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ON THE ALMOST SURE CENTRAL LIMIT THEOREMS IN THE JOINT VERSION FOR THE MAXIMA AND MINIMA OF SOME RANDOM VARIABLES

Abstract. Let: $\{X_i\}$ be a sequence of r.v.'s, and: $M_n := \max(X_1, \ldots, X_n)$, $m_n := \min(X_1, \ldots, X_n)$. Our goal is to prove the almost sure central limit theorem for the properly normalized vector $\{M_n, m_n\}$, provided: 1) $\{X_i\}$ is an i.i.d. sequence, 2) $\{X_i\}$ is a certain standardized stationary Gaussian sequence.

1. Introduction

The almost sure central limit theorem (ASCLT) has become an intensively studied subject in recent years. The simplest versions of the ASCLT have been proved in the papers of Brosamler [3], Schatte [19], Fisher [12], Lacey and Philipp [13] and Berkes and Dehling [1]. They relate to the ASCLT for the sums of independent r.v.'s. The ASCLT for sums has been later generalized by Peligrad and Shao [18], Matuła [15], [16], Mielniczuk [17] and Dudziński [8], for the sums of some weakly dependent r.v.'s. Starting from Cheng et al. [4] and Fahrner and Stadtmueller [11], the ASCLT for maxima of r.v.'s has become another popular direction of the research concerning the topic. The ASCLT for the maxima of independent r.v.'s has been proved in the mentioned papers. These results have been later extended by Csaki and Gonchigdanzan [6] and Dudziński [7] to the cases of maxima of certain stationary Gaussian sequences. Some other valuable results concerning the issue of the ASCLT's are due to Berkes and Csaki [2], Stadtmueller [20] and Dudziński [9], [10]. In Berkes and Csaki [2], the ASCLT's for the variety of functions of independent r.v.'s, such as maxima of partial sums, the Kolmogorov statistics, U-statistics etc. have been proved. In [20], the proof of the ASCLT for certain order statistics of i.i.d. r.v.'s has been given. In turn,

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the result in [9] considers the ASCLT in its joint version for the maxima and sums of some r.v.'s, while the ASCLT for the vectors of several large maxima and for some random permanents has been stated in [10].

The purpose of our paper is to prove another version of the ASCLT, which is the ASCLT in its joint version for the maxima and minima of some r.v.'s. Namely, we investigate the almost sure convergence of the following sequence of logarithmic means: $\left\{\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{M_n \leq u_n, m_n \leq v_n\right\}\right\}$, where: M_n, m_n denote the maximum and minimum of some r.v.'s $X_1,\ldots,$ X_n, respectively, and $\{u_n\}$, $\{v_n\}$ are certain numerical sequences. We shall show that the almost sure convergence occurs for such the logarithmic averages, provided $\{X_i\}$ is an i.i.d. sequence or $\{X_i\}$ is some standard normal one.

The following notations will be used in our paper: $M_n := \max(X_1,\ldots,X_n)$, $m_n := \min(X_1,\ldots,X_n)$, $M_{k,n} := \max(X_{k+1},\ldots,X_n)$, $m_{k,n} := \min(X_{k+1},\ldots,X_n)$, $r(n) := Cov(X_1, X_{1+n})$, Φ - the standard normal d.f., C_G (C_H) - the set of the continuity points of the d.f. G (H). Furthermore, $f(n) \ll g(n)$ and $f(n) \sim g(n)$ will stand for $f(n) = \mathcal{O}(g(n))$ and $f(n)/g(n) \to 1$, as $n \to \infty$, respectively, and $\xrightarrow{w}$ will denote that the convergence occurs at all the continuity points of the limit function.

2. Main results

Our first main result is the following ASCLT in the joint version for the maxima and minima of i.i.d. r.v.'s.

THEOREM 1. *Suppose that $X_1, X_2, \ldots$ is an i.i.d. sequence with the common d.f. F. Then:*

(i) If, for some numbers $0 \leq \tau \leq \infty$, $0 \leq \eta \leq \infty$, the sequences $\{u_n\}$, $\{v_n\}$ satisfy, respectively:

$$n(1-F(u_n)) \to \tau,\ nF(v_n) \to \eta,\ \textit{as } n \to \infty, \tag{1}$$

we have

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{M_n \leq u_n, m_n \leq v_n\right\} = e^{-\tau}\left(1-e^{-\eta}\right) \quad a.s., \tag{2}$$

(ii) If, for some d.f.'s G, H, the sequences $\{a_n > 0\}$, $\{b_n\}$ and $\{\alpha_n > 0\}$, $\{\beta_n\}$ are such that:

$$P\left\{a_n(M_n - b_n) \leq x\right\} \xrightarrow{w} G(x),\ P\left\{\alpha_n(m_n - \beta_n) \leq y\right\} \xrightarrow{w} H(y), \tag{3}$$

we have for all $x \in C_G$, $y \in C_H$ that

(4) $$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{a_n\left(M_n - b_n\right) \le x, \alpha_n\left(m_n - \beta_n\right) \le y\right\} = G(x) H(y) \quad a.s.$$

The second of our main results is the following ASCLT in the joint version for the maxima and minima of some stationary standard normal sequences.

THEOREM 2. *Suppose that $X_1, X_2, \ldots$ is a standardized stationary Gaussian sequence. Let the covariance function $r(n) := Cov(X_1, X_{1+n})$ satisfy*

(5) $$r(n) \log n \left(\log\log n\right)^{1+\varepsilon} \ll 1 \text{ for some } \varepsilon > 0.$$

Then:

(i) If, for some numbers $0 \le \tau < \infty$, $0 \le \eta < \infty$, the sequences $\{u_n\}$, $\{v_n\}$ satisfy, respectively:

(6) $$n\left(1 - \Phi\left(u_n\right)\right) \to \tau,\ n\Phi\left(v_n\right) \to \eta, \text{ as } n \to \infty,$$

we have

(7) $$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{M_n \le u_n, m_n \le v_n\right\} = e^{-\tau}\left(1 - e^{-\eta}\right) \quad a.s.,$$

(ii) If the sequences $\{a_n > 0\}$, $\{b_n\}$ and $\{\alpha_n > 0\}$, $\{\beta_n\}$ are such that:

(8) $$\begin{aligned} a_n &:= \left(2\log n\right)^{1/2}, \\ b_n &:= \left(2\log n\right)^{1/2} - \frac{1}{2}\left(2\log n\right)^{-1/2}\left(\log\log n + \log 4\pi\right), \end{aligned}$$

(9) $$\begin{aligned} \alpha_n &:= -\left(2\log n\right)^{1/2}, \\ \beta_n &:= -\left(2\log n\right)^{1/2} + \frac{1}{2}\left(2\log n\right)^{-1/2}\left(\log\log n + \log 4\pi\right), \end{aligned}$$

we have for all x, $y \in \mathbb{R}$ that

(10) $$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{a_n\left(M_n - b_n\right) \le x, \alpha_n\left(m_n - \beta_n\right) \le y\right\} = \exp\left(-e^{-x}\right)\left(1 - \exp\left(-e^{-y}\right)\right) \quad a.s.$$

3. Auxiliary results

In this section, we state and prove four lemmas, which will be needed for the proofs of our main results. The first lemma will be applied in the proof of Theorem 1, while the next three ones will be used in the proof of Theorem 2. Here the first of the mentioned lemmas is.

LEMMA 1. *Under the assumptions of Theorem 1 on* $X_1, X_2, \ldots,$ *and* $\{u_n\}$, $\{v_n\}$, *we have for* $k < n$ *that*

$$(11) \qquad |Cov\,(I\{M_k \le u_k, m_k > v_k\}, I\{M_n \le u_n, m_n > v_n\})| \ll k/n.$$

Proof. Let $k < n$. Notice that

$$\begin{aligned}
&|Cov\,(I\{M_k \le u_k, m_k > v_k\}, I\{M_n \le u_n, m_n > v_n\})| \\
&\quad \ll E\,|I\{M_n \le u_n, m_n > v_n\} - I\{M_{k,n} \le u_n, m_{k,n} > v_n\}| \\
&\qquad + |Cov\,(I\{M_k \le u_k, m_k > v_k\}, I\{M_{k,n} \le u_n, m_{k,n} > v_n\})|.
\end{aligned}$$

Observe that, since the $X_i's$ are independent, the second component in the last derivation is equal to zero. As moreover, the event $\{M_n \le u_n, m_n > v_n\}$ is a subset of the event $\{M_{k,n} \le u_n, m_{k,n} > v_n\}$, we obtain

$$\begin{aligned}
(12) \qquad &|Cov\,(I\{M_k \le u_k, m_k > v_k\}, I\{M_n \le u_n, m_n > v_n\})| \\
&\quad \ll E\,|I\{M_n \le u_n, m_n > v_n\} - I\{M_{k,n} \le u_n, m_{k,n} > v_n\}| \\
&\quad = P\{M_{k,n} \le u_n, m_{k,n} > v_n\} - P\{M_n \le u_n, m_n > v_n\}.
\end{aligned}$$

Thus, by (12) and an assumption that the $X_i's$ are i.i.d. r.v.'s with the common d.f. F, we get

$$\begin{aligned}
&|Cov\,(I\{M_k \le u_k, m_k > v_k\}, I\{M_n \le u_n, m_n > v_n\})| \\
&\qquad \ll P\{X_{k+1} \le u_n, \ldots, X_n \le u_n, X_{k+1} > v_n, \ldots, X_n > v_n\} \\
&-P\{X_1 \le u_n, \ldots, X_n \le u_n, X_1 > v_n, \ldots, X_n > v_n\} \\
&\qquad = P\{v_n < X_{k+1} \le u_n, \ldots, v_n < X_n \le u_n\} \\
&-P\{v_n < X_1 \le u_n, \ldots, v_n < X_{k+1} \le u_n, \ldots, v_n < X_n \le u_n\} \\
&\qquad = (F(u_n) - F(v_n))^{n-k} - (F(u_n) - F(v_n))^n.
\end{aligned}$$

This and the elementary fact that $z^{n-k} - z^n \le k/n$, if $0 \le z \le 1$, yield

$$|Cov\,(I\{M_k \le u_k, m_k > v_k\}, I\{M_n \le u_n, m_n > v_n\})| \ll k/n,$$

which is the desired result (11). ∎

We now formulate and prove three lemmas, which will be used in the proof of our second main result.

LEMMA 2. *Under the assumptions of Theorem 2 on* $X_1, X_2, \ldots,$ *and* $\{u_n\}$, $\{v_n\}$, $r(n)$, *we have for* $k < n$ *that*

$$\begin{aligned}
(13) \qquad &E\,|I\{M_n \le u_n, m_n > v_n\} - I\{M_{k,n} \le u_n, m_{k,n} > v_n\}| \\
&\qquad \ll k/n + (\log\log n)^{-(1+\varepsilon)} \ \text{for some } \varepsilon > 0.
\end{aligned}$$

Proof. Let $k < n$. Since the event $\{M_n \le u_n, m_n > v_n\}$ is a subset of the event $\{M_{k,n} \le u_n, m_{k,n} > v_n\}$, we have

$$E\left|I\{M_n \le u_n, m_n > v_n\} - I\{M_{k,n} \le u_n, m_{k,n} > v_n\}\right| \\ = P\{M_{k,n} \le u_n, m_{k,n} > v_n\} - P\{M_n \le u_n, m_n > v_n\}.$$

Therefore,

$$\begin{aligned}(14)\quad & E\left|I\{M_n \le u_n, m_n > v_n\} - I\{M_{k,n} \le u_n, m_{k,n} > v_n\}\right| \\ &= (P\{M_{k,n} \le u_n\} + P\{m_{k,n} > v_n\} - P\{\{M_{k,n} \le u_n\} \cup \{m_{k,n} > v_n\}\}) \\ &\quad - (P\{M_n \le u_n\} + P\{m_n > v_n\} - P\{\{M_n \le u_n\} \cup \{m_n > v_n\}\}) \\ &= (P\{M_{k,n} \le u_n\} - P\{M_n \le u_n\}) + (P\{m_{k,n} > v_n\} - P\{m_n > v_n\}) \\ &\quad + (P\{\{M_n \le u_n\} \cup \{m_n > v_n\}\} - P\{\{M_{k,n} \le u_n\} \cup \{m_{k,n} > v_n\}\}).\end{aligned}$$

As $\{\{M_n \le u_n\} \cup \{m_n > v_n\}\}$ is a subset of $\{\{M_{k,n} \le u_n\} \cup \{m_{k,n} > v_n\}\}$, we get

$$P\{\{M_n \le u_n\} \cup \{m_n > v_n\}\} - P\{\{M_{k,n} \le u_n\} \cup \{m_{k,n} > v_n\}\} \le 0.$$

This and (14) yield

$$\begin{aligned}(15)\quad & E\left|I\{M_n \le u_n, m_n > v_n\} - I\{M_{k,n} \le u_n, m_{k,n} > v_n\}\right| \\ &\le (P\{M_{k,n} \le u_n\} - P\{M_n \le u_n\}) + (P\{m_{k,n} > v_n\} - P\{m_n > v_n\}).\end{aligned}$$

It follows from the proof of Lemma 2.4 in Csaki and Gonchigdanzan [6] that

$$(16)\quad P\{M_{k,n} \le u_n\} - P\{M_n \le u_n\} \\ \ll k/n + (\log\log n)^{-(1+\varepsilon)} \text{ for some } \varepsilon > 0.$$

Thus, it remains to estimate the difference $(P\{m_{k,n} > v_n\} - P\{m_n > v_n\})$. Clearly, we have

$$\begin{aligned}(17)\quad & P\{m_{k,n} > v_n\} - P\{m_n > v_n\} \\ &= P\{-\max(-X_{k+1}, \dots, -X_n) > v_n\} - P\{-\max(-X_1, \dots, -X_n) > v_n\} \\ &= P\{\max(-X_{k+1}, \dots, -X_n) < -v_n\} - P\{\max(-X_1, \dots, -X_n) < -v_n\}.\end{aligned}$$

Set: $\breve{M}_n := \max(-X_1, \dots, -X_n)$, $\breve{M}_{k,n} := \max(-X_{k+1}, \dots, -X_n)$, $\breve{r}(n) := Cov(-X_1, -X_{1+n})$. It is obvious that $\breve{r}(n) = r(n)$. In addition, since the sequence $\{X_i\}$ is standard normal, the sequence $\{-X_i\}$ is standard normal as well, and $\breve{M}_{k,n}$, $\breve{M}_n$ are the maxima of some standardized stationary Gaussian r.v.'s. Due to an assumption on $\{v_n\}$ in (6) and the fact that

$$n\Phi(v_n) = n(1 - \Phi(-v_n)) \sim n\exp\left(-\frac{(-v_n)^2}{2}\right)/\sqrt{2\pi}(-v_n),$$

we get:

$$\exp\left(-\frac{(-v_n)^2}{2}\right) \sim \frac{K\sqrt{2\pi}\,(-v_n)}{n} \text{ for some } K \geq 0, \tag{18}$$
$$(-v_n) \sim (2\log n)^{1/2}.$$

Thus, by applying (17) and replacing M_n by $\breve{M}_n$ and u_n by $(-v_n)$ in the proof of Lemma 2.4 in Csaki and Gonchigdanzan [6], we obtain

$$P\{m_{k,n} > v_n\} - P\{m_n > v_n\} = P\left\{\breve{M}_{k,n} \leq -v_n\right\} - P\left\{\breve{M}_n \leq -v_n\right\} \tag{19}$$
$$\ll k/n + (\log\log n)^{-(1+\varepsilon)} \text{ for some } \varepsilon > 0.$$

The relations in (15), (16) and (19) imply

$$E\left|I\{M_n \leq u_n, m_n > v_n\} - I\{M_{k,n} \leq u_n, m_{k,n} > v_n\}\right|$$
$$\ll k/n + (\log\log n)^{-(1+\varepsilon)} \text{ for some } \varepsilon > 0,$$

which is the desired result (13). ∎

We now give the proof of the following auxiliary result.

LEMMA 3. *Under the assumptions of Theorem 2 on $X_1, X_2, \ldots$, and $\{u_n\}$, $\{v_n\}$, $r(n)$, we have for $k < n$ that*

$$\left|Cov\left(I\{M_k \leq u_k, m_k > v_k\}, I\{M_{k,n} \leq u_n, m_{k,n} > v_n\}\right)\right| \tag{20}$$
$$\ll (\log\log n)^{-(1+\varepsilon)} \textit{ for some } \varepsilon > 0.$$

Proof. Let $k < n$. We have

$$\left|Cov\left(I\{M_k \leq u_k, m_k > v_k\}, I\{M_{k,n} \leq u_n, m_{k,n} > v_n\}\right)\right|$$
$$= \left|P\{M_k \leq u_k, m_k > v_k, M_{k,n} \leq u_n, m_{k,n} > v_n\}\right.$$
$$\left. - P\{M_k \leq u_k, m_k > v_k\}\, P\{M_{k,n} \leq u_n, m_{k,n} > v_n\}\right|.$$

It is easy to check that

$$|Cov(I\{M_k \leq u_k, m_k > v_k\}, I\{M_{k,n} \leq u_n, m_{k,n} > v_n\})|$$
$$= |P\{v_k < X_1 \leq u_k, \ldots, v_k < X_k \leq u_k, v_n < X_{k+1} \leq u_n, \ldots, v_n < X_n \leq u_n\}.$$
$$-P\{v_k < X_1 \leq u_k, \ldots, v_k < X_k \leq u_k\}P\{v_n < X_{k+1} \leq u_n, \ldots, v_n < X_n \leq u_n\}|.$$

Let $\left(\tilde{X}_{k+1}, \ldots, \tilde{X}_n\right)$ denote a standard normal vector, which has the same distribution as $(X_{k+1}, \ldots, X_n)$, but is independent of $(X_1, \ldots, X_k)$. Put:

$$\{\xi_i = X_i,\ i = 1, \ldots, n\},$$
$$\left\{\eta_i = X_i,\ i = 1, \ldots, k;\ \eta_i = \tilde{X}_i,\ i = k+1, \ldots, n\right\}. \tag{21}$$

Thus, we can write that

$$\begin{aligned}&|Cov(I\{M_k \le u_k, m_k > v_k\}, I\{M_{k,n} \le u_n, m_{k,n} > v_n\})|\\ &= |P\{v_k < \xi_1 \le u_k, \dots, v_k < \xi_k \le u_k, v_n < \xi_{k+1} \le u_n, \dots, v_n < \xi_n \le u_n\}\\ &\quad -P\{v_k < \eta_1 \le u_k, \dots, v_k < \eta_k \le u_k, v_n < \eta_{k+1} \le u_n, \dots, v_n < \eta_n \le u_n\}|.\end{aligned}$$

Denote by f_1 - the n-dimensional normal joint density function of the sequence $\{\xi_1, \dots, \xi_k, \xi_{k+1}, \dots, \xi_n\}$, based on the positive definite covariance matrix Λ^1, and by f_0 - the n-dimensional normal joint density function of the sequence $\{\eta_1, \dots, \eta_k, \eta_{k+1}, \dots, \eta_n\}$, based on the positive definite covariance matrix Λ^0. By using these notations together with the last equality, we have

$$(22)\quad \begin{aligned}&|Cov(I\{M_k \le u_k, m_k > v_k\}, I\{M_{k,n} \le u_n, m_{k,n} > v_n\})|\\ &= \Bigg|\int_{v_n}^{u_n}\dots\int_{v_n}^{u_n}\int_{v_k}^{u_k}\dots\int_{v_k}^{u_k} f_1(y_1, \dots, y_k, y_{k+1}, \dots, y_n)\, dy_1 \dots dy_k dy_{k+1} \dots dy_n\\ &\quad - \int_{v_n}^{u_n}\dots\int_{v_n}^{u_n}\int_{v_k}^{u_k}\dots\int_{v_k}^{u_k} f_0(y_1, \dots, y_k, y_{k+1}, \dots, y_n)\, dy_1 \dots dy_k dy_{k+1} \dots dy_n\Bigg|.\end{aligned}$$

Set $\Lambda^h := h\Lambda^1 + (1-h)\Lambda^0$, $0 \le h \le 1$. The matrix Λ^h is positive definite with units down the main diagonal and the elements $h\Lambda^1_{ij} + (1-h)\Lambda^0_{ij}$ for $i \ne j$. Let f_h stand for the n-dimensional normal joint density function, based on the covariance matrix Λ^h, and

$$F(h) := \int_{v_n}^{u_n}\dots\int_{v_n}^{u_n}\int_{v_k}^{u_k}\dots\int_{v_k}^{u_k} f_h(y_1, \dots, y_k, y_{k+1}, \dots, y_n)\, dy_1 \dots dy_k dy_{k+1} \dots dy_n.$$

Then, due to (22),

$$(23)\quad \begin{aligned}&|Cov(I\{M_k \le u_k, m_k > v_k\}, I\{M_{k,n} \le u_n, m_{k,n} > v_n\})|\\ &\qquad = |F(1) - F(0)| = \left|\int_0^1 F'(h)\, dh\right|,\end{aligned}$$

where

$$F'(h) = \int_{v_n}^{u_n}\dots\int_{v_n}^{u_n}\int_{v_k}^{u_k}\dots\int_{v_k}^{u_k} \frac{\partial f_h(y_1, \dots, y_k, y_{k+1}, \dots, y_n)}{\partial h} dy_1 \dots dy_k dy_{k+1} \dots dy_n.$$

The density f_h depends on h only through the elements Λ^h_{ij}, $i \le j$, of the matrix Λ^h. Notice that, as $\Lambda^h_{ij} = h\Lambda^1_{ij} + (1-h)\Lambda^0_{ij}$, if $i < j$, we get

$\partial\Lambda_{ij}^h/\partial h = \Lambda_{ij}^1 - \Lambda_{ij}^0$ for $i < j$. Furthermore, since $\Lambda_{ii}^h = 1$, we obtain $\partial\Lambda_{ii}^h/\partial h = 0$. Hence

$$F'(h) = \sum_{i\le j} \int_{v_n}^{u_n} \cdots \int_{v_n}^{u_n} \int_{v_k}^{u_k} \cdots \int_{v_k}^{u_k} \frac{\partial f_h}{\partial \Lambda_{ij}^h} \frac{\partial \Lambda_{ij}^h}{\partial h} dy_1 \dots dy_k dy_{k+1} \dots dy_n$$

$$= \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{v_n}^{u_n} \cdots \int_{v_n}^{u_n} \int_{v_k}^{u_k} \cdots \int_{v_k}^{u_k} \frac{\partial f_h}{\partial \Lambda_{ij}^h} dy_1 \dots dy_k dy_{k+1} \dots dy_n.$$

By the property of the multidimensional normal density (see Cramer and Leadbetter [5], p. 26), we have $\dfrac{\partial f_h}{\partial \Lambda_{ij}^h} = \dfrac{\partial^2 f_h}{\partial y_i \partial y_j}$. Therefore,

$$F'(h) = \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{v_n}^{u_n} \cdots \int_{v_n}^{u_n} \int_{v_k}^{u_k} \cdots \int_{v_k}^{u_k} \frac{\partial^2 f_h}{\partial y_i \partial y_j} dy_1 \dots dy_k dy_{k+1} \dots dy_n.$$

Set:

$$w_m := \begin{cases} u_k, & m = 1, \dots, k, \\ u_n, & m = k+1, \dots, n, \end{cases} \qquad z_m := \begin{cases} v_k, & m = 1, \dots, k, \\ v_n, & m = k+1, \dots, n, \end{cases} \tag{24}$$

and: $\mathbf{w} := (w_1, \dots, w_n)$, $\mathbf{z} := (z_1, \dots, z_n)$, $d\mathbf{y} := dy_1 \dots dy_k dy_{k+1} \dots dy_n$.

Let in addition, $f_h\,(y_i = a, y_j = b)$ denote the function of $n-2$ variables, obtained by putting $y_i = a$, $y_j = b$ into the formula on $f_h(y_1, \dots, y_k, y_{k+1}, \dots, y_n)$. Then

$$F'(h) = \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{\mathbf{z}}^{\mathbf{w}} \cdots \int \frac{\partial^2 f_h}{\partial y_i \partial y_j} d\mathbf{y}$$

$$= \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{\mathbf{z}'}^{\mathbf{w}'} \cdots \int f_h\,(y_i = w_i, y_j = w_j)\, d\mathbf{y}'$$

$$- \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{\mathbf{z}'}^{\mathbf{w}'} \cdots \int f_h\,(y_i = w_i, y_j = z_j)\, d\mathbf{y}'$$

$$- \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{\mathbf{z}'}^{\mathbf{w}'} \cdots \int f_h\,(y_i = z_i, y_j = w_j)\, d\mathbf{y}'$$

$$+ \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{\mathbf{z}'}^{\mathbf{w}'} \cdots \int f_h\,(y_i = z_i, y_j = z_j)\, d\mathbf{y}',$$

where

$$d\mathbf{y}' = d\mathbf{y}'_{ij} := \begin{cases} dy_1 \dots dy_{i-1} dy_{i+1} \dots dy_{j-1} dy_{j+1} \dots dy_n, & 1 < i < j < n, \\ dy_2 \dots\dots\dots dy_{n-1}, & i = 1,\ j = n. \end{cases}$$

Moreover, we can dominate the last integrals by replacing $(\Lambda_{ij}^1 - \Lambda_{ij}^0)$ and $-(\Lambda_{ij}^1 - \Lambda_{ij}^0)$ by their absolute values, and by allowing the variables to run from $-\infty$ to ∞. Therefore,

$$F'(h) \le \sum_{i<j} |\Lambda_{ij}^1 - \Lambda_{ij}^0| \int_{-\infty}^{\infty} \cdots \int f_h(y_i = w_i, y_j = w_j)\, d\mathbf{y}' \tag{25}$$
$$+ \sum_{i<j} |\Lambda_{ij}^1 - \Lambda_{ij}^0| \int_{-\infty}^{\infty} \cdots \int f_h(y_i = w_i, y_j = z_j)\, d\mathbf{y}'$$
$$+ \sum_{i<j} |\Lambda_{ij}^1 - \Lambda_{ij}^0| \int_{-\infty}^{\infty} \cdots \int f_h(y_i = z_i, y_j = w_j)\, d\mathbf{y}'$$
$$+ \sum_{i<j} |\Lambda_{ij}^1 - \Lambda_{ij}^0| \int_{-\infty}^{\infty} \cdots \int f_h(y_i = z_i, y_j = z_j)\, d\mathbf{y}'.$$

Notice that

$$\int_{-\infty}^{\infty} \cdots \int f_h(y_i = w_i, y_j = w_j)\, d\mathbf{y}', \quad \int_{-\infty}^{\infty} \cdots \int f_h(y_i = w_i, y_j = z_j)\, d\mathbf{y}',$$
$$\int_{-\infty}^{\infty} \cdots \int f_h(y_i = z_i, y_j = w_j)\, d\mathbf{y}', \quad \int_{-\infty}^{\infty} \cdots \int f_h(y_i = z_i, y_j = z_j)\, d\mathbf{y}'$$

are the joint densities of two standard normal r.v.'s, evalueted at (w_i, w_j), (w_i, z_j), (z_i, w_j), (z_i, z_j), respectively, with the correlation Λ_{ij}^h. Thus:

$$\int_{-\infty}^{\infty} \cdots \int f_h(y_i = w_i, y_j = w_j)\, d\mathbf{y}' = \frac{1}{2\pi(1 - (\Lambda_{ij}^h)^2)^{1/2}} \exp\left\{ -\frac{w_i^2 - 2\Lambda_{ij}^h w_i w_j + w_j^2}{2(1 - (\Lambda_{ij}^h)^2)} \right\}, \tag{26}$$

$$\int_{-\infty}^{\infty} \cdots \int f_h(y_i = w_i, y_j = z_j)\, d\mathbf{y}' = \frac{1}{2\pi(1 - (\Lambda_{ij}^h)^2)^{1/2}} \exp\left\{ -\frac{w_i^2 - 2\Lambda_{ij}^h w_i z_j + z_j^2}{2(1 - (\Lambda_{ij}^h)^2)} \right\}, \tag{27}$$

$$\int_{-\infty}^{\infty} \cdots \int f_h(y_i = z_i, y_j = w_j)\, d\mathbf{y}' = \frac{1}{2\pi(1 - (\Lambda_{ij}^h)^2)^{1/2}} \exp\left\{ -\frac{z_i^2 - 2\Lambda_{ij}^h z_i w_j + w_j^2}{2(1 - (\Lambda_{ij}^h)^2)} \right\}, \tag{28}$$

$$\int_{-\infty}^{\infty}\cdots\int f_h\left(y_i = z_i, y_j = z_j,\right) d\mathbf{y}' = \frac{1}{2\pi(1-(\Lambda_{ij}^h)^2)^{1/2}} \exp\left\{-\frac{z_i^2 - 2\Lambda_{ij}^h z_i z_j + z_j^2}{2(1-(\Lambda_{ij}^h)^2)}\right\}. \tag{29}$$

We have $\left|\Lambda_{ij}^h\right| = \left|h\Lambda_{ij}^1 + (1-h)\,\Lambda_{ij}^0\right| \le \max(|\Lambda_{ij}^1|, |\Lambda_{ij}^0|) =: \rho_{ij}$. Clearly

$$\frac{a^2 - 2\Lambda_{ij}^h ab + b^2}{1-(\Lambda_{ij}^h)^2} = \frac{a^2 - 2\Lambda_{ij}^h ab + b^2}{(1-|\Lambda_{ij}^h|)(1+|\Lambda_{ij}^h|)} \ge \frac{a^2 - 2|\Lambda_{ij}^h|\,|a|\,|b| + b^2}{(1-|\Lambda_{ij}^h|)\,(1+\rho_{ij})} \ge \frac{a^2+b^2}{1+\rho_{ij}},$$

where the last relation follows from the fact that $\frac{a^2 - 2\,|c|\,|a|\,|b| + b^2}{1-|c|}$ reaches its minimum for $c = 0$. This and (26)–(29) imply:

$$\int_{-\infty}^{\infty}\cdots\int f_h\left(y_i = w_i, y_j = w_j\right) d\mathbf{y}' \le \frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp\left\{-\frac{w_i^2 + w_j^2}{2(1+\rho_{ij})}\right\}, \tag{30}$$

$$\int_{-\infty}^{\infty}\cdots\int f_h\left(y_i = w_i, y_j = z_j\right) d\mathbf{y}' \le \frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp\left\{-\frac{w_i^2 + z_j^2}{2(1+\rho_{ij})}\right\}, \tag{31}$$

$$\int_{-\infty}^{\infty}\cdots\int f_h\left(y_i = z_i, y_j = w_j\right) d\mathbf{y}' \le \frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp\left\{-\frac{z_i^2 + w_j^2}{2(1+\rho_{ij})}\right\}, \tag{32}$$

$$\int_{-\infty}^{\infty}\cdots\int f_h\left(y_i = z_i, y_j = z_j\right) d\mathbf{y}' \le \frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp\left\{-\frac{z_i^2 + z_j^2}{2(1+\rho_{ij})}\right\}. \tag{33}$$

By (30)–(33) and (25), we get

$$\begin{aligned} F'(h) \le {} & \sum_{i<j} |\Lambda_{ij}^1 - \Lambda_{ij}^0| \frac{1}{2\pi\left(1-\rho_{ij}^2\right)^{1/2}} \exp\left\{-\frac{w_i^2 + w_j^2}{2\,(1+\rho_{ij})}\right\} \\ & + \sum_{i<j} |\Lambda_{ij}^1 - \Lambda_{ij}^0| \frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp\left\{-\frac{w_i^2 + z_j^2}{2(1+\rho_{ij})}\right\} \\ & + \sum_{i<j} |\Lambda_{ij}^1 - \Lambda_{ij}^0| \frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp\left\{-\frac{z_i^2 + w_j^2}{2(1+\rho_{ij})}\right\} \\ & + \sum_{i<j} |\Lambda_{ij}^1 - \Lambda_{ij}^0| \frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp\left\{-\frac{z_i^2 + z_j^2}{2(1+\rho_{ij})}\right\}. \end{aligned} \tag{34}$$

Observe that, due to (5), $r(n) \to 0$, as $n \to \infty$. This and the remark before the statement of Lemma 4.3.2 in Leadbetter et al. [14] imply $\sup_{n\geq 1} |r(n)| = \delta < 1$, which yields that $\sup_{i\neq j} |\rho_{ij}| = \delta < 1$. Therefore,

$$\frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \leq \frac{1}{2\pi(1-\delta^2)^{1/2}} =: C(\delta). \tag{35}$$

Let $C = C(\delta)$. Obviously C is a positive constant, which depends only on δ. Due to (34), (35), we deduce that $F'(h)$ is not greater than the sum

$$C\sum_{i<j}|\Lambda^1_{ij}-\Lambda^0_{ij}|\exp\left\{\frac{w_i^2+w_j^2}{2(1+\rho_{ij})}\right\} + C\sum_{i<j}|\Lambda^1_{ij}-\Lambda^0_{ij}|\exp\left\{\frac{w_i^2+z_j^2}{2(1+\rho_{ij})}\right\}$$
$$+C\sum_{i<j}|\Lambda^1_{ij}-\Lambda^0_{ij}|\exp\left\{\frac{z_i^2+w_j^2}{2(1+\rho_{ij})}\right\} + C\sum_{i<j}|\Lambda^1_{ij}-\Lambda^0_{ij}|\exp\left\{\frac{z_i^2+z_j^2}{2(1+\rho_{ij})}\right\}.$$

This and the relation in (23) imply

$$\begin{aligned}
&|Cov(I\{M_k \leq u_k, m_k > v_k\}, I\{M_{k,n} \leq u_n, m_{k,n} > v_n\})| \\
&\leq C\int_0^1\sum_{i<j}|\Lambda^1_{ij}-\Lambda^0_{ij}|\exp\left\{-\frac{w_i^2+w_j^2}{2(1+\rho_{ij})}\right\}dh \\
&+ C\int_0^1\sum_{i<j}|\Lambda^1_{ij}-\Lambda^0_{ij}|\exp\left\{-\frac{w_i^2+z_j^2}{2(1+\rho_{ij})}\right\}dh \\
&+ C\int_0^1\sum_{i<j}|\Lambda^1_{ij}-\Lambda^0_{ij}|\exp\left\{-\frac{z_i^2+w_j^2}{2(1+\rho_{ij})}\right\}dh \\
&+ C\int_0^1\sum_{i<j}|\Lambda^1_{ij}-\Lambda^0_{ij}|\exp\left\{-\frac{z_i^2+z_j^2}{2(1+\rho_{ij})}\right\}dh.
\end{aligned} \tag{36}$$

By the definitions of the sequences $\{\xi_1,\ldots,\xi_n\}$, $\{\eta_1,\ldots,\eta_n\}$ in (21), we obtain that the elements Λ^1_{ij}, Λ^0_{ij}, $i<j$, of their covariance matrices Λ^1, Λ^0, are defined as follows:

$$\Lambda^1_{ij} = Cov(X_i,X_j) = r(j-i),\ i<j,$$

$$\Lambda^0_{ij} = \begin{cases} Cov(X_i,X_j) = r(j-i), & 1\leq i<j\leq k \text{ or } k+1\leq i<j\leq n, \\ 0, & i=1,\ldots,k,\ j=k+1,\ldots,n. \end{cases}$$

Hence, provided $i<j$,

$$|\Lambda^1_{ij}-\Lambda^0_{ij}| = \begin{cases} |r(j-i)|, & i=1,\ldots,k,\ j=k+1,\ldots,n, \\ 0, & \textit{otherwise}. \end{cases}$$

This, (36) and the definitions of $\{w_m\}$, $\{z_m\}$ in (24) imply

$$
\begin{aligned}
&|Cov\,(I\,\{M_k \le u_k, m_k > v_k\}\,, I\,\{M_{k,n} \le u_n, m_{k,n} > v_n\})| \\
&\qquad \le C \sum_{i=1}^{k} \sum_{j=k+1}^{n} |r\,(j-i)| \exp\left\{-\frac{u_k^2 + u_n^2}{2(1+\rho_{ij})}\right\} \\
&\qquad + C \sum_{i=1}^{k} \sum_{j=k+1}^{n} |r\,(j-i)| \exp\left\{-\frac{u_k^2 + v_n^2}{2(1+\rho_{ij})}\right\} \\
&\qquad + C \sum_{i=1}^{k} \sum_{j=k+1}^{n} |r\,(j-i)| \exp\left\{-\frac{v_k^2 + u_n^2}{2(1+\rho_{ij})}\right\} \\
&\qquad + C \sum_{i=1}^{k} \sum_{j=k+1}^{n} |r\,(j-i)| \exp\left\{-\frac{v_k^2 + v_n^2}{2(1+\rho_{ij})}\right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(37)\qquad &|Cov\,(I\,\{M_k \le u_k, m_k > v_k\}\,, I\,\{M_{k,n} \le u_n, m_{k,n} > v_n\})| \\
&\ll k \sum_{t=1}^{n-1} |r\,(t)| \exp\left\{-\frac{u_k^2 + u_n^2}{2\,(1+|r\,(t)|)}\right\} + k \sum_{t=1}^{n-1} |r\,(t)| \exp\left\{-\frac{u_k^2 + v_n^2}{2\,(1+|r\,(t)|)}\right\} \\
&\quad + k \sum_{t=1}^{n-1} |r\,(t)| \exp\left\{-\frac{v_k^2 + u_n^2}{2\,(1+|r\,(t)|)}\right\} + k \sum_{t=1}^{n-1} |r\,(t)| \exp\left\{-\frac{v_k^2 + v_n^2}{2\,(1+|r\,(t)|)}\right\} \\
&=: A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

It follows from the assumptions on $\{u_n\}$, $\{v_n\}$ in (6) and Lemma 2.1 in Csaki and Gonchigdanzan [6] that

$$
(38)\qquad A_i \ll (\log\log n)^{-(1+\varepsilon)} \text{ for some } \varepsilon > 0,\ i = 1,2,3,4.
$$

The relations in (37), (38) establish the desired result (20). ■

LEMMA 4. *Under the assumptions of Theorem 2 on* $X_1, X_2, \ldots,$ *and* $\{u_n\}$, $\{v_n\}$, $r\,(n)$, *we have for* $k < n$ *that*

$$
(39)\qquad \lim_{n\to\infty} P\,\{M_n \le u_n, m_n > v_n\} = e^{-(\tau+\eta)}, \text{ where } \tau,\ \eta \text{ satisfy (6).}
$$

Proof. Let $\{\tilde{X}_i\}$ be an i.i.d. standard normal sequence. Put: $\tilde{M}_n := \max(\tilde{X}_1, \ldots, \tilde{X}_n)$, $\tilde{m}_n := \min(\tilde{X}_1, \ldots, \tilde{X}_n)$. First, we shall estimate the term $|P\,\{M_n \le u_n, m_n > v_n\} - P\{\tilde{M}_n \le u_n, \tilde{m}_n > v_n\}|$. Observe that

$$
\begin{aligned}
&\left|P\,\{M_n \le u_n, m_n > v_n\} - P\left\{\tilde{M}_n \le u_n, \tilde{m}_n > v_n\right\}\right| = \\
&\left|P\,\{v_n < X_1 \le u_n, \ldots, v_n < X_n \le u_n\} - P\left\{v_n < \tilde{X}_1 \le u_n, \ldots, v_n < \tilde{X}_n \le u_n\right\}\right|.
\end{aligned}
$$

Thus, by applying similar methods to those used in the estimation of the term $|Cov(I\{M_k \le u_k, m_k > v_k\}, I\{M_{k,n} \le u_n, m_{k,n} > v_n\})|$ in the proof of Lemma 3, with: $\{\xi_i = X_i,\ i = 1, \dots, n\}$, $\left\{\eta_i = \tilde{X}_i,\ i = 1, \dots, n\right\}$, $\{w_m = u_n, m = 1, \dots, n\}$, $\{z_m = v_n, m = 1, \dots, n\}$, we get, for some positive constant C, that

$$\begin{aligned}
&\left|P\{M_n \le u_n, m_n > v_n\} - P\left\{\tilde{M}_n \le u_n, \tilde{m}_n > v_n\right\}\right| \\
&\qquad \le C\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |r(j-i)| \exp\left\{-\frac{u_n^2+u_n^2}{2(1+|r(j-i)|)}\right\} \\
&\qquad + C\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |r(j-i)| \exp\left\{-\frac{u_n^2+v_n^2}{2(1+|r(j-i)|)}\right\} \\
&\qquad + C\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |r(j-i)| \exp\left\{-\frac{v_n^2+u_n^2}{2(1+|r(j-i)|)}\right\} \\
&\qquad + C\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |r(j-i)| \exp\left\{-\frac{v_n^2+v_n^2}{2(1+|r(j-i)|)}\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
(40)\quad &\left|P\{M_n \le u_n, m_n > v_n\} - P\left\{\tilde{M}_n \le u_n, \tilde{m}_n > v_n\right\}\right| \\
&\ll n\sum_{t=1}^{n-1} |r(t)| \exp\left\{-\frac{u_n^2+u_n^2}{2(1+|r(t)|)}\right\} + n\sum_{t=1}^{n-1} |r(t)| \exp\left\{-\frac{u_n^2+v_n^2}{2(1+|r(t)|)}\right\} \\
&+ n\sum_{t=1}^{n-1} |r(t)| \exp\left\{-\frac{v_n^2+u_n^2}{2(1+|r(t)|)}\right\} + n\sum_{t=1}^{n-1} |r(t)| \exp\left\{-\frac{v_n^2+v_n^2}{2(1+|r(t)|)}\right\}.
\end{aligned}$$

By (6), we have:

$$\exp\left(-\frac{u_n^2}{2}\right) \sim \frac{K_1\sqrt{2\pi}u_n}{n}, \quad \exp\left(-\frac{(-v_n)^2}{2}\right) \sim \frac{K_2\sqrt{2\pi}(-v_n)}{n},$$

for some K_1, $K_2 \ge 0$, $u_n \sim (2\log n)^{1/2}$, $(-v_n) \sim (2\log n)^{1/2}$. This, (40) and Lemma 2.1 in Csaki and Gonchigdanzan [6] yield

$$(41)\quad \left|P\{M_n \le u_n, m_n > v_n\} - P\left\{\tilde{M}_n \le u_n, \tilde{m}_n > v_n\right\}\right| \ll (\log\log n)^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$.

Since $(\log \log n)^{-(1+\varepsilon)} \to 0$, as $n \to \infty$, it follows from (41) that

$$\lim_{n\to\infty} \left| P\{M_n \le u_n, m_n > v_n\} - P\left\{\tilde{M}_n \le u_n, \tilde{m}_n > v_n\right\}\right| = 0. \tag{42}$$

Moreover, due to Theorem 1.8.2 in Leadbetter et al. [14], we obtain that, provided τ, η satisfy (6), then

$$\lim_{n\to\infty} P\left\{\tilde{M}_n \le u_n, \tilde{m}_n > v_n\right\} = e^{-(\tau+\eta)}. \tag{43}$$

Due to (42), (43)

$$\lim_{n\to\infty} P\{M_n \le u_n, m_n > v_n\} = \lim_{n\to\infty} P\left\{\tilde{M}_n \le u_n, \tilde{m}_n > v_n\right\} = e^{-(\tau+\eta)},$$

which is the result (39), we wished to prove. ∎

We now give the proofs of our main results. They make an extensive use of the earlier proved Lemmas 1–4.

Proof of Theorem 1(i). First, we will show that, under the assumptions of Theorem 1(i),

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(I\{M_n \le u_n, m_n > v_n\} - P\{M_n \le u_n, m_n > v_n\}\right) \underset{N\to\infty}{\longrightarrow} 0 \quad a.s. \tag{44}$$

Put $\mu_n := I\{M_n \le u_n, m_n > v_n\}$. In order to prove (44), we will estimate the variance

$$Var\left(\sum_{n=1}^{N}\frac{1}{n}\mu_n\right) \le \sum_{n=1}^{N}\frac{1}{n^2}Var(\mu_n) + 2\sum_{1\le k<n\le N}\frac{1}{kn}\left|Cov(\mu_k,\mu_n)\right| =: B_1 + B_2. \tag{45}$$

Obviously

$$B_1 \ll \sum_{n=1}^{N}\frac{1}{n^2} < \infty. \tag{46}$$

Thus, it remains to estimate the component B_2 in (45). By Lemma 1, we have

$$B_2 \ll \sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\frac{1}{kn}\frac{k}{n} = \sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\frac{1}{n^2} \ll \sum_{k=1}^{N-1}\frac{1}{k} \ll \log N. \tag{47}$$

Due to (45)–(47)

$$Var\left(\sum_{n=1}^{N}\frac{1}{n}I\{M_n \le u_n, m_n > v_n\}\right) \ll \log N. \tag{48}$$

This and Lemma 3.1 in Csaki and Gonchigdanzan [6] imply (44). It follows from Theorem 1.8.2 in Leadbetter et al. [14] that

$$\lim_{n\to\infty} P\left\{M_n \le u_n, m_n > v_n\right\} = e^{-(\tau+\eta)}, \text{ where } \tau,\ \eta \text{ satisfy } (1). \tag{49}$$

Since (44), (49) hold, then, by the regularity property of logarithmic means, we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{M_n \le u_n, m_n > v_n\right\} = e^{-(\tau+\eta)} \quad a.s. \tag{50}$$

Furthermore, it follows from Theorem 1.1 in Csaki and Gonchigdanzan [6] that, provided τ satisfies (1),

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{M_n \le u_n\right\} = e^{-\tau} \quad a.s. \tag{51}$$

Due to (50), (51)

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{M_n \le u_n, m_n \le v_n\right\} \tag{52}$$

$$= \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{M_n \le u_n\right\} - \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{M_n \le u_n, m_n > v_n\right\}$$

$$= e^{-\tau} - e^{-(\tau+\eta)} = e^{-\tau}\left(1 - e^{-\eta}\right) \quad a.s.$$

Derivations in (52) imply the result (2) in our assertion. ∎

P r o o f o f T h e o r e m 1(ii). Under the assumptions in (3), the result (4) follows immediately from Theorem 1(i), by identifying u_n, v_n with $x/a_n + b_n$, $y/\alpha_n + \beta_n$, and τ, η with $-\log G(x)$, $-\log(1 - H(y))$, respectively. ∎

P r o o f o f T h e o r e m 2(i). The idea of this proof is similar to the idea of the proof of Theorem 1(i). First, we will show that, under the assumptions of Theorem 2(i),

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left(I\left\{M_n \le u_n, m_n > v_n\right\} - P\left\{M_n \le u_n, m_n > v_n\right\}\right) \underset{N\to\infty}{\longrightarrow} 0 \quad a.s. \tag{53}$$

Put $\eta_n := I\left\{M_n \le u_n, m_n > v_n\right\}$. In order to prove (53), we will estimate the variance

$$Var\left(\sum_{n=1}^{N} \frac{1}{n}\eta_n\right) \le \sum_{n=1}^{N} \frac{1}{n^2} Var(\eta_n) + 2\sum_{1\le k<n\le N} \frac{1}{kn} \left|Cov(\eta_k, \eta_n)\right| \tag{54}$$

$$=: D_1 + D_2.$$

Clearly

$$D_1 \ll \sum_{n=1}^{N} \frac{1}{n^2} < \infty. \tag{55}$$

Thus, we only need to estimate the component D_2 in (54). We have

$$\begin{aligned} |Cov(\eta_k,\eta_n)| &= |Cov(I\{M_k \le u_k, m_k > v_k\}, I\{M_n \le u_n, m_n > v_n\})| \\ &\ll E\,|I\{M_n \le u_n, m_n > v_n\} - I\{M_{k,n} \le u_n, m_{k,n} > v_n\}| \\ &\quad + |Cov(I\{M_k \le u_k, m_k > v_k\}, I\{M_{k,n} \le u_n, m_{k,n} > v_n\})|. \end{aligned}$$

Thus, by Lemmas 2, 3,

$$|Cov(\eta_k,\eta_n)| \ll k/n + (\log\log n)^{-(1+\varepsilon)} \text{ for some } \varepsilon > 0.$$

This and the definition of D_2 in (54) imply

$$D_2 \ll \sum_{k=1}^{N-1} \sum_{n=k+1}^{N} \frac{1}{kn}\frac{k}{n} + \sum_{1\le k<n\le N} \frac{1}{kn}(\log\log n)^{-(1+\varepsilon)} =: D_{21} + D_{22}. \tag{56}$$

By the estimation of B_2 in the proof of Theorem 1(i), we get

$$D_{21} \ll \log N. \tag{57}$$

We have the following estimate for D_{22} in (56)

$$\begin{aligned} D_{22} &\ll \sum_{n=3}^{N} \frac{1}{n(\log\log n)^{1+\varepsilon}} \sum_{k=1}^{n-1} \frac{1}{k} \ll \sum_{n=3}^{N} \frac{\log n}{n(\log\log n)^{1+\varepsilon}} \\ &\ll \frac{\log N}{(\log\log N)^{1+\varepsilon}} \sum_{n=3}^{N} \frac{1}{n} \ll \frac{(\log N)^2}{(\log\log N)^{1+\varepsilon}} \text{ for some } \varepsilon > 0. \end{aligned} \tag{58}$$

Due to (56)–(58)

$$D_2 \ll \frac{(\log N)^2}{(\log\log N)^{1+\varepsilon}} \text{ for some } \varepsilon > 0. \tag{59}$$

Thus, by (54), (55) and (59),

$$Var\left(\sum_{n=1}^{N} \frac{1}{n} I\{M_n \le u_n, m_n > v_n\}\right) \ll \frac{(\log N)^2}{(\log\log N)^{1+\varepsilon}} \text{ for some } \varepsilon > 0. \tag{60}$$

This and Lemma 3.1 in Csaki and Gonchigdanzan [6] imply (53).

Moreover, it follows from Lemma 4 that

$$\lim_{n\to\infty} P\{M_n \le u_n, m_n > v_n\} = e^{-(\tau+\eta)}, \text{ where } \tau,\ \eta \text{ satisfy (6)}. \tag{61}$$

Since (60), (61) hold, then, by the regularity property of logarithmic means, we obtain that

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\{M_n\le u_n, m_n>v_n\}=e^{-(\tau+\eta)}\quad a.s. \tag{62}$$

Furthermore, it follows from Theorem 1.1 in Csaki and Gonchigdanzan [6] that

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\{M_n\le u_n\}=e^{-\tau}\quad a.s. \tag{63}$$

By using (62), (63) and proceeding analogously as in (52), we immediately get

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\{M_n\le u_n, m_n\le v_n\}=e^{-\tau}\left(1-e^{-\eta}\right)\quad a.s.,$$

which is the result (7), we wished to prove. ∎

Proof of Theorem 2(ii). It is easy to check that, provided a_n, b_n, α_n, β_n are defined such as in (8), (9), assumption (6) holds with: $u_n := x/a_n+b_n$, $\tau := e^{-x}$, $v_n := y/\alpha_n+\beta_n$, $\eta := e^{-y}$. Therefore, the almost sure convergence in (10) follows from Theorem 2(i), by identifying u_n, v_n with $x/a_n + b_n$, $y/\alpha_n + \beta_n$, and τ, η with e^{-x}, e^{-y}, respectively. ∎

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