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ON SOME FORMS OF WEAKLY CONTINUOUS FUNCTIONS IN BITOPOLOGICAL SPACES

Abstract. As a generalization of weakly continuous functions, we introduce the notion of (i, j)-weakly m-continuous functions in bitopological spaces and obtain unified characterizations and properties of certain forms of weakly continuous functions in bitopological spaces.

1. Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of modifications of continuity in topological and bitopological spaces. Maheshwari and Prasad [16] and Bose [2] introduced the concepts of semi-open sets and semi-continuity in bitopological spaces. Jelić [7], Kar and Bhatacharyya [9] and Khedr et al. [13] introduced the concepts of preopen sets and precontinuity in bitopological spaces. The notion of α-open sets (or feebly open sets) and α-continuity (or feeble continuity) in bitopological spaces were stdied in [8], [18] and [14]. The notions of β-open sets and β-continuity in bitopological spaces were studied in [13].

In 1961, Levine [15] introduced the concept of weakly continuous functions in topological spaces. Bose and Sinha [4] extended the notion of weakly continuous functions to bitopological spaces. Weak semi-continuity in bitopological spaces are introduced and studied in [12]. Some properties of weak semi-continuity (or quasi-continuity) in bitopological spaces are studied in [23], [28] and [5]. Recently, the present authors [19] have introduced and studied weakly precontinuous functions between bitopological spaces. The present authors [24], [25] introduced the notions of minimal

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structure, m-spaces and m-continuity. In [27], they introduced and studied the notions of weakly m-continuous functions. In the present paper, by using these concepts we obtain the unified definitions and properties of weak forms of continuity in bitopological spaces.

2. Preliminaries

Throughout the present paper, (X, τ_1, τ_2) (resp. (X, τ)) denote a bitopological (resp. topological) space. Let (X, τ) be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. The closure of A and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively, for $i = 1, 2$.

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-regular open* [1] (resp. *(i, j)-regular closed*) if $A = i\mathrm{Int}(j\mathrm{Cl}(A))$ (resp. $A = i\mathrm{Cl}(j\mathrm{Int}(A))$), where $i \neq j, i, j = 1, 2$,

(2) (i, j)-*semi-open* [2] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, i, $j = 1$, 2,

(3) (i, j)-*preopen* [7] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, i, $j = 1$, 2,

(4) (i, j)-α-*open* [8], [10] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, $i, j = 1, 2$,

(5) (i, j)-*semi-preopen* (briefly (i, j)-*sp-open*) [13] if there exists an (i, j)-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j, i, j = 1, 2$.

The family of all (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-*sp*-open) sets of (X, τ_1, τ_2) is denoted by $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$).

Definition 2.2. The complement of an (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-*sp*-open) set is said to be (i, j)-*semi-closed* (resp. (i, j)-*preclosed,* (i, j)-α-*closed,* (i, j)-*sp-closed*).

Definition 2.3. The (i, j)-*semi-closure* [16] (resp. (i, j)-*preclosure* [13], (i, j)-α-*closure* [18], (i, j)-*sp-closure* [13]) of A is defined by the intersection of (i, j)-semi-closed (resp. (i, j)-preclosed, (i, j)-α-closed, (i, j)-*sp*-closed) sets containing A and is denoted by (i, j)-$\mathrm{sCl}(A)$ (resp. (i, j)-$\mathrm{pCl}(A)$, (i, j)-$\alpha\mathrm{Cl}(A)$, (i, j)-$\mathrm{spCl}(A)$).

Definition 2.4. The (i, j)-*semi-interior* (resp. (i, j)-*preinterior,* (i, j)-α-*interior* , (i, j)-*sp-interior*) of A is defined by the union of (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-*sp*-open) sets contained in A and is denoted by (i, j)-$\mathrm{sInt}(A)$ (resp. (i, j)-$\mathrm{pInt}(A)$, (i, j)-$\alpha\mathrm{Int}(A)$, (i, j)-$\mathrm{spInt}(A)$).

Definition 2.5. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. A point x of X is said to be in (i, j)-θ-*closure* [10] of A, denoted by

(i,j)-$\mathrm{Cl}_\theta(A)$, if $A \cap j\mathrm{Cl}(U) \neq \emptyset$ for every τ_i-open set U containing x, where $i, j = 1, 2$ and $i \neq j$.

A subset A of X is said to be *(i,j)-θ-closed* if $A = (i,j)$-$\mathrm{Cl}_\theta(A)$. A subset A of X is said to be *(i,j)-θ-open* if $X - A$ is (i,j)-θ-closed. The *(i,j)-θ-interior* of A, denoted by (i,j)-$\mathrm{Int}_\theta(A)$, is defined as the union of all *(i,j)-θ-open* sets contained in A. Hence $x \in (i,j)$-$\mathrm{Int}_\theta(A)$ if and only if there exists a τ_i-open set U containing x such that $x \in U \subset j\mathrm{Cl}(U) \subset A$.

LEMMA 2.1. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*
(1) $X - (i,j)$-$\mathrm{Int}_\theta(A) = (i,j)$-$\mathrm{Cl}_\theta(X - A)$,
(2) $X - (i,j)$-$\mathrm{Cl}_\theta(A) = (i,j)$-$\mathrm{Int}_\theta(X - A)$.

LEMMA 2.2. (Kariofillis [10]). *Let (X, τ_1, τ_2) be a bitopological space. If U is a τ_j-open set of X, then (i,j)-$\mathrm{Cl}_\theta(U) = i\mathrm{Cl}(U)$.*

DEFINITION 2.6. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i,j)-weakly continuous* [4] (resp. *(i,j)-weakly semi-continuous* [12], *(i,j)-weakly precontinuous* [19]) if for each $x \in X$ and each σ_i-open set V of Y containing $f(x)$, there exists a τ_i-open (resp. (i,j)-semi-open, (i,j)-preopen) set U containing x such that $f(U) \subset j\mathrm{Cl}(V)$.

DEFINITION 2.7. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i,j)-weakly α-continuous* (resp. *(i,j)-weakly sp-continuous*) if for each $x \in X$ and each σ_i-open set V of Y containing $f(x)$, there exists an (i,j)-α-open (resp. (i,j)-*sp*-open) set U containing x such that $f(U) \subset j\mathrm{Cl}(V)$.

REMARK 2.1. Since every τ_i-open set is (i,j)-semi-open (resp. (i,j)-preopen), every (i,j)-weakly continuous function is (i,j)-weakly semi-continuous (resp. (i,j)-weakly precontinuous) for $i \neq j$ and $i, j = 1, 2$.

3. Minimal structures and weak m-continuity

DEFINITION 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (or briefly *m-structure*) [23], [24] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

DEFINITION 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [17] as follows:

(1) m_X-Cl$(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) m_X-Int$(A) = \cup\{U : U \subset A, U \in m_X\}$.

REMARK 3.1. (1) Let (X, τ_1, τ_2) be a bitopological space. Then the families (i,j)SO(X), (i,j)PO(X), $(i,j)\alpha(X)$ and (i,j)SPO(X) are all m-structures on X.

(2) Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. If $m_X = (i,j)$SO(X) (resp. (i,j)PO(X), $(i,j)\alpha(X)$, (i,j)SPO(X)), then we have

(a) m_X-Cl$(A) = (i,j)$-sCl(A) (resp. (i,j)-pCl(A), (i,j)-αCl(A), (i,j)-spCl(A)),

(b) m_X-Int$(A) = (i,j)$-sInt(A) (resp. (i,j)-pInt(A), (i,j)-αInt(A), (i,j)-spInt(A)).

LEMMA 3.1 (Maki et al. [17]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) m_X-Cl$(X - A) = X - m_X$-Int(A) and m_X-Int$(X - A) = X - m_X$-Cl(A),

(2) If $(X - A) \in m_X$, then m_X-Cl$(A) = A$ and if $A \in m_X$, then m_X-Int$(A) = A$,

(3) m_X-Cl$(\emptyset) = \emptyset, m_X$-Cl$(X) = X$, m_X-Int$(\emptyset) = \emptyset$ and m_X-Int$(X) = X$,

(4) If $A \subset B$, then m_X-Cl$(A) \subset m_X$-Cl(B) and m_X-Int$(A) \subset m_X$-Int(B),

(5) $A \subset m_X$-Cl(A) and m_X-Int$(A) \subset A$,

(6) m_X-Cl$(m_X$-Cl$(A)) = m_X$-Cl(A) and m_X-Int$(m_X$-Int$(A)) = m_X$-Int(A).

LEMMA 3.2 (Popa and Noiri [23]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in m_X$-Cl(A) if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

DEFINITION 3.3. A minimal structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [17] if the union of any family of subsets belonging to m_X belongs to m_X.

REMARK 3.2. Let (X, τ_1, τ_2) be a bitopological space. Then the families (i,j)SO(X), (i,j)PO(X), $(i,j)\alpha(X)$ and (i,j)SPO(X) are all m-structures on X satisfying property $\mathcal{B}$ by Theorem 2 of [16] (resp. Theorem 4.2 of [9] or Theorem 3.2 of [13], Theorem 5 of [18], Theorem 3.2 of [13]).

LEMMA 3.3 (Popa and Noiri [26]). *Let (X, m_X) be an m-space and A a subset of X. Then, the following properties are equivalent:*

(1) m_X has property $\mathcal{B}$,

(2) if m_X-Int$(A) = A$, then $A \in m_X$,
(3) if m_X-Cl$(A) = A$, then A is m_X-closed.

DEFINITION 3.4. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *weakly m-continuous* [27] (resp. *m-continuous* [25]) if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset \mathrm{Cl}(V)$ (resp. $f(U) \subset V$).

DEFINITION 3.5. A function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-*weakly m-continuous* if for each $x \in X$ and each $V \in \sigma_i$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset j\mathrm{Cl}(V)$.

THEOREM 3.1. *For a function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*
(1) f is (i, j)-weakly m-continuous;
(2) m_X-Cl$(f^{-1}(j\mathrm{Int}(i\mathrm{Cl}(B)))) \subset f^{-1}(i\mathrm{Cl}(B))$ for every subset B of Y;
(3) m_X-Cl$(f^{-1}(j\mathrm{Int}(F))) \subset f^{-1}(F)$ for every (i, j)-regular closed set F of Y;
(4) m_X-Cl$(f^{-1}(V)) \subset f^{-1}(i\mathrm{Cl}(V))$ for every σ_j-open set V of Y;
(5) $f^{-1}(V) \subset m_X$-Int$(f^{-1}(j\mathrm{Cl}(V)))$ for every σ_i-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \in X - f^{-1}(i\mathrm{Cl}(B))$. Then $f(x) \in Y - i\mathrm{Cl}(B)$ and there exists a σ_i-open set V of Y containing $f(x)$ such that $V \cap B = \emptyset$. Therefore, $V \cap j\mathrm{Int}(i\mathrm{Cl}(B)) = \emptyset$ and hence $j\mathrm{Cl}(V) \cap j\mathrm{Int}(i\mathrm{Cl}(B)) = \emptyset$. Therefore, there exists an m_X-open set U containing x such that $f(U) \subset j\mathrm{Cl}(V)$. Hence, we have $U \cap f^{-1}(j\mathrm{Int}(i\mathrm{Cl}(B))) = \emptyset$ and $x \in X - m_X$-Cl$(f^{-1}(j\mathrm{Int}(i\mathrm{Cl}(B))))$ by Lemma 3.2. Thus, we obtain m_X-Cl$(f^{-1}(j\mathrm{Int}(i\mathrm{Cl}(B)))) \subset f^{-1}(i\mathrm{Cl}(B))$.

(2) $\Rightarrow$ (3): Let F be an (i,j)-regular closed set of Y. Then $F = i\mathrm{Cl}(j\mathrm{Int}(F))$ and we have

$$\begin{aligned} m_X\text{-Cl}(f^{-1}(j\mathrm{Int}(F))) &= m_X\text{-Cl}(f^{-1}(j\mathrm{Int}(i\mathrm{Cl}(j\mathrm{Int}(F))))) \\ &\subset f^{-1}(i\mathrm{Cl}(j\mathrm{Int}(F))) = f^{-1}(F). \end{aligned}$$

(3) $\Rightarrow$ (4): Let V be a σ_j-open set of Y. Then $i\mathrm{Cl}(V)$ is (i,j)-regular closed. Then we obtain m_X-Cl$(f^{-1}(V)) \subset m_X$-Cl$(f^{-1}(j\mathrm{Int}(i\mathrm{Cl}(V)))) \subset f^{-1}(i\mathrm{Cl}(V))$.

(4) $\Rightarrow$ (5): Let V be a σ_i-open set of Y. Then $Y - j\mathrm{Cl}(V)$ is σ_j-open and we have m_X-Cl$(f^{-1}(Y - j\mathrm{Cl}(V))) \subset f^{-1}(i\mathrm{Cl}(Y - j\mathrm{Cl}(V)))$ and hence $X - m_X$-Int$(f^{-1}(j\mathrm{Cl}(V))) \subset X - f^{-1}(i\mathrm{Int}(j\mathrm{Cl}(V))) \subset X - f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) \subset m_X$-Int$(f^{-1}(j\mathrm{Cl}(V)))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be a σ_i-open set containing $f(x)$. We have $x \in f^{-1}(V) \subset m_X$-Int$(f^{-1}(j\mathrm{Cl}(V)))$. Then there exists $U \in m_X$ contining

x such that $x \in U \subset f^{-1}(j\text{Cl}(V))$. Hence $f(U) \subset j\text{Cl}(V))$. This shows that f is (i,j)-weakly m-continuous.

THEOREM 3.2. *For a function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (i, j)-weakly m-continuous;

(2) $f(m_X\text{-Cl}(A)) \subset (i,j)\text{-Cl}_\theta(f(A))$ for every subset A of X;

(3) $m_X\text{-Cl}(f^{-1}(B)) \subset f^{-1}((i,j)\text{-Cl}_\theta(B))$ for every subset B of Y;

(4) $m_X\text{-Cl}(f^{-1}(j\text{Int}((i,j)\text{-Cl}_\theta(B)))) \subset f^{-1}((i,j)\text{-Cl}_\theta(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Assume that f is (i,j)-weakly m-continuous. Let A be any subset of X, $x \in m_X\text{-Cl}(A)$ and V be a σ_i-open set of Y containing $f(x)$. Then, there exists an m_X-open set U containing x such that $f(U) \subset j\text{Cl}(V)$. Since $x \in m_X\text{-Cl}(A)$, by Lemma 3.2 we obtain $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U) \cap f(A) \subset j\text{Cl}(V) \cap f(A)$. Therefore, we obtain $f(x) \in (i,j)\text{-Cl}_\theta(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then we have $f(m_X\text{-Cl}(f^{-1}(B))) \subset (i,j)\text{-Cl}_\theta(f(f^{-1}(B))) \subset (i,j)\text{-Cl}_\theta(B)$ and hence $m_X\text{-Cl}(f^{-1}(B)) \subset f^{-1}((i,j)\text{-Cl}_\theta(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Since $(i,j)\text{-Cl}_\theta(B))$ is σ_i-closed in Y, by using Lemma 2.2

$$\begin{aligned} m_X\text{-Cl}(f^{-1}(j\text{Int}((i,j)\text{-Cl}_\theta(B)))) &\subset f^{-1}((i,j)\text{-Cl}_\theta(j\text{Int}((i,j)\text{-Cl}_\theta(B)))) \\ &= f^{-1}(i\text{Cl}(j\text{Int}((i,j)\text{-Cl}_\theta(B)))) \subset f^{-1}(i\text{Cl}((i,j)\text{-Cl}_\theta(B))) \\ &= f^{-1}((i,j)\text{-Cl}_\theta(B)). \end{aligned}$$

(4) $\Rightarrow$ (1): Let V be any σ_j-open set of Y. Then by Lemma 2.2, $V \subset j\text{Int}(i\text{Cl}(V)) = j\text{Int}((i,j)\text{-Cl}_\theta(V))$ and we have $m_X\text{-Cl}(f^{-1}(V)) \subset m_X\text{-Cl}(f^{-1}(j\text{Int}((i,j)\text{-Cl}_\theta(V)))) \subset f^{-1}((i,j)\text{-Cl}_\theta(V)) = f^{-1}(i\text{Cl}(V))$. Thus we obtain $m_X\text{-Cl}(f^{-1}(V)) \subset f^{-1}(i\text{Cl}(V))$. It follows from Theorem 3.1 that f is (i,j)-weakly m-continuous.

REMARK 3.3. If $\sigma = \sigma_1 = \sigma_2$, then by Theorems 3.1 and 3.2, we obtain the results established in Theorems 3.1 and 3.2 of [27].

THEOREM 3.3. *For a function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (i, j)-weakly m-continuous;

(2) $m_X\text{-Cl}(f^{-1}(V)) \subset f^{-1}(i\text{Cl}(V))$ for every (j,i)-preopen set V of Y;

(3) $f^{-1}(V \subset m_X\text{-Int}(f^{-1}(j\text{Cl}(V)))$ for every (i, j)-preopen set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any (j,i)-preopen set of Y. Suppose that $x \notin f^{-1}(i\text{Cl}(V))$. Then there exists a σ_i-open set W containing $f(x)$ such that $W \cap V = \emptyset$. Hence we have $i\text{Cl}(W \cap V) = \emptyset$. Since V is (j,i)-preopen, we have

$$V \cap j\mathrm{Cl}(W) \subset j\mathrm{Int}(i\mathrm{Cl}(V)) \cap j\mathrm{Cl}(W) \subset j\mathrm{Cl}(j\mathrm{Int}(i\mathrm{Cl}(V)) \cap W) \subset j\mathrm{Cl}(i\mathrm{Cl}(V)) \cap W) \subset j\mathrm{Cl}(i\mathrm{Cl}(V \cap W)) = j\mathrm{Cl}(\emptyset) = \emptyset.$$

Since f is (i,j)-weakly m-continuous and W is a σ_i-open set containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset j\mathrm{Cl}(W)$. Then $f(U) \cap V = \emptyset$ and hence $U \cap f^{-1}(V) = \emptyset$. This shows that $x \notin m_X$-$\mathrm{Cl}(f^{-1}(V))$. Therefore, we obtain m_X-$\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(i\mathrm{Cl}(V))$.

(2) $\Rightarrow$ (3): Let V be any (i,j)-preopen set of Y. By (2), we have

$$f^{-1}(V) \subset f^{-1}(i\mathrm{Int}(j\mathrm{Cl}(V))) = X - f^{-1}(i\mathrm{Cl}(Y - j\mathrm{Cl}(V))) \subset X - m_X\text{-}\mathrm{Cl}(f^{-1}(Y - j\mathrm{Cl}(V))) = m_X\text{-}\mathrm{Int}(f^{-1}(j\mathrm{Cl}(V))).$$

(3) $\Rightarrow$ (1): Let V be any σ_i-open set of Y. Then V is (i,j)-preopen in Y and $f^{-1}(V) \subset m_X$-$\mathrm{Int}(f^{-1}(j\mathrm{Cl}(V)))$. By Theorem 3.1, f is (i,j)-weakly m-continuous.

REMARK 3.4. If $\sigma = \sigma_1 = \sigma_2$, then by Theorem 3.3, we obtain the results established in Theorem 3.3 of [27].

DEFINITION 3.6. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = m(\tau_i, \tau_j)$ an m-structure on X determined by τ_1 and τ_2. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i,j)-weakly m-continuous* if a function $f : (X, m_{ij}) \to (Y, \sigma_1, \sigma_2)$ is (i,j)-weakly m-continuous, equivalently if for each $x \in X$ and each $V \in \sigma_i$ containing $f(x)$, there exists $U \in m_{ij}$ containing x such that $f(U) \subset j\mathrm{Cl}(V)$.

REMARK 3.5. 1) A bitopological space (X, τ_1, τ_2) having an m-structure $m_{ij} = m(\tau_i, \tau_j)$ determined by τ_1 and τ_2 is briefly called a space (X, τ_1, τ_2) with an m-structure m_{ij}.

2) If $m_{ij} = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$), we obtain the definition of (i,j)-weakly semi-continuous (resp. (i,j)-weakly precontinuous) functions.

By Definition 3.6 and Theorems 3.1-3.3, we obtain the following theorems.

THEOREM 3.4. *Let (X, τ_1, τ_2) be a bitopological space with an m-structure m_{ij} on X. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (i, j)-weakly m-continuous;

(2) m_{ij}-$\mathrm{Cl}(f^{-1}(j\mathrm{Int}(i\mathrm{Cl}(B)))) \subset f^{-1}(i\mathrm{Cl}(B))$ for every subset B of Y;

(3) m_{ij}-$\mathrm{Cl}(f^{-1}(j\mathrm{Int}(F))) \subset f^{-1}(F)$ for every (i, j)-regular closed set F of Y;

(4) m_{ij}-$\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(i\mathrm{Cl}(V))$ for every σ_j-open set V of Y;

(5) $f^{-1}(V) \subset m_{ij}$-$\mathrm{Int}(f^{-1}(j\mathrm{Cl}(V)))$ for every σ_i-open set V of Y.

THEOREM 3.5. *Let (X, τ_1, τ_2) be a bitopological space with an m-structure m_{ij} on X. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (i, j)-weakly m-continuous;

(2) $f(m_{ij}\text{-Cl}(A)) \subset (i,j)\text{-Cl}_\theta(f(A))$ for every subset A of X;

(3) $m_{ij}\text{-Cl}(f^{-1}(B)) \subset f^{-1}((i,j)\text{-Cl}_\theta(B))$ for every subset B of Y;

(4) $m_{ij}\text{-Cl}(f^{-1}(j\text{Int}((i,j)\text{-Cl}_\theta(B)))) \subset f^{-1}((i,j)\text{-Cl}_\theta(B))$ for every subset B of Y.

THEOREM 3.6. *Let (X, τ_1, τ_2) be a bitopological space with an m-structure m_{ij} on X. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (i, j)-weakly m-continuous;

(2) $m_{ij}\text{-Cl}(f^{-1}(V)) \subset f^{-1}(i\text{Cl}(V))$ for every (j,i)-preopen set V of Y;

(3) $f^{-1}(V) \subset m_{ij}\text{-Int}(f^{-1}(j\text{Cl}(V)))$ for every (i, j)-preopen set V of Y.

REMARK 3.6. 1) If $m_{ij} = (i,j)\text{SO}(X)$ (resp. $(i,j)\text{PO}(X)$), by Theorems 3.4 and 3.5 we obtain the results in Theorems 3.2 and 3.3 of [12] and Theorems 3.1 and 3.2 of [23] (resp. Theorems 3.1 and 3.2 of [19]).

2) If $m_{ij} = \text{SO}(X)$, by Theorem 3.6 we obtain the results in Theorem 3.1 of [27].

3) If $\tau = \tau_1 = \tau_2$ and $m_{ij} = \text{SO}(X)$, then by Theorem 3.4 we obtain the results in Proposition 2.2 of [6].

4) If $\tau = \tau_1 = \tau_2$ and $m_{ij} = \tau$, then by Theorem 3.4 we obtain the results in Lemma 3.1 of [2] and also by Theorem 3.6 we obtain the results of Theorem 3.6 of [2].

4. Weak m-continuity and m-continuity

DEFINITION 4.1. A function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is said to be *m-i-continuous* if $f : (X, m_X) \to (Y, \sigma_i)$ is m-continuous.

LEMMA 4.1. *For a function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is m-i-continuous;

(2) $f^{-1}(V) = m_X\text{-Int}(f^{-1}(V))$ for every σ_i-open set V of Y;

(3) $f^{-1}(F) = m_X\text{-Cl}(f^{-1}(F))$ for every σ_i-closed set F of Y.

Proof. The proof follows from Definition 4.1 and Theorem 3.1 of [25].

DEFINITION 4.2. A bitopological space (X, τ_1, τ_2) is said to be *(i,j)-regular* [11] if for each $x \in X$ and each τ_i-open set U containing x, there exists a τ_i-open set V such that $x \in V \subset j\text{Cl}(V) \subset U$.

LEMMA 4.2 (Popa and Noiri [28]). *If a bitopological space (X, τ_1, τ_2) is (i, j)-regular, then $(i,j)\text{-Cl}_\theta(F) = F$ for every τ_i-closed set F.*

THEOREM 4.1. *Let (Y, σ_1, σ_2) be an (i,j)-regular bitopological space. For a function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is m-i-continuous;

(2) $f^{-1}((i,j)\text{-Cl}_\theta(B)) = m_X\text{-Cl}(f^{-1}((i,j)\text{-Cl}_\theta(B)))$ for every subset B of Y;

(3) f is (i,j)-weakly m-continuous;

(4) $f^{-1}(F) = m_X\text{-Cl}(f^{-1}(F))$ for every (i,j)-θ-closed set F of Y;

(5) $f^{-1}(V) = m_X\text{-Int}(f^{-1}(V))$ for every (i,j)-θ-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Since $(i,j)\text{-Cl}_\theta(B)$ is σ_i-closed in Y, it follows from Lemma 4.1 that $f^{-1}((i,j)\text{-Cl}_\theta(B)) = m_X\text{-Cl}(f^{-1}((i,j)\text{-Cl}_\theta(B)))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then by (2) and Lemma 3.1 we have

$$m_X\text{-Cl}(f^{-1}(B)) \subset m_X\text{-Cl}(f^{-1}((i,j)\text{-Cl}_\theta(B))) = f^{-1}((i,j)\text{-Cl}_\theta(B)).$$

Hence $m_X\text{-Cl}(f^{-1}(B)) \subset f^{-1}((i,j)\text{-Cl}_\theta(B))$. It follows from Theorem 3.2 that f is (i,j)-weakly m-continuous.

(3) $\Rightarrow$ (4): Let F be any (i,j)-θ-closed set of Y. Then by Theorem 3.2, $m_X\text{-Cl}(f^{-1}(F)) \subset f^{-1}((i,j)\text{-Cl}_\theta(F)) = f^{-1}(F)$. By Lemma 3.1, $f^{-1}(F) = m_X\text{-Cl}(f^{-1}(F))$.

(4) $\Rightarrow$ (5): Let V be any (i,j)-θ-open set of Y. By (4), $X - f^{-1}(V) = f^{-1}(Y - V) = m_X\text{-Cl}(f^{-1}(Y - V)) = X - m_X\text{-Int}((f^{-1}(V))$. Hence $f^{-1}(V) = m_X\text{-Int}(f^{-1}(V))$.

(5) $\Rightarrow$ (1): Since Y is (i,j)-regular, by Lemma 4.2 $(i,j)\text{-Cl}_\theta(B) = B$ for every σ_i-closed set B of Y and hence every σ_i-open set is (i,j)-θ-open. Therefore, $f^{-1}(V) = m_X\text{-Int}(f^{-1}(V))$ for every σ_i-open set V of Y. By Lemma 4.1, f is m-i-continuous.

REMARK 4.1. If $\sigma = \sigma_1 = \sigma_2$, then by Theorem 4.1, we obtain the results established in Theorem 4.1 of [27].

THEOREM 4.2. *Let (X, τ_1, τ_2) be a bitopological space with an m-structure m_{ij} and (Y, σ_1, σ_2) an (i,j)-regular bitopological space. Then, for a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is m-i-continuous;

(2) $f^{-1}((i,j)\text{-Cl}_\theta(B)) = m_{ij}\text{-Cl}(f^{-1}((i,j)\text{-Cl}_\theta(B)))$ for every subset B of Y;

(3) f is (i,j)-weakly m-continuous;

(4) $f^{-1}(F) = m_{ij}\text{-Cl}(f^{-1}(F))$ for every (i,j)-θ-closed set F of Y;

(5) $f^{-1}(V) = m_{ij}\text{-Int}(f^{-1}(V))$ for every (i,j)-θ-open set V of Y.

REMARK 4.2. If $m_{ij} = (i,j)\text{SO}(X)$ (resp. $(i,j)\text{PO}(X)$), then by Theorem 4.2 we obtain the results in Theorem 3.2 of [27] (resp. Theorem 3.3 of [19]).

DEFINITION 4.3. A function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is said to have *the (i, j)-m-interiority condition* if m_X-Int$(f(j\text{Cl}(f^{-1}(V)))) \subset f^{-1}(V)$ for every σ_i-open set V of Y.

THEOREM 4.3. *If a function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-weakly m-continuous and satisfies the (i, j)-m-interiority condition, then f is m-i-continuous.*

Proof. Let V be any σ_i-open set of Y. Since f is (i,j)-weakly m-continuous, by Theorem 3.1 $f^{-1}(V) \subset m_X$-Int$(f^{-1}(j\text{Cl}(V)))$. By the weak (i,j)-m-interiority condition of f, we have m_X-Int$(f(j\text{Cl}(f^{-1}(V)))) \subset f^{-1}(V)$. Therefore, $f^{-1}(V) \subset m_X$-Int$(f^{-1}(j\text{Cl}(V))) \subset m_X$-Int$(f^{-1}(V)) \subset f^{-1}(V)$. By Lemma 4.1, f is m-i-continuous.

DEFINITION 4.4. Let (X, τ_1, τ_2) be a bitopological space with an m-structure m_{ij}. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have *the (i, j)-m-interiority condition* if $f : (X, m_{ij}) \to (Y, \sigma_1, \sigma_2)$ has the (i,j)-m-interiority condition.

By Theorem 4.3 and Definition 4.4, we have the following theorem.

THEOREM 4.4. *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-weakly m-continuous and satisfies the (i, j)-m-interiority condition, then f is m-i-continuous.*

REMARK 4.3. If $m_{ij} = (i,j)\text{SO}(X)$ (resp. $(i,j)\text{PO}(X)$), then by Theorem 4.4 we obtain the results in Theorem 4.2 of [28] (resp. Theorem 4.2 of [19]).

5. Some properties of (i,j)-weak m-continuity

DEFINITION 5.1. A bitopological space (X, τ_1, τ_2) is said to be *pairwise Urysohn* [3] if for each distinct points x, y of X there exist a τ_i-open set U and a τ_j-open set V such that $x \in U, y \in V$ and $j\text{Cl}(U) \cap i\text{Cl}(V) = \emptyset$ for $i \neq j, i, j = 1, 2$.

DEFINITION 5.2. A bitopological space (X, τ_1, τ_2) with m-structures m_{ij} and m_{ji} is said to be *pairwise m-Hausdorff* or *pairwise m-T_2* if for each pair of distinct points x and y of X, there exist an m_{ij}-open set U containing x and an m_{ji}-open set V containing y such that $U \cap V = \emptyset$ for $i \neq j, i, j = 1, 2$.

REMARK 5.1. If $m_{ij} = \tau_i$ and $m_{ji} = \tau_j$, then we obtain the definition of pairwise T_2-space [11].

THEOREM 5.1. *Let (X, τ_1, τ_2) be a bitopological space with m-structures m_{ij} and m_{ji} and (Y, σ_1, σ_2) a pairwise Urysohn space. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise weakly m-continuous injection, then (X, τ_1, τ_2) is pairwise m-T_2.*

Proof. Let x and y be any distinct points of X. Then $f(x) \neq f(y)$. Since Y is pairwise Urysohn, there exist a σ_i-open set U and a σ_j-open set V such that $f(x) \in U, f(y) \in V$ and $j\mathrm{Cl}(U) \cap i\mathrm{Cl}(V) = \emptyset$. Hence $f^{-1}(j\mathrm{Cl}(U)) \cap f^{-1}(i\mathrm{Cl}(V)) = \emptyset$. Therefore, m_{ij}-$\mathrm{Int}(f^{-1}(j\mathrm{Cl}(U))) \cap m_{ji}$-$\mathrm{Int}(f^{-1}(i\mathrm{Cl}(V))) = \emptyset$. Since f is pairwise weakly m-continuous, by Theorem 3.4 $x \in f^{-1}(U) \subset m_{ij}$-$\mathrm{Int}(f^{-1}(j\mathrm{Cl}(U)))$ and $y \in f^{-1}(V) \subset m_{ji}$-$\mathrm{Int}(f^{-1}(i\mathrm{Cl}(V)))$. Hence, there exist $U_x \in m_{ij}$ and $V_y \in m_{ji}$ such that $x \in U_x \subset m_{ij}$-$\mathrm{Int}(f^{-1}(j\mathrm{Cl}(U)))$ and $y \in V_y \subset m_{ji}$-$\mathrm{Int}(f^{-1}(i\mathrm{Cl}(V)))$. Hence $U_x \cap V_y = \emptyset$. This implies that (X, τ_1, τ_2) is pairwise m-T_2.

REMARK 5.2. If $m_{ij} = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$), then by Theorem 5.1 we obtain the results in Theorem 2.5 of [12] (resp. Theorem 2.1 of [19]).

DEFINITION 5.3. A bitopological space (X, τ_1, τ_2) is said to be *pairwise connected* [22] if it cannot be expressed as the union of two nonempty disjoint sets U and V such that U is τ_i-open and V is τ_j-open, where $i, j = 1, 2$ and $i \neq j$.

DEFINITION 5.4. A bitopological space (X, τ_1, τ_2) with two m-structures m_{ij} and m_{ji} is said to be *pairwise m-connected* if it cannot be expressed as the union of two nonempty disjoint sets U and V such that U is m_{ij}-open and V is m_{ji}-open, where $i, j = 1, 2$ and $i \neq j$.

THEOREM 5.2. *Let (X, τ_1, τ_2) be a bitopological space with two m-structures m_{ij} and m_{ji} having property $\mathcal{B}$. If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise weakly m-continuous surjection and (X, τ_1, τ_2) is pairwise m-connected, then (Y, σ_1, σ_2) is pairwise connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not pairwise connected. Then, there exist a σ_i-open set U and a σ_j-open set V such that $U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset$ and $U \cup V = Y$. Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Moreover $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. Since f is pairwise weakly m-continuous, by Theorem 3.4 we have $f^{-1}(U) \subset m_{ij}$-$\mathrm{Int}(f^{-1}(j\mathrm{Cl}(U)))$ and $f^{-1}(V) \subset m_{ji}$-$\mathrm{Int}(f^{-1}(i\mathrm{Cl}(V)))$. Since U and V are σ_j-closed and σ_i-clsoed, respectively, we have $f^{-1}(U) \subset m_{ij}$-$\mathrm{Int}(f^{-1}(U))$ and $f^{-1}(V) \subset m_{ji}$-$\mathrm{Int}(f^{-1}(V))$. Hence by Lemma 3.1 $f^{-1}(U) = m_{ij}$-$\mathrm{Int}(f^{-1}(U))$ and $f^{-1}(V) = m_{ji}$-$\mathrm{Int}(f^{-1}(V))$. By Lemma 3.3, $f^{-1}(U)$ is m_{ij}-open and $f^{-1}(V)$ is m_{ji}-open in (X, τ_1, τ_2). This shows that (X, τ_1, τ_2) is not pairwise m-connected.

REMARK 5.3. If $m_{ij} = (i,j)\mathrm{SO}(X)$ and $m_{ji} = (j,i)\mathrm{SO}(X)$ (resp. $m_{ij} = (i,j)\mathrm{PO}(X)$ and $m_{ji} = (j,i)\mathrm{PO}(X)$), then by Theorem 5.2 we obtain the results in Theorem 2.4 of [12] (resp. Theorem 6.2 of [19]).

DEFINITION 5.5. An m-space (X, m_X) is said to be *m-compact* [27] if every cover of X by m_X-open sets has a finite subcover. A subset K of X is said to be *m-compact* if every cover of K by m_X-open sets has a finite subcover.

DEFINITION 5.6. Let (X, τ_1, τ_2) be a bitopological space with an m-structure m_{ij}. A subset K of X is said to be *m-compact* if K is m-compact in (X, m_{ij}).

DEFINITION 5.7. A subset K of a bitopological space (X, τ_1, τ_2) is said to be *(i, j)-quasi H-closed relative to X* [1] if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of K by τ_i-open sets of X, there exists a finite subset Δ_0 of Δ such that $K \subset \cup\{j\text{Cl}(U_\alpha) : \alpha \in \Delta_0\}$.

THEOREM 5.3. *If $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-weakly m-continuous and K is an m-compact set in (X, m_X), then f(K) is (i, j)-quasi H-closed relative to (Y, σ_1, σ_2).*

Proof. Let K be m-compact in X and $\{V_\alpha : \alpha \in \Delta\}$ any cover of $f(K)$ by σ_i-open sets of (Y, σ_1, σ_2). For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is (i, j)-weakly m-continuous, there exists $U_x \in m_X$ containing x such that $f(U_x) \subset j\text{Cl}(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of K by sets of m_X. Since K is m-compact, there exist a finite number of points, say $x_1, x_2, \ldots, x_n$ in K such that $K \subset \cup\{U_{x_k} : x_k \in K, k = 1, 2, \ldots, n\}$. Therefore, we obtain $f(K) \subset \cup\{f(U_{x_k}) : x_k \in K, k = 1, 2, \ldots, n\} \subset \cup\{j\text{Cl}((V_{\alpha(x_k)}) : x_k \in K, k = 1, 2, \ldots, n\}$. This shows that $f(K)$ is (i, j)-quasi H-closed relative to (Y, σ_1, σ_2).

REMARK 5.4. If $\sigma = \sigma_1 = \sigma_2$, then by Theorem 5.3, we obtain the result established in Theorem 5.5 of [27].

THEOREM 5.4. *Let (X, τ_1, τ_2) be a bitopological space with an m-structures m_{ij}. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-weakly m-continuous and K is an m-compact set of (X, τ_1, τ_2), then f(K) is (i, j)-quasi H-closed relative to (Y, σ_1, σ_2).*

Proof. This is an immediate consequence of Theorem 5.3.

DEFINITION 5.8. Let (X, m_X) be an m-space and A be a subset of X. The *m_X-frontier* of A, m_X-Fr(A), is defined as follows: $m_X\text{-Fr}(A) = m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) = m_X\text{-Cl}(A) - m_X\text{-Int}(A)$.

THEOREM 5.5. *Let (X, m_X) be an m-space and (Y, σ_1, σ_2) a bitopological space. The set of all points x of X at which a function $f : (X, m_X) \to (Y, \sigma_1, \sigma_2)$ is not (i, j)-weakly m-continuous is identical with the union of all m_X-frontiers of the inverse images of the σ_j-closure of σ_i-open sets of Y contining f(x).*

Proof. Let x be a point of X at which f is not (i, j)-weakly m-continuous. Then, there exists a σ_i-open set V of Y containing $f(x)$ such that $U \cap (X - f^{-1}(j\mathrm{Cl}(V))) \neq \emptyset$ for every m_X-open set U of X containing x. By Lemma 3.2, $x \in m_X$-$\mathrm{Cl}(X - f^{-1}(j\mathrm{Cl}(V)))$. Since $x \in f^{-1}(j\mathrm{Cl}(V))$, we have $x \in m_X$-$\mathrm{Cl}(f^{-1}(j\mathrm{Cl}(V)))$ and hence $x \in m_X$-$\mathrm{Fr}(f^{-1}(j\mathrm{Cl}(V)))$.

Conversely, if f is (i, j)-weakly m-continuous at x, then for each σ_i-open set V of Y containing $f(x)$, there exists an m_X-open set U containing x such that $f(U) \subset j\mathrm{Cl}(V)$ and hence $x \in U \subset f^{-1}(j\mathrm{Cl}(V))$. Therefore, we obtain that $x \in m_X$-$\mathrm{Int}(f^{-1}(j\mathrm{Cl}(V)))$. This contradicts that $x \in m_X$-$\mathrm{Fr}(f^{-1}(j\mathrm{Cl}(V)))$.

REMARK 5.5. If $\sigma = \sigma_1 = \sigma_2$, then by Theorem 5.5, we obtain the result established in Theorem 5.7 of [27].

THEOREM 5.6. *Let (X, τ_1, τ_2) be a bitopological space with an m-structure m_{ij}. The set of all points x of X at which a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not (i, j)-weakly m-continuous is identical with the union of all m_{ij}-frontiers of the inverse images of the σ_j-closure of σ_i-open sets of Y contining f(x).*

REMARK 5.6. If $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$), then by Theorem 5.6 we obtain the results in Theorem 4.3 of [28] (resp. Theorem 4.3 of [19]).

6. New forms of (i, j)-weakly continuous functions

There are many modifications of open sets in topological spaces. In order to define some new modifications of open sets in a bitopological space, let recall θ-open sets and δ-open sets due to Veličko [29]. Let (X, τ) be a topological space. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of a subset A of X if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the *θ-closure* (resp. *δ-closure*) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$). If $A = \mathrm{Cl}_\theta(A)$ (resp. $A = \mathrm{Cl}_\delta(A)$), then A is said to be *θ-closed* (resp. *δ-closed*) [29]. The complement of a θ-closed (resp. δ-closed) set is said to be *θ-open* (resp. *δ-open*). The union of all θ-open (resp. δ-open) sets contained in A is called the *θ-interior* (resp. *δ-interior*) of A and is denoted by $\mathrm{Int}_\theta(A)$ (resp. $\mathrm{Int}_\delta(A)$).

DEFINITION 6.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-δ-semi-open* [20] if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-δ-preopen* [21] if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-δ-semi-preopen* (simply *(i, j)-δ-sp-open*) if there exists an (i, j)-δ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

DEFINITION 6.2. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i,j)-θ-semi-open* if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i,j)-θ-preopen* if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i,j)-θ-semi-preopen* (simply *(i,j)-θ-sp-open*) if there exists an (i,j)-θ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

Let (X, τ_1, τ_2) be a bitopological space. The family of (i,j)-δ-semi-open (resp. (i,j)-δ-preopen, (i,j)-δ-*sp*-open, (i,j)-θ-semi-open, (i,j)-θ-preopen, (i,j)-θ-*sp*-open) sets of (X, τ_1, τ_2) is denoted by $(i,j)\delta\mathrm{SO}(X)$ (resp. $(i,j)\delta\mathrm{PO}(X)$, $(i,j)\delta\mathrm{SPO}(X)$, $(i,j)\theta\mathrm{SO}(X)$, $(i,j)\theta\mathrm{PO}(X)$, $(i,j)\theta\mathrm{SPO}(X)$).

REMARK 6.1. Let (X, τ_1, τ_2) be a bitopological space. The family $(i,j)\delta\mathrm{SO}(X)$, $(i,j)\delta\mathrm{PO}(X)$, $(i,j)\delta\mathrm{SPO}(X)$, $(i,j)\theta\mathrm{SO}(X)$, $(i,j)\theta\mathrm{PO}(X)$ and $(i,j)\theta\mathrm{SPO}(X)$ are all m-structures with property $\mathcal{B}$.

For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we can define many new types of (i,j)-weakly m-continuous functions. For example, in case $m_{ij} = (i,j)\delta\mathrm{SO}(X)$ (resp. $(i,j)\delta\mathrm{PO}(X)$, $(i,j)\delta\mathrm{SPO}(X)$, $(i,j)\theta\mathrm{SO}(X)$, $(i,j)\theta\mathrm{PO}(X)$, $(i,j)\theta\mathrm{SPO}(X)$) we can define new types of (i,j)-weakly m-continuous functions as follows:

DEFINITION 6.3. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i,j)-weakly δ-semi-continuous* (resp. *(i,j)-weakly δ-precontinuous*, *(i,j)-weakly δ-sp-continuous*) if $f : (X, m_{ij}) \to (Y, \sigma_1, \sigma_2)$ is (i,j)-weakly m-continuous, where $m_{ij} = (i,j)\delta\mathrm{SO}(X)$ (resp. $(i,j)\delta\mathrm{PO}(X)$, $(i,j)\delta\mathrm{SPO}(X)$).

DEFINITION 6.4. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i,j)-weakly θ-semi-continuous* (resp. *(i,j)-weakly θ-precontinuous*, *(i,j)-weakly θ-sp-continuous*) if $f : (X, m_{ij}) \to (Y, \sigma_1, \sigma_2)$ is (i,j)-weakly m-continuous, where $m_{ij} = (i,j)\theta\mathrm{SO}(X)$ (resp. $(i,j)\theta\mathrm{PO}(X)$, $(i,j)\theta\mathrm{SPO}(X)$).

Conclusion. We can apply the characterizations established in Section 3 and several properties obtained in Section 5 to the functions defined in Definition 2.7, Definitions 6.3 and 6.4 and also to functions defined by using any m-structure $m_{ij} = m(\tau_1, \tau_2)$ determined by τ_1 and τ_2 in a bitopological space (X, τ_1, τ_2).

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