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EQUATIONAL BASES FOR k-NORMAL IDENTITIES

Abstract. The depth of a term may be used as a measurement of complexity of identities. For any natural number k, an identity $u \approx v$ is called k-normal if $u = v$ or both u and v have depth at least k. For any variety V, the k-normalization of V is the variety $N_k(V)$ defined by all k-normal identities of V. We describe a process to produce from a basis for V a basis for $N_k(V)$, for any variety V which has an idempotent term; when the type of V is finite and V is finitely based, this results in a finite basis for $N_k(V)$ as well. This process encompasses several known examples, for varieties of bands and lattices, and allows us to give a new basis for the normalization of the variety PL of pseudo-complemented lattices.

1. Introduction

In this paper we develop a technique for producing from a basis for a variety V a basis for the k-normalization $N_k(V)$ of V, when V is a variety with an idempotent term. When V is finitely based and of finite type, our basis for $N_k(V)$ is also finite. In this section we introduce the definitions and background needed. Our basis construction is described and verified in Section 2, and in Section 3 we illustrate this process with some examples, including a previously unknown basis for the normalization of the variety of pseudo-complemented lattices.

Throughout this paper we consider a type $\tau = (n_i)_{i \in I}$ of algebras and identities, with f_i an n_i-ary operation symbol of type τ for each $i \in I$, and we make the assumption that our type contains no nullary operation symbols. A number of structural properties of identities, including regularity, normality, externality and P-compatibility of identities have been much studied (see for instance [2], [10], [11], [12], [7], [4], [15], [16]. We consider here a generalization of the property of normality, using the complexity of terms

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and identities. This approach was used by Volkov in [17], using the length of a term (the total number of occurrences of variable symbols in the term) to measure complexity, and generalized to arbitrary complexity measures in [8]. Here we use the depth of terms as our complexity measurement. For any term t of type τ, we denote by $d(t)$ the depth of t, defined inductively by

(i) $d(t) = 0$, if t is a variable;

(ii) $d(t) = 1 + max\{d(t_j) : 1 \leq j \leq n_i\}$, if t is a composite term of the form $t = f_i(t_1, \ldots, t_{n_i})$.

When a term is portrayed by a tree diagram, with the nodes corresponding to operation symbols in the term and the leaves to variable symbols, the depth of the term t corresponds to the length of the longest path from root to leaves in the tree diagram for t.

Let $k \geq 0$ be any natural number. An identity $s \approx t$ of type τ is called *k-normal* (with respect to the depth measurement of complexity) if either s and t are equal, or $d(t)$, $d(s) \geq k$. It was proved in [8] that the set of all k-normal identities of type τ forms an equational theory. For any variety V, the set of all k-normal identities of V is also an equational theory, and the variety $N_k(V)$ determined by this set is called the *k-normalization of V*. In the special case that $N_k(V) = V$, we say that V is a *k-normal variety*; this occurs when every identity of V is a k-normal identity. Otherwise, V is a proper subvariety of $N_k(V)$, and $N_k(V)$ is the least k-normal variety to contain V. The variety $N_k(TR)$, where TR is the trivial variety of type τ, is the smallest k-normal variety, and has been described in [9]. In the case that $k = 1$, all of these concepts coincide with the usual concepts of a normal identity and variety and the normalization $N(V)$ of a variety V.

We note here that the concept of k-normal variety is related to Volkov's definition of a k-nilpotent variety in [17]. However, k-normality and k-nilpotence are different properties. First, Volkov used the length of a term as the complexity measurement, rather than the depth. More importantly, he defined a variety to be k-nilpotent if it satisfies *all* k-normal identities of its type. In this case, the variety $N_k(TR)$ is the largest k-nilpotent variety, and all k-nilpotent varieties are subvarieties of it. In our definition, a variety is k-normal if all of its identities are k-normal, making $N_k(TR)$ the smallest k-normal variety, and any variety which contains it a k-normal variety. Thus the variety $N_k(TR)$ is the only variety which is both k-nilpotent and k-normal. It is known that the k-normalization $N_k(V)$ of a variety V is equal to $N_k(TR) \vee V$, and by Volkov's theorem this is finitely based iff V is finitely based. However this is for the k-normalization based on length of words rather than depth, although our technique of inflating terms to produce a basis is similar to his proofs.

An algebraic characterization of the algebras in $N_k(V)$ by means of the algebras in V was given by Denecke and Wismath in [9], using the concept of a k-choice algebra. They showed that any algebra in $N_k(V)$ is a homomorphic image of a k-choice algebra constructed from an algebra in V. In this paper we return to the equational approach, by constructing a (finite) basis for $N_k(V)$ from a (finite) basis for V.

2. The basis construction

Let $k \geq 1$ be a fixed natural number. We consider a variety V with an idempotent term t: that is, we assume that there exists some term t, of some arity n, such that $t(x, \ldots, x) \approx x$ holds in V. By replacing occurrences of x in this identity by $t(x, \ldots, x)$ if necessary, we may assume that t has depth at least k. We start with an equational basis Σ_V for V.

Our basis construction, and the proof that it does give a basis, use the idea of inflation of terms. There are two obvious methods one could use to inflate terms using the special term t. The first method is to apply t to entire terms: for any term u of type τ, we can form the term $t(u)$, and we note that $t(u)$ has depth at least k and that $u \approx t(u)$ is satisfied in V. Alternatively, we could inflate any term u by inflating each variable individually. That is, we define a map $u \to \overline{u}$ inductively on the set of all terms of type τ, by $\overline{u} = t(u)$ if u is a variable, and $\overline{u} = f_i(\overline{u_1}, \ldots, \overline{u_{n_i}})$ if u is a compound term of the form $f_i(u_1, \ldots, u_{n_i})$.

In the remainder of this paper we shall assume that $d(t) \geq k$, so that both $t(u)$ and $\overline{u}$ have depth $\geq k$ for all terms u. We now define some sets of identities to use for our new basis for $N_k(V)$.

DEFINITION 1. i) Let $\overline{\Sigma_V}$ be the set consisting of all identities $\overline{u} \approx \overline{v}$ where $u \approx v$ is any identity in the basis Σ_V.

ii) Let Σ_{w_1} be the set of weak idempotent identities of the form $s(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_{p-1}, t(x_p), x_{p+1}, \ldots, x_n)$, for every n-ary term s of depth k and every $1 \leq p \leq n$.

iii) Let Σ_{w_2} be the set of weak idempotent identities of the form $s(x_1, \ldots, x_n) \approx t(s(x_1, \ldots, x_n))$, for every n-ary term s of depth k.

iv) Let $\Sigma_{N_k(V)}$ be the set of identities $\overline{\Sigma_V} \cup \Sigma_{w_1} \cup \Sigma_{w_2}$.

THEOREM 1. *Let V be a variety of type τ with an idempotent term t, and let $k \geq 1$. Then $\Sigma_{N_k(V)} = \overline{\Sigma_V} \cup \Sigma_{w_1} \cup \Sigma_{w_2}$ is a basis for the variety $N_k(V)$.*

COROLLARY 1. *Let $k \geq 1$. If V is a finitely based variety of a finite type τ, and V has an idempotent term, then $N_k(V)$ is also finitely based.*

We prove this theorem and its corollary by a series of lemmas.

LEMMA 1. *Let V be a variety with an idempotent term t, and let $\Sigma_{N_k(V)} = \overline{\Sigma_V} \cup \Sigma_{w_1} \cup \Sigma_{w_2}$ be defined as above. Then for every n-ary term s with $d(s) \geq k$ and every $1 \leq p \leq n$, the following identities are consequences of $\Sigma_{N_k(V)}$:*

(i) $s(x_1, \ldots, x_{p-1}, t(x_p), x_{p+1}, \ldots, x_n) \approx s(x_1, x_2, \ldots, x_n)$,
(ii) $t(s(x_1, x_2, \ldots, x_n)) \approx s(x_1, x_2, \ldots, x_n)$.

Proof. (i) We begin by assuming that the variables of s are distinct. If the variable x_p terminates a string in s of length $l \leq k$, then there exists a term v of depth k and some terms $h_1, h_2, \ldots, h_e$ such that $s = v(h_1, h_2, \ldots, h_e, x_1, \ldots, x_n)$ and x_p is not a variable of any of the terms $h_1, h_2, \ldots, h_e$. By Σ_{w_1} we have $s \approx v(h_1, h_2, \ldots, h_e, x_1, \ldots, t(x_p), \ldots, x_n) = s(x_1, \ldots, t(x_p), \ldots, x_n)$.

Now assume that x_p terminates a string of length $l \gneqq k$. There exists an m-ary term r and an n-ary term s_p with $d(s_p) \geq k$ such that the length of the string in s_p terminating with the variable x_p has length exactly k and $s = r(s_p, x_1, \ldots, x_n)$. Now there also exists a term u of depth k and terms $s_{p_1}, s_{p_2}, \ldots, s_{p_m}$ such that $s_p \approx u(s_{p_1}, s_{p_2}, \ldots, s_{p_m}, x_1, \ldots, x_n)$ and x_p is not a variable of any of the terms $s_1, s_2, \ldots, s_m$. By Σ_{w_1} we have $s_p \approx u(s_1, s_2, \ldots, s_m, x_1, \ldots, t(x_p), \ldots, x_n)$, which again gives us $s \approx s(x_1, \ldots, t(x_p), \ldots, x_n)$ as a consequence of $\Sigma_{N_k(V)}$.

If the variable x_p occurs w times in s, then the w occurrences of x_p can be replaced by the distinct variables $x_{p_1}, x_{p_2}, \ldots, x_{p_w}$ to obtain the term

$$s' = s'(x_1, \ldots, x_{p-1}, x_{p_1}, x_{p_2}, \ldots, x_{p_w}, x_{p+1}, \ldots, x_n).$$

Repeating the above procedure for each x_{p_i} we obtain

$$\begin{aligned} s'(x_1, x_2, \ldots, x_n) &\approx s'(x_1, \ldots, x_{p-1}, t(x_{p_1}), x_{p_2}, \ldots, x_{p_w}, x_{p+1}, \ldots, x_n) \\ &\approx s'(x_1, \ldots, x_{p-1}, t(x_{p_1}), t(x_{p_2}), \ldots, x_{p_w}, x_{p+1}, \ldots, x_n) \\ &\approx s'(x_1, \ldots, x_{p-1}, t(x_{p_1}), t(x_{p_2}), \ldots, t(x_{p_w}), x_{p+1}, \ldots, x_n), \end{aligned}$$

and replacing each of $x_{p_1}, x_{p_2}, \ldots, x_{p_w}$ with x_p gives us $s(x_1, \ldots, x_{p-1}, t(x_p), x_{p+1}, \ldots, x_n) \approx s(x_1, x_2, \ldots, x_n)$ as a consequence of $\Sigma_{N_k(V)}$.

(ii) Since s has depth $\geq k$, there exists some m-ary term r of depth k and n-ary terms $s_1, s_2, \ldots, s_m$ such that $s = r(s_1, s_2, \ldots, s_m)$. By Σ_{w_2}, we have $s = r(s_1, s_2, \ldots, s_m) \approx t(r(s_1, s_2, \ldots, s_m)) = t(s)$. □

LEMMA 2. *For every n-ary term s, the basis $\Sigma_{N_k(V)} = \overline{\Sigma_V} \cup \Sigma_{w_1} \cup \Sigma_{w_2}$ yields the identity*

$$t(s(x_1, x_2, \ldots, x_n)) \approx s(t(x_1), \ldots, t(x_n)).$$

Proof.

$$\begin{aligned} t(s(x_1, x_2, \ldots, x_n)) &\approx t(s(t(x_1), \ldots, t(x_n))) && \text{by Lemma 1(i)} \\ &\approx s(t(x_1), \ldots, t(x_n)) && \text{by Lemma 1(ii).} \end{aligned}$$

□

LEMMA 3. *For every term u of type τ, the identity $\overline{u} \approx t(u)$ follows from $\Sigma_{N_k(V)}$.*

Proof. We proceed by induction on the complexity of the term u. In the base case, when u is a variable, the claim follows from the definition of the mapping $u \to \overline{u}$. Inductively, when $u = f_i(u_1, \ldots, u_{n_i})$ for some $i \in I$, we have

$$\begin{aligned} \overline{u} = f_i(\overline{u_1}, \ldots, \overline{u_{n_i}}) & \quad \text{by the definition of } \overline{u} \\ &\approx f_i(t(u_1), \ldots, t(u_{n_i})) \quad \text{by the induction hypothesis} \\ &\approx t(f_i(u_1, \ldots, u_{n_i})) \quad \text{by Lemma 2} \\ &\approx t(u). \end{aligned}$$

□

LEMMA 4. *For any term u of depth $\geq k$, the identity $u \approx \overline{u}$ can be deduced from $\Sigma_{N_k(V)}$.*

Proof. This claim follows immediately from Lemma 3 and Lemma 1(ii). □

For the next lemma we need some additional notation regarding substitution of a term for a variable. We will denote by $u(x/p)$ the term obtained by substitution of the term p for every occurrence of the variable x in the term u.

LEMMA 5. *Let u and p be any terms, and let x be any variable. Then the identities $u(x/\overline{p}) \approx \overline{u}(x/p) \approx \overline{u}(x/\overline{p}) \approx \overline{u(x/p)}$ are consequences of $\Sigma_{N_k(V)}$.*

Proof. Assuming that $x = x_1$ for notational convenience, we have

$$\begin{aligned} \overline{u(x/p)} &\approx u(\overline{p}, \overline{x}_2, \ldots, \overline{x}_n) \text{ by the definition of } \overline{u} \\ &\approx u(\overline{p}, x_2, \ldots, x_n) \text{ by Lemma 1(i).} \end{aligned}$$

This shows that $\overline{u(x/p)} \approx u(x/\overline{p})$ is a consequence of $\Sigma_{N_k(V)}$. Also,

$$\begin{aligned} \overline{u}(x/p) &= u(\overline{x}, \overline{x_2}, \ldots, \overline{x_n})(x/p) \\ &= u(t(x), \overline{x_2}, \ldots, \overline{x_n})(x/p) \text{ since } x \text{ is a variable} \\ &= u(t(p), \overline{x_2}, \ldots, \overline{x_n}) \\ &\approx u(\overline{p}, \overline{x_2}, \ldots, \overline{x_n}) \text{ by Lemma 3,} \end{aligned}$$

so that $\overline{u}(x/p) \approx \overline{u(x/p)} \approx u(x/\overline{p})$ is a consequence of $\Sigma_{N_k(V)}$. This also implies that $\overline{u}(x/\overline{p}) \approx \overline{\overline{u}}(x/p)$ and $\overline{\overline{u}}(x/p) \approx \overline{u}(x\overline{/p)})$ since $\overline{\overline{u}} \approx \overline{u}$ by Lemma 1(i), so we have $u(x/\overline{p}) \approx \overline{u}(x/p) \approx \overline{u}(x/\overline{p}) \approx \overline{u(x/p)}$ as a consequence of $\Sigma_{N_k(V)}$. □

LEMMA 6. *For any k-normal identity $p \approx q$ of V, we can inflate any deduction of $p \approx q$ from the basis Σ_V into a deduction of $\overline{p} \approx \overline{q}$ from $\Sigma_{N_k(V)}$.*

Proof. Since $p \approx q$ is an identity of V, there exists a deduction of $p \approx q$ from the basis Σ_V using Birkhoff's five rules of deduction. We will call this deduction the *given deduction.* We take the given deduction and replace each step $u_j \approx w_j$ by $\overline{u_j} \approx \overline{w_j}$, to produce a sequence of identities called the *derived list.* We shall show that the derived list is a deduction of $\overline{p} \approx \overline{q}$ from $\Sigma_{N_k(V)}$, by verifying that the justification for each step j in the derived list is the same as the justification for step j in the given deduction. Consider the identity $u_j \approx w_j$ at any step j in the given deduction. If step j was an instance of an identity from Σ_V, then step j in the derived list is an instance of the corresponding identity from $\overline{\Sigma_V} \subseteq \Sigma_{N_k(V)}$. If step j was an instance of the reflexive, symmetric, or transitive rules of deduction, then clearly step j in the derived list is an instance of the same rule.

Now suppose that step j in the given deduction was an instance of Rule 4, the compatibility rule, on some previous steps of the deduction. Then step j has the form $f_i(u_1, \ldots, u_{n_i}) \approx f_i(w_1, \ldots, w_{n_i})$, deduced from some earlier steps $u_j \approx w_j$ for $1 \leq j \leq n_i$. According to our construction of the derived list, step j in the derived list is $\overline{f_i(u_1, \ldots, u_{n_i})} \approx \overline{f_i(w_1, \ldots, w_{n_i})}$. By definition of the $u \to \overline{u}$ operator, this is equal to $f_i(\overline{u_1}, \ldots, \overline{u_{n_i}}) \approx f_i(\overline{w_1}, \ldots, \overline{w_{n_i}})$, which can be deduced from the corresponding earlier steps $\overline{u_j} \approx \overline{w_j}$ in the derived list, again by Rule 4.

Finally, suppose that step j in the given deduction was an instance of deduction Rule 5, the substitution rule, on some previous line $u \approx w$. Then step j in the given deduction has the form $u(x/z) \approx w(x/z)$ for some term z, and step j in the derived list is $\overline{u(x/z)} \approx \overline{w(x/z)}$. When we apply the substitution rule to the earlier step $\overline{u} \approx \overline{w}$ in the derived list, we obtain $\overline{u}(x/z)) \approx \overline{w}(x/z)$. That these two identities are equivalent under $\Sigma_{N_k(V)}$ then follows from Lemma 5. Thus, the derived list is a deduction of $\overline{p} \approx \overline{q}$ from $\Sigma_{N_k(V)}$. □

Using these lemmas we can now prove Theorem 1. We need to show that any identity $p \approx q$ which holds in V and is k-normal can be deduced from the purported basis set $\Sigma_{N_k(V)}$. From Lemma 4, it follows that we can deduce each of $p \approx \overline{p}$ and $q \approx \overline{q}$ from $\Sigma_{N_k(V)}$. From Lemma 6, we can also deduce $\overline{p} \approx \overline{q}$, and hence we can indeed deduce $p \approx q$ from $\Sigma_{N_k(V)}$ as required.

The process used for Theorem 1 is a bit redundant, since it inflates all the identities from the original basis Σ_V, even those identities which are already k-normal. The basis construction could clearly be stream-lined by inflating only those identities from Σ_V which are not k-normal, and keeping those which are k-normal in their original form.

We illustrate this streamlining in more detail in the special case that $k = 1$, where the k-normalization of V coincides with the usual normalization of V, denoted by $N(V)$. In this case the weak idempotent identities in Σ_{w_1} and Σ_{w_2} involve only fundamental terms of the form $f_i(x_1, \ldots, x_{n_i})$. This means that the sets Σ_{w_1} and Σ_{w_2} can be replaced by the set of identities $f_i(x_1, \ldots, x_{n_i}) \approx f_i(x_1, \ldots, x_{p-1}, t(x_p), \ldots, x_{n_i}) \approx t(f_i(x_1, \ldots, x_{n_i}))$, for every index $i \in I$ and every $1 \leq p \leq n_i$.

A further simplification to our basis construction can be made in the case that $k = 1$ when the special term t used to inflate identities also has depth 1. In this case, t has the form $f_{i_0}(x, \ldots, x)$ for some fixed index $i_0 \in I$. Thus to construct the basis $\Sigma_{N(V)}$, we can use the following four-step process on the identities $u \approx v$ in a given basis Σ_V for V, as described in [12]:

1) If $u \approx v$ is a normal identity, then we put $u \approx v$ in $\Sigma_{N(V)}$.

2) If $u \approx v$ is a non-normal identity but not an idempotent identity $f_j(x, \ldots, x) \approx x$ for any operation symbol f_j, then one of u and v, say v, is a variable, and we put the identity $u \approx t(v)$ in $\Sigma_{N(V)}$.

3) If an operation symbol $f_j \neq f_{i_0}$ is idempotent in V, then we add $f_{i_0}(x, \ldots, x) \approx f_j(x, \ldots, x)$ to $\Sigma_{N(V)}$.

4) For every $j \in I$ and every $1 \leq p \leq n_j$, we add to $\Sigma_{N(V)}$ the following *weak idempotent* identities:

$$f_j(x_1, \ldots, x_{n_j}) \approx t(f_j(x_1, \ldots, x_{n_j})) \approx f_j(x_1, \ldots, x_{p-1}, t(x_p), x_{p+1}, \ldots, x_{n_j}).$$

This results in a simplified version of Theorem 1, for the case $k = 1$, as described in [12]:

THEOREM 2. *([12]) Let V be a variety with an idempotent term. Given a basis Σ_V for V, the set $\Sigma_{N(V)}$ constructed as above is a basis for $N(V)$.*

3. Examples

In this section we illustrate Theorems 1 and 2 with a number of examples, reviewing some known basis results and providing a new basis in the case of the variety of pseudo-complemented lattices.

EXAMPLE 1. We illustrate Theorem 2 for the type $(2, 2)$ variety L of lattices. This variety has a well-known basis Σ_L consisting of the associative, commutative, absorption and idempotent laws for meet and join. Bases have been constructed for the varieties obtained from L using a number of different equational properties. Well-known bases for $N(L)$ and the externalization of L are described in [2], [12] and [5]. A basis for the 2-normalization $N_2(L)$ was given in [3]. The regularization $Reg(L)$ of L was shown to be the variety QL of quasilattices by Padmanabhan in [13].

Following the process used for Theorem 2, with the idempotent term $t(x) = x \vee x$ of depth 1, yields the following well-known basis for $N(L)$ (see [12], [2]):

commutativity	$x_1 \vee x_2 \approx x_2 \vee x_1,$	$x_1 \wedge x_2 \approx x_2 \wedge x_1$
associativity	$x_1 \vee (x_2 \vee x_3) \approx (x_1 \vee x_2) \vee x_3$	
	$x_1 \wedge (x_2 \wedge x_3) \approx (x_1 \wedge x_2) \wedge x_3$	
weak absorption	$x_1 \vee (x_1 \wedge x_2) \approx x_1 \vee x_1,$	$x_1 \wedge (x_1 \vee x_2) \approx x_1 \wedge x_1$
weak idempotence	$x_1 \vee (x_2 \vee x_2) \approx x_1 \vee x_2,$	$x_1 \wedge (x_2 \wedge x_2) \approx x_1 \wedge x_2$
	$x_1 \vee x_1 \approx x_1 \wedge x_1$	

EXAMPLE 2. In [6] bases were constructed for $N_k(V)$ for any subvariety V of the type (2) variety B of bands, or idempotent semigroups. To illustrate Theorem 1 we construct here a basis for the variety $N_2(B)$. Following the convention of denoting the binary operation of type (2) by juxtaposition, we start with the standard basis for B,

$$\Sigma_B = \{x_1x_1 = x_1,\ x_1(x_2x_3) = (x_1x_2)x_3\}.$$

Then letting $t(x) = (xx)x$, we get identities

$$((x_1x_1)x_1)((x_1x_1)x_1) \approx (x_1x_1)x_1,$$
$$((x_1x_1)x_1)(((x_2x_2)x_2)((x_3x_3)x_3)) \approx ((x_1x_1)x_1)((x_2x_2)x_2))((x_3x_3)x_3),$$
$$((x_1x_1)x_1)(x_2x_3) \approx x_1(x_2x_3),$$
$$x_1(((x_2x_2)x_2)x_3) \approx x_1(x_2x_3),$$
$$x_1(x_2((x_3x_3)x_3)) \approx x_1(x_2x_3),\ \ldots,$$
$$(x_1x_2)x_3 \approx (((x_1x_2)x_3)((x_1x_2)x_3))((x_1x_2)x_3),\ldots,$$

where the first two identities are from $\overline{\Sigma_B}$, the identities on the next three lines are from Σ_{w_1}, and the last line gives identities from Σ_{w_2}.

The basis $\Sigma_{N_2(B)} = \overline{\Sigma_B} \cup \Sigma_{w_1} \cup \Sigma_{w_2}$ can now be refined to the basis exhibited in [6], consisting of associativity and the two identities $xyz \approx x^2yz \approx xyz^2$. It is clear that associativity is a consequence of our basis $\Sigma_{N_2(B)}$, and hence we may follow the custom of omitting brackets in our identities. We can deduce from our basis both $x^3 \approx x^6$ and $x^3 \approx x^5$, and hence $x^3 \approx x^a$ for all $a \geq 4$. Then we get $x^2yz \approx x(xyz) \approx x(x^3yz) \approx x^4yz \approx x^3yz \approx xyz$, and similarly for $xyz^2 \approx xyz$. Conversely, the identities $xyz \approx x^2yz \approx xyz^2$ plus associativity yield our basis $\Sigma_{N_2(B)}$ as well.

EXAMPLE 3. As another example we construct a basis for the variety $N(G)$ where G is the variety of all groups, considered as algebras of type $(2,1)$. We start with the basis

$$\Sigma_G = \{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1^{-1}(x_1x_2) \approx (x_2x_1)x_1^{-1}, x_1^{-1}(x_1x_2) \approx x_2\}$$

for G. As our idempotent term we use $t(x) = x^{-1}(xx)$, which satisfies $t(x) \approx x$. Following the construction for Theorem 1, we keep the first two identities of the basis Σ_G, and change the third one to $x_1^{-1}(x_1x_2) \approx t(x_2)$. Then we add the weak idempotent identities $xt(y) \approx xy \approx t(x)y$ and $t(x)^{-1} \approx x^{-1} \approx t(x^{-1})$, for the two operation symbols of our type.

EXAMPLE 4. In our final example we produce a new basis for the normalization of the type $(2,2,1)$ variety PL of pseudo-complemented lattices or p-algebras. This variety has been studied in [1], and a basis for the regularization of PL has been given by Penner in [14]. Penner showed that just as the regularization of the variety L of lattices is the variety QL of quasi-lattices, so the regularization of the variety PL of pseudo-complemented lattices consists of all pq-algebras.

The variety PL of all pseudo-complemented lattices has a basis Σ_{PL} consisting of the eight lattice axioms from Σ_L plus the following four axioms:

$$\begin{aligned} &\text{S1.} \quad x_1 \wedge (x_2 \wedge x_2^*) \approx x_2 \wedge x_2^* \\ &\text{S2.} \quad x_1 \wedge (x_1 \wedge x_2)^* \approx x_1 \wedge x_2^* \\ &\text{S3.} \quad x_1 \wedge (x_2 \wedge x_2^*)^* \approx x_1 \\ &\text{S4.} \quad (x_1 \wedge x_1^*)^{**} \approx x_1 \wedge x_1^*. \end{aligned}$$

In order to exhibit an equational basis for $N(PL)$, we carry out the four-step construction from Theorem 2, using $t(x) = x \vee x$. On the identities from the basis for L, this results in the set $\Sigma_{N(L)}$ already described in Example 1. In addition, we keep the normal identities S1, S2 and S4, and replace S3 with its normalized version. Then we add the additional weak idempotent identities needed for the additional unary operation *: $x^* \approx (x \vee x)^* \approx x^* \vee x^*$. Thus we need the following five identities:

$$\begin{aligned} &\text{N1.} \quad (x_2 \wedge x_2^*) \wedge x_1 \approx x_2 \wedge x_2^* \\ &\text{N2.} \quad x_1 \wedge (x_1 \wedge x_2)^* \approx x_1 \wedge x_2^* \\ &\text{N3.} \quad x_1 \wedge (x_2 \wedge x_2^*)^* \approx x_1 \vee x_1 \\ &\text{N4.} \quad (x_2 \wedge x_2^*)^{**} \approx x_2 \wedge x_2^* \\ &\text{N5.} \quad (x_1 \vee x_1)^* \approx x_1^* \approx x_1^* \vee x_1^*. \end{aligned}$$

Note that by N1 we have both $(x_2 \wedge x_2^*) \wedge (x_1 \wedge x_1^*) \approx x_2 \wedge x_2^*$ and $(x_1 \wedge x_1^*) \wedge (x_2 \wedge x_2^*) \approx x_1 \wedge x_1^*$ so by commutativity it follows that $x_2 \wedge x_2^* \approx x_1 \wedge x_1^*$. This leads to the following new result:

THEOREM 3. $\Sigma_{N(PL)} = \Sigma_{N(L)} \cup \{N1, N2, N3, N4, N5\}$ *is an equational basis for* $N(PL)$.

Bases for $N_k(PL)$ for $k > 1$ may also be produced, using the method of Theorem 1.

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