ be the set of all endomorphisms of an algebra α on A. Given a set X, let $b: X \to A$ and consider the function $\boldsymbol{r}_b: \mathbb{E}_\alpha \to A^X$, defined by $\boldsymbol{r}_b(h) = h \cdot b$, for $h \in \mathbb{E}_\alpha$, namely $\boldsymbol{r}_b$ "samples each h at" b by providing each $x \in X$ with the value $h(b(x))$. When a function $b: X \to A$ serves to define such a sampling of endomorphisms, we call it a *frame of* α. If this sampling represents every endomorphism by any sample and conversely, namely if we get that

$$\boldsymbol{r}_b: \mathbb{E}_\alpha \rightarrowtail\!\!\!\!\twoheadrightarrow A^X, \tag{0}$$

then every structure on $\mathbb{E}_\alpha$ defines another on A^X and we will say that

- $\boldsymbol{r}_b$ is a *(natural) analytic representation of* $\mathbb{E}_\alpha$, while X is its *dimension set* and the cardinality of X is its *dimension,*
- A^X is the set of the *(square universal) matrices of* α *with respect to* b, while every value $M(x)$ of a matrix $M: X \to A$ is its *column at* $x \in X$,
- b is a *basis* or *(universal) reference frame* of α, while its values $b(x)$ are *reference elements* or *selectors* that form the *basis set* $B \subseteq A$ for $b: X \twoheadrightarrow B$,
- the $\boldsymbol{r}_b$-image $\circ: A^X \times A^X \to A^X$ of functional composition on $\mathbb{E}_\alpha \subseteq A^A$ is its *matrix product* (that clearly has b as unit),
- b and the function $\chi: A \to A^{A^X}$, defined by (0) from the functional application of endomorphisms, as $\chi_a(\boldsymbol{r}_b(h)) = h(a)$ for $h \in \mathbb{E}_\alpha \subseteq A^A$ and $a \in A$, form the *Menger system derived from* α, with respect to the *frame of selectors* b, of dimension set X and that
- $\circ$ and b form the *monoid of the matrices of* α under $\boldsymbol{r}_b$ or with respect to b, which is isomorphic onto the endomorphism one by definition.

Notice that $A = \emptyset$ by (0) implies $X = \emptyset$, whereas for a singleton A every set X satisfies (0). In the former case we say that *the carrier (of the algebra) is trivial;* in the latter that *the algebra is trivial.* When the algebra is not trivial, $X = \emptyset$ iff $\mathbb{E}_\alpha = \{\boldsymbol{i}_A\}$. It does when all algebra elements are constants. This also implies that

(1) $X = \emptyset$ iff $\circ : 1 \times 1 \twoheadrightarrow 1$ or iff $\chi_a(M) = a$ for all $M : X \to A$, $a \in A$.

See **0.2** of [11] or of [10] for the differences between our Menger systems and the finitary ones of Universal Algebra as in [0]. See **0.5** of [11] for the equivalence between our basis definition and the conventional ones. See **0.6** of [11] for an example concerning the usual vector space.

1. Analytic monoids and constant generators

1.0. Definitions. Let X and A be two sets. Possibly, X can be a natural number $n = \{0, \dots, n-1\}$. Among the functions in A^X we consider the constant ones. For $a \in A \neq \emptyset$ we denote the one with value a by $\boldsymbol{k}_a$:

(2) $$\boldsymbol{k}_a(x) = a,$$

for all $x \in X \neq \emptyset$. Also, this always defines a *constant generating function* $\boldsymbol{k} : A \to A^X$. In fact, for $X = \emptyset$ and $A \neq \emptyset$ there only are the trivial cases $\boldsymbol{k}_a = \emptyset$ and for $A = \emptyset$ the case $\boldsymbol{k} = \emptyset$.

On A^X consider a binary operation $\circ : A^X \times A^X \to A^X$ (with infix notation) and assume it has a "right $\boldsymbol{K}$–preserved unit", viz. a function $U : X \to A$ with

(3) $$M \circ \boldsymbol{k}_{U(x)} = \boldsymbol{k}_{M(x)}$$

for all $M : X \to A$ and $x \in X$, that also is a "$\boldsymbol{K}$–restricted left unit", viz.

(4) $$U \circ \boldsymbol{k}_a = \boldsymbol{k}_a$$

for all $a \in A$, and satisfies a "$\boldsymbol{K}$–restricted associativity",

(5) $$(M \circ L) \circ \boldsymbol{k}_a = M \circ (L \circ \boldsymbol{k}_a),$$

for all $L, M : X \to A$ and all $a \in A$. Then, we will say that $\circ$ and U define an *analytic monoid of dimension set X on A* with the carrier A^X and that U is its *unit.* As shown in **2.1** of [8], (3), the *dimensionality* axiom, generalizes the idea that a Kronecker delta is diagonal, namely that each reference vector lies in its axis.

The requirement that $U : X \to A$ implies that for an empty A one cannot have an analytic monoid, unless X too is empty. In the latter case, the carrier is singleton, whatever A may be, and it also is iff A is, whatever X may be. On the contrary, when the carrier has at least two elements, we – as usual – will say that the analytic monoid is *non trivial.*

As far as such set–theoretical cases are concerned, only the first, $A = X = \emptyset$, is completely trivial and only it will allow us to skip definitions and proofs concerning the corresponding analytic monoids: most trivial analytic monoids are not trivial set–theoretically. In fact, even the null dimension case, $X = \emptyset$, determines a single analytic monoid, the trivial one with carrier $1 = \{\emptyset\}$, that *is on every* set A, since $A^\emptyset = \{\emptyset\}$ whatever A is.

Notice also that our three defining conditions are not the three *equational* conditions for monoids, and that (3) involves the dimension set. The first and last of the following properties motivate the name "analytic monoid", which still denotes a mathematical structure different from abstract monoids. (See [9] for details.)

1.1. Recalled properties. From **1.7** of [7] we recall that

(Monoid) *$\circ$ and U form a monoid on A^X;*

(χ–definability) $M \circ \boldsymbol{k}_a = \boldsymbol{k}_{(M \circ \boldsymbol{k}_a)(y)}$ *, for all $M : X \to A$, $a \in A$ and $y \in X$;*

(Analytic) *$\circ$ and U define an analytic monoid on A iff they form the monoid of the matrices of some algebra on A under the analytic representation $\boldsymbol{r}_U$ as in* **0.2**.

1.2. Definitions. We called the second property *χ–definability,* because it allows us to define a function $\chi : A \to A^{A^X}$, by

$$\chi_a(M) = \begin{cases} a & \text{when } X = \emptyset; \\ (M \circ \boldsymbol{k}_a)(x) & \text{for any } x \in X \neq \emptyset, \end{cases} \tag{6}$$

for all $M : X \to A$ and $a \in A$. This determines an algebra, made of constant–arity operations $\chi_a : A^X \to A$ indexed by the very carrier. We call such an algebra, together with U or without it, the *Menger system derived from* our analytic monoid on A. In fact, **1.5** (C) will show that, given any χ, U is unique. Given $\circ$, if $X \neq \emptyset$, then A and this Menger system are unique. When necessary, we will identify χ as the *algebra of* the Menger system.

We will also consider another analytic monoid of dimension set Y on B, denoted by $\diamond : B^Y \times B^Y \to B^Y$ and $V : Y \to B$, together with its derived Menger system $\xi : B \to B^{B^Y}$. Hereinafter, we will refer to them as the former and latter monoid respectively. By the Analytic property in **1.1** we can refer to their elements as the former and latter matrices respectively. The same for the derived Menger systems, their elements or operations, their "matrices" of arguments and so on.

Then, from (6) we respectively get

$$L \circ \boldsymbol{k}_a = \boldsymbol{k}_{\chi_a(L)}, \quad \text{for all } a \in A \text{ and } L : X \to A \text{ and} \tag{7}$$

$$M \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_{\xi_b(M)}, \quad \text{for all } b \in B \text{ and } M : Y \to B, \tag{8}$$

where $\boldsymbol{k} : A \to A^X$ and $\boldsymbol{\kappa} : B \to B^Y$ respectively denote the former and latter constant generators, both defined as in (2). Hence,

$$\boldsymbol{\kappa}_b(y) = b \ \text{ for all } b \in B,\ y \in Y \neq \emptyset, \tag{9}$$

while (3) – (5) become

(10) $M \diamond \boldsymbol{\kappa}_{V(y)} = \boldsymbol{\kappa}_{M(y)}$, for all $y \in Y$ and $M : Y \to B$,

(11) $V \diamond \boldsymbol{\kappa}_b = \boldsymbol{\kappa}_b$, for all $b \in B$, and

(12) $(M \diamond L) \diamond \boldsymbol{\kappa}_b = M \diamond (L \diamond \boldsymbol{\kappa}_b)$, for all $L, M : Y \to B$ and $b \in B$.

Similarly, when we consider two of the Menger systems derived from based algebras, defined in **0.2**, we denote the former and latter analytic representations by

(13) $$\boldsymbol{r}'_U : \mathcal{E} \twoheadrightarrow A^X \quad \text{and} \quad \boldsymbol{r}''_V : \mathcal{F} \twoheadrightarrow B^Y,$$

where $\mathcal{E} \subseteq A^A$ and $\mathcal{F} \subseteq B^B$ respectively denote the set of the endomorphisms of the former algebra and the one of the latter. Therefore, by (0)

(14) $e \in \mathcal{E}$ iff there is $L : X \to A$ such that $e(a) = \chi_a(L)$, for all $a \in A$

and

(15) $f \in \mathcal{F}$ iff there is $M : Y \to B$ such that $f(b) = \xi_b(M)$, for all $b \in B$.

1.3. Definitions. Given any two functions $U : X \to A$ and $V : Y \to B$, we will also define two *Menger systems,* without deriving them from either an algebra or an analytic monoid, by assigning two functions $\chi : A \to A^{A^X}$ and $\xi : B \to B^{B^Y}$ respectively, which satisfy three conditions each. As this disregards their representation use, often we will call any of them the *algebra of a Menger system.* We will still call U and V the *units* or *frames of selectors.*

The three defining conditions for the former Menger system are:

(16) $\chi_{U(x)}(L) = L(x)$, for all $L : X \to A$ and $x \in X$;

(17) $\chi_a(U) = a$, for all $a \in A$ and

(18) $\chi_{\chi_a(L)}(M) = \chi_a(M \circ L)$, for all $a \in A$ and $L, M : X \to A$,

where $\circ : A^X \times A^X \to A^X$ here denotes the composition defined by χ in (22). The three for the latter are:

(19) $\xi_{V(y)}(M) = M(y)$, for all $M : Y \to B$ and $y \in Y$;

(20) $\xi_b(V) = b$, for all $b \in B$ and

(21) $\xi_{\xi_b(M)}(L) = \xi_b(L \diamond M)$, for all $b \in B$ and $L, M : Y \to B$,

where $\diamond : B^Y \times B^Y \to B^Y$ here denotes the composition defined by ξ in (23).

(22) $(M \circ L)_x = \chi_{L(x)}(M)$, for all $L, M : X \to A$ and $x \in X$ and

(23) $(M \diamond L)_y = \xi_{L(y)}(M)$, for all $L, M : Y \to B$ and $y \in Y$.

The cases $X, A = \emptyset$ are the same as the ones for analytic monoids in **1.0** and (1) continues to hold by (17): when $X = \emptyset$, $\chi: A \mapsto\!\!\twoheadrightarrow A^1$ merely is the generator of singleton constants, while (22) defines $\circ: 1 \times 1 \mapsto\!\!\twoheadrightarrow 1$ trivially.

By the property (Menger to monoid) of **1.4** such compositions together with U or V respectively will define two analytic monoids that we call the *analytic monoids derived from* the corresponding Menger systems. The algebras of such systems also define endomorphism monoids. By the property (Endomorphism) of **1.4** we still denote their carriers by $\mathcal{E}$ and $\mathcal{F}$ respectively, e.g. $\mathcal{E} = \{e: A \to A \mid e(\chi_a(L)) = \chi_a(e \cdot L)$ for all $a \in A$ and $L: X \to A\}$.

1.4. Recalled properties. (See proofs either in **1.5** of [11] or in **1.4** of [10].) It does not matter how we define analytic monoids and Menger systems nor how they rise, namely

(Algebra to Menger) *the Menger system derived from a based algebra is a Menger system;* conversely,

(Menger to algebra) *every Menger system is derived from an algebra that can be the one of the Menger system, when derived with respect to its unit;*

(Menger to monoid) *the analytic monoid derived from a Menger system is an analytic monoid;*

(Monoid to Menger) *any Menger system derived from an analytic monoid is a Menger system;*

(Monoid loop) *every analytic monoid is derived from the Menger system derived from it;*

(Menger loop) *every Menger system is derived from the analytic monoid derived from it;*

(Endomorphism) *the algebra of the Menger system derived from an algebra keeps its set of endomorphisms.*

1.5. Corollaries.

(A) *$\circ$ and U form an analytic monoid iff they define the monoid derived from some Menger system and iff they form the monoid of the matrices of its algebra with respect to its unit.*

(B) *The algebras χ and ξ of two Menger systems derived from the same algebra α, with respect to possibly different reference frames, have the same endomorphisms: $\mathcal{E} = \mathcal{F} \subseteq A^A$.*

(C) *The algebra of a Menger system determines its frame of selectors.*

Proofs. (A) and (C) See either **1.6** of [11] or **1.5** of [10].

(B) By the property (Endomorphism) of **1.4** $\mathcal{E} = \mathbb{E}_\alpha = \mathcal{F}$. *Q.E.D.*

1.6. Definitions. Consider our constant generators $\boldsymbol{k}: A \to A^X$ and $\boldsymbol{\kappa}: B \to B^Y$. When $C \subseteq A^X$ and $D \subseteq B^Y$ denote the two corresponding sets of constant functions, unless $A = X = \emptyset$, we get two bijections,

(24) $\quad \boldsymbol{k}: A \twoheadrightarrow C$, when $X \neq \emptyset$, and $\boldsymbol{\kappa}: B \twoheadrightarrow D$, when $Y \neq \emptyset$,

or two constants: $X, Y = \emptyset$ respectively imply $C, D = 1 = \{\emptyset\}$, $\boldsymbol{k}: A \twoheadrightarrow C$ and $\boldsymbol{\kappa}: B \twoheadrightarrow D$. Also, $C = A^X$ iff A or X is at most singleton. Likewise for D.

We say that a function $t: A^X \to B^Y$ *retypes* $\boldsymbol{K}$, when for all $f: X \to A$ $t(f)$ is constant iff f is. This is the same as to require that $t \cdot \boldsymbol{i}_C$ is onto D. We also say that a bijection $t: A^X \twoheadrightarrow B^Y$ for $X, Y \neq \emptyset$ *depicts elements as constants,* when there exists a bijection $g: A \twoheadrightarrow B$ such that

$$t \cdot \boldsymbol{k} = \boldsymbol{\kappa} \cdot g \ . \tag{25}$$

(This cannot extend to the cases $X, Y = \emptyset$, where t does not determines A and/nor B, as it should become a property of A and/or B, not of t.)

1.7. Lemmata.

(A) *When $X, Y \neq \emptyset$, a bijection $t: A^X \twoheadrightarrow B^Y$ retypes $\boldsymbol{K}$ iff it depicts elements as constants.*

(B) *When a bijection $t: A^X \twoheadrightarrow B^Y$ retypes $\boldsymbol{K}$ and A has at least two elements, if X is singleton, then Y is.*

Proofs. (A) (Only if) As $t \cdot \boldsymbol{i}_C: C \twoheadrightarrow D$, $g = \boldsymbol{\kappa}^{-1} \cdot t \cdot \boldsymbol{k}$ provides us the required bijection by (24). (If) Since by (25) $t \cdot \boldsymbol{i}_C = t \cdot \boldsymbol{k} \cdot \boldsymbol{k}^{-1} = \boldsymbol{\kappa} \cdot g \cdot \boldsymbol{k}^{-1}$, the function $t \cdot \boldsymbol{i}_C$ is onto D, as required.

(B) When X is singleton, $C = A^X$ and it has at least two elements. Then, $t \cdot \boldsymbol{i}_C = t$ is onto both $B^Y \supseteq D$ and D. This implies that $D = B^Y$ and that it has at least two elements. No B with less than two elements can do it. Hence, Y too is singleton. *Q.E.D.*

1.8. Definition. When $X, Y \neq \emptyset$ and $t: A^X \twoheadrightarrow B^Y$ retypes $\boldsymbol{K}$, we say that t $\boldsymbol{K}$*-induces* the above $g: A \twoheadrightarrow B$. Clearly, (25) defines at most one g. By (9) and (25) the $\boldsymbol{K}$-induced bijection is defined by $g(a) = \boldsymbol{\kappa}_{g(a)}(y) = t(\boldsymbol{k}_a)(y)$, for all $a \in A$ and every $y \in Y$.

2. Flocks and dilatations

2.0. Definitions. Universal transformations will require to generalize some simple notions that we know from vector spaces to any based universal algebra. We say that $c \in A$ is a *flock combiner of* χ or *of* the Menger system of χ, when

(26) $$\chi_c(\boldsymbol{k}_a) = a, \text{ for all } a \in A.$$

Then, the element of a singleton A is a flock combiner. Hence, for $X = \emptyset$ by (1) and (6) $c \in A$ is a flock combiner iff A is singleton. Yet, things are less trivial for nontrivial dimensions as we know from vector spaces (see details in **1.7** of [11]).

Flock combiners define a *(universal) flock* $\Phi'_L \subseteq A$ *with respect to* χ by $\Phi'_L = \{\chi_c(L) \mid c \text{ is a flock combiner}\}$ from any matrix $L : X \to A$. By **1.4** (Menger to algebra), when we derive χ from a given algebra, we say that such a $\chi_c(L)$ is the *L–combination* of flock combiner c *with respect to* U and that Φ'_L is the *L–flock with respect to* U. (A flock in a vector space can also use flock combiners from vector spaces of a different dimension, e.g. in order to state that all the space is a flock, and this occurs on other algebras as in 1.2 (B) of [12], yet here we will not use this generalization.)

When L is our reference frame U, we will also say that flock Φ'_U is the *reference flock of* χ or *with respect to* U; likewise we define the reference flock Φ''_V of ξ. In **2.1** (C) this allows us to see combiners as combinations.

2.1. Recalled corollaries. (Proofs either in **1.8** of [11] or in **2.1** of [10].)

(A) *Bases are made of flock combiners,* $U : X \to \Phi'_U$ *and* $V : Y \to \Phi''_V$.

(B) In general, *each column of any matrix is a matrix combination,* $L : X \to \Phi'_L$ *for all* $L : X \to A$.

(C) *The set of all flock combiners is the reference flock.*

(D) *The flocks of non trivial constants are the singletons of their values:* $\Phi'_{\boldsymbol{k}(a)} = \{a\}$ *for all* $a \in A$ *with* $X \neq \emptyset$.

2.2. Definitions. Flock combiners are a case of a *dilatation indicator* defined in the former Menger system as an element $c \in A$ such that $\chi_c \cdot \boldsymbol{k} : A \to A$ is any endomorphism $e \in \mathcal{E}$ of χ. Then, e and its matrix $S = e \cdot U : X \to A$ are respectively called a *dilatation* and a *(universal) scalar* of χ (see **3.2** of [8], [6] and **5.1** of [5]), while c is called an *indicator of* e or *of* S.

In fact, (26) states that e is the identity on A (which always is in $\mathcal{E}$), namely flock combiners merely are the indicators of the identity. They also are general dilatation indicators up to the dilatations themselves, as **2.4** (A) will show.

The above dilatations are not all the ones of a Menger χ. When $X = \emptyset$, we say that $\boldsymbol{i}_A$ and its matrix $S = \emptyset$ are *the dilatation* and *the scalar* of $\chi : A \to A^1$ respectively, even for a non singleton A, namely even when there are not dilatation indicators. This is a split definition, yet it comes from the unsplit one in **2.5** of [6] for general universal algebras.

The latter uses unary elementary functions ("term operations"), not indicators, in order to define a dilatation as an isotropic endomorphism, without any splitting. (Such a unarity formalizes the isotropy condition for endomorphisms that concerns their "geometric" dimensions as in **5.1** of [5].) This does not matter till X has at least one element: any X–ary elementary functions is a χ_c, as shown in **6.3** and **6.7** of [5], and we get any dilatation as $\chi_c \cdot \boldsymbol{k}$, for some indicator c.

On the contrary, when $X = \emptyset$, every elementary function χ_a is a nullary constant. Unless A is singleton, no nullary function can replace the identity. Yet, the identity, the only endomorphism, always satisfies the recalled isotropy. Then, when the general definition applies to the algebra of a Menger system, both indicator defined dilatations and (in the last case) an identity without indicators can rise.

Anyway, the characterization in **2.4** (C) of scalars will avoid any splitting, as the recalled definition of general dilatation did. This characterization formally disregards any indicator and any dilatation. It also is fully analytic in the sense that it uses the multiplication of an analytic monoid to state a "$\boldsymbol{K}$-restricted" commutativity.

Indicators are not formally necessary to define scalars. They serve to determine the "amount" of a dilatation by an element, instead of by a matrix, as a scalar does. This will allow *dilatations to relate with carrier bijections.* Yet, while a dilatation has a single matrix, in general it has a set of indicators, possibly an empty one. I_e will denote the set of indicators of dilatation e.

Our split definition introduces scalars by dilatations also in order to show easily that universal scalars do correspond to the scalars we know from vector spaces as in **1.7** of [11]. As shown in the following, even the properties of indicators are extensions of the ones of flock combiners.

$F \subseteq A^X$ and $G \subseteq B^Y$ will respectively denote the sets of scalars of χ and ξ. $\Delta \subseteq \mathcal{E}$ and $\Gamma \subseteq \mathcal{F}$ will respectively denote the corresponding sets of dilatations. By **2.4** (F) and (G) in both cases such sets carry monoids that we respectively call the *scalar monoid* and the *dilatation monoid* of the corresponding reference frames, Menger systems or analytic monoids. Clearly, for $X = \emptyset$ they are fairly trivial, since $F = \{\emptyset\}$ and $\Delta = \{\boldsymbol{i}_A\}$.

2.3. Lemma. *c is a dilatation indicator in the former Menger system iff there exists $L : X \to A$ such that $\chi_c(\boldsymbol{k}_a) = \chi_a(L)$ for all $a \in A$. Likewise in the latter Menger system: d is iff there exists $M : Y \to B$ such that $\xi_d(\boldsymbol{\kappa}_b) = \xi_b(M)$ for all $b \in B$. Such an L and M are the scalars of the corresponding dilatations.*

Proof. The (iff) parts come from (14) and (15), while the scalar observations from (13) by (16) and (19), e.g. $(e \cdot U)(x) = ((\chi_c \cdot \boldsymbol{k}) \cdot U)(x) = \chi_c(\boldsymbol{k}_{U(x)}) = \chi_{U(x)}(L) = L(x)$ for all $x \in X \neq \emptyset$, while for $X = \emptyset$ it is trivial, $L = \emptyset$. *Q.E.D.*

2.4. Recalled properties.
(A) *For every scalar $S : X \to A$ of χ, the value $c = \chi_u(S)$ of its dilatation at any flock combiner $u \in \Phi'_U$ is an indicator of S.* (Proved in **2.3** of [11].)

(B) *For every scalar $S : X \to A$ of χ, each column S_x for $x \in X \neq \emptyset$ is a dilatation indicator of S: for all $e \in \Delta$, $e \cdot U : X \to I_e$.* (Proved in **2.4** ibid.)

(C) *A matrix $S : X \to A$ is a scalar of χ iff $S \circ \boldsymbol{k}_a = \boldsymbol{k}_a \circ S$ for all $a \in A$.* (Proof in **2.5** ibid..)

(D) *The product of a matrix $L : X \to I_e$ of indicators of a dilatation $e \in \Delta$ times one $M : X \to I_f$ for an $f \in \Delta$ is a matrix $M \circ L : X \to I_{e \cdot f}$ of indicators of the commuted corresponding composition.* (Proof in **2.5** ibid..)

(E) *For every scalar $S : X \to A$ of χ, let $c = \chi_u(S)$ be the value of its dilatation at any $u \in A$, then, if c is an indicator of S and the dilatation is one to one, u is a flock combiner, $u \in \Phi'_U$.* (Proof in **2.6** ibid..)

(F) *Scalars form a submonoid of the analytic monoid.* (Proof in **2.6** ibid..)

(G) *Dilatations form a submonoid of the endomorphism monoid and the scalar monoid is the isomorphic image of the dilatation monoid under the analytic representation.* (Proof in **2.6** ibid..)

(H) *The product of matrices of flock combiners is a matrix of flock combiners: $\circ' : \Phi_U^X \times \Phi_U^X \to \Phi_U^X$, where $\circ' = \circ \cdot \boldsymbol{i}_{\Phi_U^X \times \Phi_U^X}$ denotes this restriction of the product.* (Proof in **2.6** ibid..)

2.5. Example. Given a non trivial vector space with two reference frames $U, V : X \to A$, let F denote the carrier of its field, namely the set of its "vector-space scalars". In the former analytic monoid, consider the function $D : \mathrm{F} \to A^X$ that provides each number $s \in \mathrm{F}$ with its diagonal matrix $D_s = \overline{s} \cdot U$, where $\overline{s} : A \to A$ is the multiplication by s, namely $D_s(x) = sU_x$ for each $x \in X$. Likewise, in the latter consider $D' : \mathrm{F} \to A^X$ with $D'_s = \overline{s} \cdot V$.

We claim that $D : \mathrm{F} \rightarrowtail\!\!\!\twoheadrightarrow F$ *is an isomorphism from the monoid of the field product onto the scalar monoid of U, which determines dilatations that do not depend on the choice of the analytic representations in* **2.4** (G)*:*

$$\chi_a(D_s) = sa = \xi_a(D'_s), \quad \text{for all } a \in A \text{ and } s \in \mathrm{F}. \tag{27}$$

Proof. Since $\overline{s} \in \mathcal{E}$, (27) follows from (Endomorphism) in **1.4**, (17) and (20): e.g. $\chi_a(D_s) = \chi_a(\overline{s} \cdot U) = \overline{s}(\chi_a(U)) = \overline{s}(a) = sa$. See the proof in **2.7** (A) of [11] for the required isomorphism. *Q.E.D.*

The recalled proof uses the commutativity of the field product, which is not the minor property one could painlessly get rid of, as **2.6** (B) will show. "Scalars" in a skew field may not be universal scalars for its moduli.

2.6. Recalled theorems. (Proofs in **2.8** of [11].)

(A) *The set of indicators c of a bijective dilatation $e = \chi_c \cdot \boldsymbol{k} : A \rightarrowtail\!\!\!\twoheadrightarrow A$ with scalar $S = e \cdot U$ is the flock of the S–combinations: $I_e = \Phi'_S$.*

(B) *The scalar monoid is commutative.*

3. Descriptions

3.0. Definitions. The notion of a transformation in **3.3** will use some set-theoretical properties of preliminary notions, which also concern the crucial counterexample **3.6**. Given the two Menger systems of **1.3** and a bijection

$$g : A \rightarrowtail\!\!\!\twoheadrightarrow B \ , \tag{28}$$

consider the relation $t \subseteq A^X \times B^Y$ defined for all $L : X \to A$ and $M : Y \to B$ by $\langle L, M \rangle \in t$ iff for all $a \in A$

$$g(\chi_a(L)) = \xi_{g(a)}(M) \ . \tag{29}$$

An example of such a relation t is the one of a transformation of the matrices, for a linear or semi-linear transformation $g : A \rightarrowtail\!\!\!\twoheadrightarrow B$, of two based vector spaces, where $Y = X$ and $\langle L, M \rangle \in t$ iff

$$M = g \cdot L \ , \tag{30}$$

which implies $t : A^X \rightarrowtail\!\!\!\twoheadrightarrow B^Y$.

Notice that in general, whenever we consider two based algebras deriving our two Menger systems as in **1.4**, the *choice of the bases*, not an a priori assumption like (30), determines t from g. The counterexample in **3.6** (A) will show that (29) does not imply (30) nor its generalization $M = g \cdot L \cdot l^{-1}$ for any $l : X \rightarrowtail\!\!\!\twoheadrightarrow Y$. Besides, in the proof of **3.1** (B) we will see that the mere requirement that (29) holds for certain a's ensures that for each L in the domain of t there only is one way to get M. The general formula that expresses such one way and replaces (30) is (32).

If t relates every former matrix L with some latter matrix M and t^{-1}, conversely, every M with some L, then we will say that g *totally induces* t and we denote the function relating the g's to the t's by $T \subseteq B^A \times \mathsf{P}(A^X \times B^Y)$. We will also call the function $V' = g^{-1} \cdot V : Y \to A$ the *(algebraically) converse* *basis (with respect to g).* Here, V' need not to be a converse of V in the sense that $\langle V', V\rangle \in t$. We are merely recalling the restricted notion of t in (30) or (33) that comes from Algebra.

3.1. Lemmata. *If $g: A \rightarrowtail\!\!\twoheadrightarrow B$ totally induces t as above, then*

(A) (for when one of the sets of matrices is singleton) *trivial dimensions must coexist, $X = \emptyset$ iff $Y = \emptyset$, or both Menger systems have trivial algebras, hence in both cases*

(31) $$A^X = \{U\} \text{ iff } B^Y = \{V\},$$

(B) *the induced relation is a bijection, $t = T_g : A^X \rightarrowtail\!\!\twoheadrightarrow B^Y$ and,*

(C) *given χ, T_g depends only on V, through the converse basis $V' : Y \to A$:*

(32) $$(T_g(L))_y = g(\chi_{V'(y)}(L)) \text{ for all } L : X \to A \text{ and } y \in Y.$$

Proofs. (A) The coexistence of algebra triviality comes from (28). Then, consider dimension triviality with non trivial algebras. As observed in **1.3**, when $X = \emptyset$, χ is the generator of singleton constants $k : A \rightarrowtail\!\!\twoheadrightarrow A^1$. Then, $g(a) = \xi_{g(a)}(M)$ in (29). Since g is onto B, any M behaves as V in (20). By (23), **1.1** (Monoid) and **1.4** $M = V$, because a left unit of a monoid is its only unit. Hence, the total induction assumption implies $B^Y = \{V\}$. As B is not singleton, this implies $Y = \emptyset$. Conversely, for $Y = \emptyset$ we consider g^{-1}.

((B) and (C)) When A and B are singleton, both A^X and B^Y are. Hence both statements easily follow from (29) and (31). When $X = \emptyset$, by (A) the induced relation is the singleton function $t : 1 \rightarrowtail\!\!\twoheadrightarrow 1$ and (32) holds trivially. Otherwise, we can assume that both $X, Y \neq \emptyset$ and, hence, $A \neq \emptyset$.

Let us show that $t : A^X \to B^Y$. From **3.0**, for all $\langle L, M\rangle \in t$, (29) holds in particular for each $a = V'(y) = g^{-1}(V(y))$ with $y \in Y$. Hence, for all $y \in Y$ by (19) and (28), $M(y) = \xi_{V(y)}(M) = \xi_{g(g-1(V(y)))}(M) = \xi_{g(a)}(M) = g(\chi_a(L)) = g(\chi_{V'(y)}(L))$. Then, $M = T_g(L)$ as in (32) and $t = T_g$.

Since $t : A^X \twoheadrightarrow B^Y$ comes from the total induction assumption, now we only have to show $t : A^X \rightarrowtail B^Y$. This, easily follows after building the converse of (32). In fact, (29) by (28) becomes its converse: $g^{-1}(\xi_b(M)) = g^{-1}(g(\chi_{g^{-1}(b)}(L))) = \chi_{g^{-1}(b)}(L)$ for all $b = g(a) \in B$. This defines t^{-1}, which is totally induced by g^{-1}, since t was by g. From this converse of (29) we get the converse of (32), by using the converse basis $U' : X \to B$ with respect to g^{-1}: for all $M : Y \to B$ and $x \in X$, $(t^{-1}(M))_x = g^{-1}(\xi_{U'(x)}(M))$. This redefines t^{-1} as a function. Hence, t is one to one. *Q.E.D.*

3.2. Corollaries. *When $g: A \rightarrowtail\!\!\twoheadrightarrow B$ totally induces t as above:*

(A) *t preserves the frames of selectors, $t(U) = V$ and,*

(B) *if* $t = T_g : A^X \mapsto B^Y$ $\boldsymbol{K}$*-induces* g *as in* **1.8** *or if the two dimensions are trivial* (namely, if t retypes $\boldsymbol{K}$ as in **1.6**), *then* g *preserves reference flocks in both ways:* $c \in \Phi'_U$ *iff* $g(c) \in \Phi''_V$.

Proofs. (A) This follows from **3.1** (B) and (31), when either of reference frames is empty or either algebra is trivial. Otherwise, from (32) by (17), **3.0** and (28) $(t(U))_y = g(\chi_{V'(y)}(U)) = g(V'(y)) = g(g^{-1}(V(y))) = V(y)$ for all $y \in Y \neq \emptyset$. Hence, $t(U) = V$.

(B) In the trivial case by **3.1** (A) both reference flocks are either empty or the singleton carriers, as observed in **2.0**. Hence, the conclusion follows from (28).

Assume $X, Y \neq \emptyset$. Let $c \in \Phi'_U$, namely by **2.1** (C) $\chi_c \cdot \boldsymbol{k} = \boldsymbol{i}_A$. Then, for all $a \in A$, $g(\chi_c(\boldsymbol{k}_a)) = g(a)$ and by (29) and (25) $g(a) = \xi_{g(c)}(t(\boldsymbol{k}_a)) = \xi_{g(c)}(\boldsymbol{\kappa}_{g(a)})$. Since $g : A \twoheadrightarrow B$, we take $b = g(a)$ and get $\xi_{g(c)}(\boldsymbol{\kappa}_b) = b$ for all $b \in B$, namely $g(c) \in \Phi''_V$. Clearly, we can reverse all these implications. *Q.E.D.*

3.3. Definitions. Assume that g totally induces t and preserves both reference flocks, $a \in \Phi'_U$ iff $g(a) \in \Phi''_V$. Then, given χ and ξ, g and t induce each other as in the next characterization **3.4** (B) and we will say that our $g : A \mapsto B$ is a *description of* χ *by* ξ (see (35)) or *from* U *to* V or also a *description from the former monoid onto the latter.*

Lastly, as **3.1** (B) has shown that the induced relation t is a function, we will say that t is *a matrix transformation induced by* g or *the matrix transformation induced by* it *from* χ *to* ξ or also, in case a single algebra derives both χ and ξ, *the matrix transformation induced by* g *from* U *to* V.

Notice that, if $X = Y = \emptyset$ or both algebras are trivial as in **3.1** (A), then every bijection $g : A \mapsto B$ is a description by (1) and **2.0**. In such a case, $t : \{U\} \mapsto \{V\}$ is the only matrix transformation.

Consider our two general Menger systems, but with the same dimension set: χ with basis $U : X \to A$ and ξ with basis $V : X \to B$. We say that $n : A \mapsto B$ is an *(element) renaming of* χ *by* ξ or that it *renames* χ *by* ξ *elementwise*, when it is a description of χ by ξ performing its matrix transformation $t = T_n$ columnwise, namely

(33) $\qquad t(M) = n \cdot M \qquad$ and

(34) $\qquad n(\chi_a(M)) = \xi_{n(a)}(n \cdot M)$, for all $a \in A$ and $M : X \to A$.

Clearly, we could easily extend such *renaming descriptions* from case $Y = X$ to the case of a bijection $l : X \mapsto Y$, yet hereinafter we will omit such seeming extensions. (Our choice in **0.2** of the bases as functions allows us to permute the selectors.) Notice also that, in case of an automorphism

$n : A \mapsto\!\!\!\twoheadrightarrow A$ of an algebra deriving our Menger systems, (34) rewrites as $\chi = \xi \cdot n$ because of the property (Endomorphism) in **1.4**.

Since in (34) χ_a and $\xi_{n(a)}$ are isomorphic, an element renaming is a (simple) case of general isomorphism [1]. Contrary to the case of vector spaces, **3.6** (A) will show a description that is not a renaming. A characterization of renamings will appear in **3.1** (A) of the continuation of this work.

3.4. Corollaries. *Let $g : A \mapsto\!\!\!\twoheadrightarrow B$ be a description of χ by ξ as above, then*

(A) *the converse basis set is made of flock combiners $V' = g^{-1} \cdot V : Y \to \Phi'_U$;*

(B) *when $X, Y \neq \emptyset$, $t = T_g : A^X \mapsto\!\!\!\twoheadrightarrow B^Y$ $\boldsymbol{K}$–induces g as in* **1.8** (then, T_g $\boldsymbol{K}$–induces g iff g preserves both reference flocks, because of **3.2** (B));

(C) *we can compute the operations of the (algebra of the) latter Menger system by the former,*

(35) $\xi_b(M) = g(\chi_{g^{-1}(b)}(t^{-1}(M)))$, *for all $b \in B$ and $M : Y \to B$,*

while we preserve the former operations as

(36) $g(\chi_a(L)) = \xi_{g(a)}(t(L))$, *for all $a \in A$ and $L : X \to A$;*

(D) *descriptions define an equivalence relation among Menger systems, namely*

(Symmetry) $g^{-1} : B \mapsto\!\!\!\twoheadrightarrow A$ *is a description of ξ by χ, while t^{-1} is its matrix transformation,*

(Transitivity) *if $h : B \mapsto\!\!\!\twoheadrightarrow C$ is a description of ξ by another Menger system γ on C, then $h \cdot g : A \mapsto\!\!\!\twoheadrightarrow C$ is of χ by γ with the composition of their matrix transformations;*

(E) *the set of descriptions between Menger systems derived from the same algebra α on A forms a (sub)group under the functional composition on A^A.*

Proofs. (A) It is trivial for $X, Y = \emptyset$, otherwise the preservation of the reference flocks in **3.3** implies it. In fact, $g(V'(y)) = V(y) \in \Phi''_V$ for each $y \in Y$ by **2.1** (A). Then, $V'(y) \in \Phi'_U$ for all $y \in Y$, as $c \in \Phi'_U$ iff $g(c) \in \Phi''_V$.

(B) By **1.7** (A) we can show (25). By (A) we can take any $c = V'(y) \in \Phi'_U$ for $y \in Y \neq \emptyset$ in (26) and by (32) and (9) get $((t \cdot \boldsymbol{k})(a))_y = (t(\boldsymbol{k}_a))_y = g(\chi_{V'(y)}(\boldsymbol{k}_a)) = g(a) = \boldsymbol{\kappa}_{g(a)}(y)$ for all $a \in A$ and $y \in Y$, namely $t \cdot \boldsymbol{k} = \boldsymbol{\kappa} \cdot g$.

(C) Take any $a \in A$ such that $g(a) = b$ and get (35) from (29) by **3.1** (B) and (28). To get (36), merely use **3.1** (B) on (29).

(D) (Symmetry) Total induction is symmetric as already observed in the proof of **3.1** (B) and the same holds for the preservation of the reference flocks by definition **3.3**.

(Transitivity) By **3.1** (B) both g and h induce bijections, $t = T_g : A^X \twoheadrightarrow B^Y$ and say $t' = T'_h : B^Y \rightarrowtail\twoheadrightarrow C^Z$. This implies $t' \cdot t : A^X \rightarrowtail\twoheadrightarrow C^Z$. Hence, to get the transitivity of total induction, we only have to prove that $h(g(\chi_a(L))) = \gamma_{h(g(a))}(t'(t(L)))$, for all $a \in A$ and $L : X \to A$. This easily follows from (36), used twice: $h(g(\chi_a(L))) = h(\xi_{g(a)}(t(L))) = \gamma_{h(g(a))}(t'(t(L)))$. Lastly, the transitivity of the preservation of reference flocks is trivial.

(E) The closure under composition was just proved in (D) (Transitivity), the composition inverse in (D) (Symmetry). As the composition was the functional one, we get the required group with unit $\boldsymbol{i}_A$. This unit is the renaming description that corresponds to $\xi = \chi$ or to $V = U$ with $t = \boldsymbol{i}_{A^X}$, yet **3.6** will show that, given $g = \boldsymbol{i}_A$, sometimes also other ξ's, V and t's can do. *Q.E.D.*

3.5. Example. (A) Given two Menger systems, by **1.4** (Menger to algebra) one might consider the isomorphisms between the algebras deriving them. In **3.3** we did not require that $g : A \rightarrowtail\twoheadrightarrow B$ be such an isomorphism nor later we proved it was. To check that this requirement is not granted consider the classical example for semi-linear transformations [13].

Let $\chi = \xi$, with $A = \mathrm{F}^3$, be the Menger system for the complex vector space on the complex field F with the Kronecker frame of selectors (versors) $U = V : 3 \to A$, namely $\chi_a(L)$ is the usual product of vector a times matrix $L : 3 \to A$. When we define $g : A \rightarrowtail\twoheadrightarrow A$ as the componentwise complex conjugation, we have a bijection that is not an automorphism of the space (nor of χ), such that

$$(37) \qquad g \cdot U = U.$$

Then, (32) by (19) defines $t : A^3 \to A^3$ as $(t(L))_y = g(\chi_{U(y)}(L)) = g(L(y))$ for $y = 0, 1, 2$, namely $t(L) = g \cdot L$ for all matrices $L : 3 \to A$. Because of our usual χ and ξ and of this $t : A^3 \twoheadrightarrow A^3$, any L and $M = t(L)$ easily satisfy (29) for all vectors $a : 3 \to \mathrm{F}$.

This implies that the relation induced by g in (29) contains our t, that it is totally induced and, by **3.1** (B), that it is t. Also, g preserves the reference flock, since $\sum_i c_i = 1$ iff $\sum_i [g(c)]_i = 1$ for all $c : 3 \to \mathrm{F}$. Therefore, g is a description, nay a renaming as in (34), but not an isomorphism.

(B) Notice that, while the choice of the reference frames determines a single automorphism $g : A \rightarrowtail\twoheadrightarrow A$, it does not for "self-descriptions" $g : A \rightarrowtail\twoheadrightarrow A$, in spite of the dependence found in **3.1** (C). In fact, the self-description g of (A) extends the identity on the reference vectors as in (37), yet by **3.4** (E) also $g = \boldsymbol{i}_A$ does and clearly its matrix transformation is $t = \boldsymbol{i}_{A^3}$. In each case both the description and its matrix transformation differ with respect to the other case.

Here, both descriptions are renamings that still *determine the latter reference frame* by (33) and its matrix transformation by **3.1** (C). To identify a "transformation", we do not need neither matrix transformations nor reference frames and, since the latter only determines isomorphisms, descriptions alone can replace them. Yet, to define such a description we still need some condition not involving matrix transformations.

In the continuation of this work **3.5** (B) will show that in vector spaces our renamings are the semi-linear transformations. Then, such a condition there is an equation that involves vector-space scalars. Their dilatations do not depend on any its reference frame, as formalized in **2.5**. This will explain why in such spaces abstract representation-free theories work.

In that continuation **3.4** will show that we can define our universal descriptions too by scalars through a condition formally identical to the one of semi-linear transformations. Yet, the next example will also show that universal scalars are representation dependent. Then, the very condition, used to get rid of reference frames in vector spaces, will prove that they become mandatory in general.

3.6. Example. (A) We show that outside vector spaces there are descriptions that are not renamings. We exhibit a description between two Menger systems of a different dimension that are derived from the same algebra. We first show the existence of such an algebra and we introduce it through some algebraic conventions that later we will replace by set-theoretical ones.

Let us consider a possible algebra on a carrier A with five operations, $\boldsymbol{f}_0, \boldsymbol{f}_1 : (A \times A) \times A \to A$ and $\boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{g}_2 : A \times A \to A$, that satisfy nine equations: for all $x : 2 \to A$, $y : 3 \to A$ and $z \in A$

$$(38) \qquad \begin{cases} \boldsymbol{f}_0(\boldsymbol{g}_0(x_0,x_1),\boldsymbol{g}_1(x_0,x_1),\boldsymbol{g}_2(x_0,x_1)) = & x_0 \ , \\ \boldsymbol{f}_1(\boldsymbol{g}_0(x_0,x_1),\boldsymbol{g}_1(x_0,x_1),\boldsymbol{g}_2(x_0,x_1)) = & x_1 \ , \end{cases}$$

$$(39) \qquad \begin{cases} \boldsymbol{g}_0(\boldsymbol{f}_0(y_0,y_1,y_2),\boldsymbol{f}_1(y_0,y_1,y_2)) = & y_0 \ , \\ \boldsymbol{g}_1(\boldsymbol{f}_0(y_0,y_1,y_2),\boldsymbol{f}_1(y_0,y_1,y_2)) = & y_1 \ , \\ \boldsymbol{g}_2(\boldsymbol{f}_0(y_0,y_1,y_2),\boldsymbol{f}_1(y_0,y_1,y_2)) = & y_2 \ , \end{cases}$$

$$(40) \qquad \begin{cases} \boldsymbol{f}_0(z,z,z) & = \boldsymbol{f}_1(z,z,z) \quad \text{and} \\ \boldsymbol{g}_0(z,z) = & \boldsymbol{g}_1(z,z) = \boldsymbol{g}_2(z,z) = z \ . \end{cases}$$

The natural correspondences $(A \times A) \times A \simeq A^3$ and $A \times A \simeq A^2$ allow us to replace our $\boldsymbol{f} : 2 \to A^{(A\times A)\times A}$ and $\boldsymbol{g} : 3 \to A^{A\times A}$ by the functions

$$(41) \qquad f' : 2 \to A^{A^3} \quad \text{and} \quad g' : 3 \to A^{A^2},$$

such that $f'_j(y) = \boldsymbol{f}_j(y_0,y_1,y_2)$ and $g'_i(x) = \boldsymbol{g}_i(x_0,x_1)$ for all $j \in 2$, $i \in 3$, $x : 2 \to A$ and $y : 3 \to A$. If $\mathbf{C}_{f'}$ and $\mathbf{C}_{g'}$ denote the functions $\mathbf{C}_{f'} : A^3 \to A^2$ and $\mathbf{C}_{g'} : A^2 \to A^3$ such that $(\mathbf{C}_{f'}(y))_j = f'_j(y)$ and $(\mathbf{C}_{g'}(x))_i = g'_i(x)$ for all such j, i, x and y, then we can rewrite (38) and (39) respectively as

(42) $$\mathbf{C}_{f'} \cdot \mathbf{C}_{g'} = \boldsymbol{i}_{A^2} \text{ and } \mathbf{C}_{g'} \cdot \mathbf{C}_{f'} = \boldsymbol{i}_{A^3} .$$

Therefore, we got two functions $f'' = \mathbf{C}_{f'}$ and $g'' = \mathbf{C}_{g'}$ that are the one the inverse of the other. Conversely, any $f'' : A^3 \rightarrowtail\!\!\!\!\twoheadrightarrow A^2$ and $g'' : A^2 \rightarrowtail\!\!\!\!\twoheadrightarrow A^3$, with $f'' = g''^{-1}$, define an f' and a g' as in (41), such that (42) holds. Hence, they also define an $\boldsymbol{f}$ and a $\boldsymbol{g}$ that satisfy the first five equations (38) and (39).

To check that all nine equations, (38), (39) and (40), are consistent, let us define a non trivial algebra without an empty carrier, satisfying them, by defining such f'' and g'' in a way compatible with (40). To do it, we take A to be the set of natural numbers, as usual.

Let C and D denote the subsets of constants in A^2 and A^3, $C = \{x : 2 \to A \mid x_0 = x_1\}$ and $D = \{y : 3 \to A \mid y_0 = y_1 = y_2\}$. Namely, when we set $X = 2$ and $Y = 3$ in (2) and (9), $C = \{\boldsymbol{k}_z \mid z \in A\}$ and $D = \{\boldsymbol{\kappa}_z \mid z \in A\}$. Then, we set $\bar{C} = A^2 \setminus C$ and $\bar{D} = A^3 \setminus D$ to get two bi-partitions $\{C, \bar{C}\}$ and $\{D, \bar{D}\}$ such that $A^2 = C \cup \bar{C}$ and $A^3 = D \cup \bar{D}$.

Clearly, we got two pairs of denumerable sets that allow us to take the bijection $d : C \rightarrowtail\!\!\!\!\twoheadrightarrow D$, such that $d(\boldsymbol{k}_z) = \boldsymbol{\kappa}_z$ for all $z \in A$, and to choose some bijection $e : \bar{C} \rightarrowtail\!\!\!\!\twoheadrightarrow \bar{D}$. Then, if we set $f'' = d^{-1} \cup e^{-1}$ and $g'' = d \cup e$, we get the required bijections, $f'' : A^3 \rightarrowtail\!\!\!\!\twoheadrightarrow A^2$ and $g'' : A^2 \rightarrowtail\!\!\!\!\twoheadrightarrow A^3$.

In fact, since $f'' = g''^{-1}$, we get the first five equations. Moreover, by (2) $\boldsymbol{f}_0(z, z, z) = f'_0(\boldsymbol{\kappa}_z) = (f''(\boldsymbol{\kappa}_z))_0 = (d^{-1}(\boldsymbol{\kappa}_z))_0 = \boldsymbol{k}_z(0) = z = \boldsymbol{k}_z(1) = (d^{-1}(\boldsymbol{\kappa}_z))_1 = (f''(\boldsymbol{\kappa}_z))_1 = f'_1(\boldsymbol{\kappa}_z) = \boldsymbol{f}_1(z, z, z)$ for all $z \in A$, while the remaining equations follows from (9) in the same way: e.g. $\boldsymbol{g}_0(z, z) = g'_0(\boldsymbol{k}_z) = (g''(\boldsymbol{k}_z))_0 = (d(\boldsymbol{k}_z))_0 = \boldsymbol{\kappa}_z(0) = z = \boldsymbol{\kappa}_z(1) = (d(\boldsymbol{k}_z))_1 = (g''(\boldsymbol{k}_z))_1 = g'_1(\boldsymbol{k}_z) = \boldsymbol{g}_1(z, z)$ for all $z \in A$.

Since there is such an algebra satisfying all nine equations, there also is a free one, satisfying them with a doubleton basis set. Since any such a free algebra satisfies the first five equations (38) and (39), by the theorem of A. Goetz and C. Ryll-Nardzewski [2] as in **5** §31 of [3] it has bases with two and more reference elements.

Now, let A and $U : 2 \to A$ respectively denote the carrier of such an algebra and one of these doubleton bases. Then, the former of (13) with $X = 2$ represents the set of endomorphisms $\mathcal{E}$ of this based algebra, as well as of the algebra of the Menger system with respect to any basis as in **1.5** (B). This will serve to prove that *the identity is a description between the Menger systems derived with respect to U and to another basis with three reference elements.*

P r o o f. Notice that the natural correspondence $A \times A \simeq A^2$ rewrites the endomorphic condition $h(\boldsymbol{g}_i(x_0, x_1)) = \boldsymbol{g}_i(h(x_0), h(x_1))$ for $h \in \mathcal{E}$ and x:

$2 \to A$ as $h(g'_i(x)) = g'_i(h \cdot x)$. Set $V = g''(U)$. Then $V : 3 \to A$ gets $\boldsymbol{r}''_V : \mathcal{E} \to A^3$, where we claim that $\boldsymbol{r}''_V$ is a bijection as in the latter of (13) with $B = A$, $\mathcal{F} = \mathcal{E}$ and $Y = 3$. In fact, since $(\boldsymbol{r}''_V(h))_i = (h \cdot g''(U))_i = h((g''(U))_i) = h(g'_i(U)) = g'_i(h \cdot U) = g'_i(\boldsymbol{r}'_U(h)) = (g''(\boldsymbol{r}'_U(h)))_i$ for all $h \in \mathcal{E}$ and $i \in 3$, it is the composition of two bijections:

$$\boldsymbol{r}''_V = g'' \cdot \boldsymbol{r}'_U. \tag{43}$$

This composition also allows the Menger system $\chi : A \to A^{A^2}$ with respect to U to redefine the one $\xi : A \to A^{A^3}$ with respect to V as

$$\xi_a = \chi_a \cdot f'', \qquad \text{for all } a \in A. \tag{44}$$

In fact, from (43) by (42) we get $\boldsymbol{r}'_U = g''^{-1} \cdot \boldsymbol{r}''_V = f'' \cdot \boldsymbol{r}''_V$. Then, by (15), **1.5** (B) and **0.2**, for all $M = \boldsymbol{r}''_V(e) : 3 \to A$ and $a \in A$, $\xi_a(M) = e(a) = \chi_a(\boldsymbol{r}'_U(e)) = \chi_a(f''(\boldsymbol{r}''_V(e))) = \chi_a(f''(M))$.

By (42) we can also rewrite (44) as $\chi_a = \xi_a \cdot g''$. Therefore, $g = \boldsymbol{i}_A : A \twoheadrightarrow A$ totally induces a relation containing g''. Since $g'' : A^2 \twoheadrightarrow A^3$, by **3.1** (B) this relation is g'' itself, $T_g = g''$. Moreover, g'' $\boldsymbol{K}$–induces g as in (25), because our free algebra satisfies (40): $((g'' \cdot \boldsymbol{k})(z))_i = (g''(\boldsymbol{k}_z))_i = g'_i(\boldsymbol{k}_z) = \boldsymbol{g}_i(z, z) = z = \boldsymbol{\kappa}_z(i) = ((\boldsymbol{\kappa} \cdot g)(z))_i$ for all $z \in A$ and $i \in 3$. Hence, by **3.2** (B) the identity $g = \boldsymbol{i}_A$ is a description of χ by ξ and g'' is its matrix transformation. *Q.E.D.*

(B) Such a g also is an automorphism as well as a general one as in **2** of [1]. Yet, in both cases the renaming condition (33) fails to provide U with the new basis $V = g''(U)$ we found. It gets U again, by the matrix transformation $\boldsymbol{i}_{A^2} : A^2 \twoheadrightarrow A^2$. Descriptions alone can miss some "transformation". This also shows that the dependence in **3.1** (C) of T on V is effective.

In **3.5** (B) were the bases to miss some "transformation". To get all of them in general vector spaces, linear transformations had to generalize into semi-linear ones. Therefore, also now we need a further generalization: our descriptions *with their matrix transformations* might be one.

Both semi-linear transformations and general isomorphisms are representation-free notions. If one could disregard the reference systems even in the universal case, then one would say that the present algebra, not just its former representation, has 2 as a dimension, whereas it does not. In fact, the description we found changes 2 into 3, in spite of the preservations in **3.4** (C). It does, *because it is from U to V.*

The coordinate-free (or in general representation-free) approach worked in vector spaces, also because of a yet unproved (see **3.5** (A) of the continuation of this work) property: all transformations can change reference frames only by the renaming condition. Anyway, we found that in the universal case this fails. Universal generality conflicts against this approach.

3.7. Transformation groupoids. Given a free algebra, **3.4** (E) provides its "self-descriptions" with a group, alike to the one of linear or semi-linear transformations for a vector space. Yet, contrary to the vector space case, its elements can miss some essential "transformation", as remarked in **3.6** (B) for the identity description and a dimension invariance. Therefore, a possible future theory of the universal transformations, introduced by this work, should use a structure different from this group.

If one wants to keep our notion of description, *that corresponds to the vector space transformation,* one of the possible structures might be a category: the one with bases as objects and with the triples $\langle U, g, V\rangle$, where g is a description from U to V, as morphisms. Since in **0.2** a basis identifies its Menger system, this structure can avoid the missing transformation problem because of **3.1** (C). Moreover, the proof of **3.4** (D) (Transitivity) still proves associativity, while the units are trivial. By **3.4** (D) (Symmetry) we get a groupoid.

Still, another category might deserve the name of *transformation groupoid:* the one where the matrix transformations replace the descriptions in the former category. By **3.4** (B) matrix transformations identify descriptions. They form a groupoid under functional composition (with the restriction of the source and target objects) for the same reasons as before. Yet, now we can see the objects as the units of the monoids of the matrices by **1.1** (Analytic). This better agrees with the finding in **0.4** of [8] that the objects naturally rising from the applications are (universal) matrices rather than their algebras.

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