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ON THREE-DIMENSIONAL LOCALLY ϕ-RECURRENT QUASI-SASAKIAN MANIFOLDS

Abstract. The object of the present paper is to study three-dimensional locally ϕ-recurrent quasi-Sasakian manifolds.

1. Introduction

On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Z. Olszak [4] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [5]. Next he has proved that if the manifold is additionally conformally flat with $\beta = constant$, then (a) the manifold is locally a product of R and a 2-dimensional Kahlerian space of constant Gauss curvature (the cosympletic case), or, (b) the manifold is of constant positive curvature (the non-cosympletic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure). An example of a three-dimensional quasi-Sasakian structure being conformally flat with non-constant structure function is also described in [5].

In 1977, T. Takahashi [6] introduced the notion of locally ϕ-symmetric Sasakian manifolds and studied their interesting properties. In [3] the notion of local ϕ-symmetry has been generalized as the notion of locally ϕ-recurrent. In the present paper we wish to apply the concept of locally ϕ-recurrent on three dimensional quasi-Sasakian manifolds. The present paper is organized as follows. Section 1 is the introductory section. Section 2 contains some basic and preliminary results related with three dimensional quasi-Sasakian manifolds. In Section 3 we investigate the nature of the characteristic vector field of the manifolds. The nature of the curvature tensor of a three-dimensional locally ϕ-recurrent quasi-Sasakian manifold have been studied

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in Section 6. In the last section we have constructed an example to illustrate the results obtained in Section 4.

2. Preliminaries

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact structure (ϕ, ξ, η, g), where ϕ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that [1], [2]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in T(M) \tag{2.2}$$

Then also

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi). \tag{2.3}$$

Let Φ be a fundamental 2-form defined by

$$\Phi(X, Y) = g(X, \phi Y), \quad X, Y \in T(M).$$

M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η, g) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$), which was first introduced by Blair [2]. The normality condition gives that the induced almost contact structure on $M \times R$ is integrable or equivalently, the torsion tensor field $N[\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M. The rank of a quasi Sasakian structure is always odd [2], it is equal to 1 if the structure is cosymplectic and it is equal to $(2n+1)$ if the structure is Sasakian.

An almost contact medtric manifold M of dimension three is quasi-Sasakian if and only if [4]

$$\nabla_X \xi = -\beta\phi X, \quad X \in T(M), \tag{2.4}$$

for a certain function β on M such that $\xi\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of M. Clearly such a quasi-Sasakian manifold is cosympletic if and only if $\beta = 0$. As a consequence of (2.4) we have [4]

$$(\nabla_X \phi)(Y) = \beta(g(X, Y)\xi - \eta(Y)X), \quad X, Y \in T(M), \tag{2.5}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y). \tag{2.6}$$

Let M be a three-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by [5]

$$S(Y, Z) = \left(\frac{r}{2} - \beta^2\right) g(Y, Z) + \left(3\beta^2 - \frac{r}{2}\right)\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y), \tag{2.7}$$

where r is the scalar curvature of M.

In a three-dimensional Riemannian manifold we always have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y) \tag{2.8}$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. Now as a consequence of (2.7), we get for the Ricci operator Q

$$QY = \left(\frac{r}{2} - \beta^2\right)Y + \left(3\beta^2 - \frac{r}{2}\right)\eta(Y)\xi + \eta(Y)(\phi grad\beta) - d\beta(\phi Y)\xi, \tag{2.9}$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X) = g(gradf, X)$. From (2.7) we have

$$S(Y,\xi) = 2\beta^2\eta(Y) - d\beta(\phi Y). \tag{2.10}$$

Moreover, as a consequence of (2.8) – (2.10) we find

$$R(X,Y)\xi = \beta^2(\eta(Y)X - \eta(X)Y) + d\beta(\phi X)Y - d\beta(\phi Y)X, \tag{2.11}$$

for $X, Y \in T(M)$. From (2.8)

$$\begin{aligned}\eta(R(X,Y)Z) &= g(Y,Z)\eta(QX) - g(X,Z)\eta(QY) \\ &+S(Y,Z)\eta(X) - S(X,Z)\eta(Y) \\ &-\frac{r}{2}(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)).\end{aligned}$$

For X, Y, Z orthogonal to ξ we obtain from above

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(QX) - g(X,Z)\eta(QY). \tag{2.12}$$

3. Nature of the characteristic vector field of locally ϕ-recurrent three-dimensional quasi-Sasakian manifolds

DEFINITION 3.1. A quasi-Sasakian manifold is said to be locally ϕ-recurrent if there exists a non zero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z, \tag{3.1}$$

for the vector fields X, Y, Z orthogonal to ξ.
If the 1-form vanishes, then the manifold reduces to a locally ϕ-symmetric manifold. From (3.1) and (2.1) we obtain

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z. \tag{3.2}$$

From (3.2) and second Bianchi identity we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0. \tag{3.3}$$

By virtue of (2.12) we obtain from (3.3)

$$A(W)[g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)] \tag{3.4}$$

$$+ A(X)[g(W,Z)\eta(QY) - g(Y,Z)\eta(QW)] + A(Y)[g(X,Z)\eta(QW) - g(W,Z)\eta(QX)] = 0.$$

Putting $Y = Z = e_i$ in (4.3) and taking summation over i, where $i = 1,2,3$ and $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold M, we obtain

$$A(W)[3\eta(QX) - g(X,e_i)\eta(Qe_i)] + A(X)[g(W,e_i)\eta(Qe_i) - 3\eta(QW)] + A(e_i)[g(X,e_i)\eta(QW) - g(W,e_i)\eta(QX)] = 0. \tag{3.5}$$

Now the Ricci operator Q is symmetric. So

$$g(X,e_i)\eta(Qe_i) = \eta(QX). \tag{3.6}$$

Similarly

$$g(W,e_i)\eta(Qe_i) = \eta(QW). \tag{3.7}$$

Again

$$A(e_i)g(X,e_i)\eta(QW) = A(X)\eta(QW). \tag{3.8}$$

Similarly,

$$A(e_i)g(W,e_i)\eta(QX) = A(W)\eta(QX). \tag{3.9}$$

Using (3.6)-(3.9) in (3.5) we obtain,

$$A(W)[3\eta(QX) - \eta(QY)] + A(X)[\eta(QW) - 3\eta(QW)] + A(X)[\eta(QW) - A(W)\eta(QX)] = 0, \tag{3.10}$$

or,

$$A(X)\eta(QW) - A(W)\eta(QX) = 0.$$

Putting $X = \xi$ we get from the above

$$A(\xi)S(W,\xi) - A(W)S(\xi,\xi) = 0.$$

Using (2.10) we have from the above

$$A(\xi)S(W,\xi) - 2\beta^2 A(W) = 0.$$

Again using (2.10) we see that

$$A(\xi)[2\beta^2\eta(W) - d\beta(\phi W)] = 2\beta^2 A(W),$$

or,

$$-A(\xi)d\beta(\phi W) = 2\beta^2 A(W),$$

where W is orthogonal to ξ. Putting $W = \xi$ from above we obtain, $2\beta^2 A(\xi) = 0$ which implies that $g(\xi,\rho) = 0$.

Thus we can state the following theorem:

THEOREM 3.1. *In a locally ϕ-recurrent quasi-Sasakian manifold of dimension three the characteristic vector field ξ and the vector field ρ associated to the 1-form A are orthogonal.*

4. Nature of the curvature tensor in three-dimensional locally ϕ-recurrent quasi-Sasakian manifold

From (2.8) we have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y).$$

Putting $Z = \xi$ and using (2.10) we have

$$R(X,Y)Z = \eta(Y)QX - \eta(X)QY + 2\beta^2\eta(Y)X - d\beta(\phi Y)X - 2\beta^2\eta(X)Y + d\beta(\phi X)Y - \frac{r}{2}(\eta(Y)X - \eta(X)Y),$$

or,

$$R(X,Y)\xi = \eta(Y)QX - \eta(X)QY + \left(2\beta^2 - \frac{r}{2}\right)(\eta(Y)X - \eta(X)Y) + d\beta(\phi X)Y - d\beta(\phi Y)X.$$

Using (2.11) we have from above

$$\beta^2(\eta(Y)X - \eta(X)Y) + d\beta(\phi X)Y - d\beta(\phi Y)X = \eta(Y)QX - \eta(X)QY + (2\beta^2 - \frac{r}{2})(\eta(Y)X - \eta(X)Y) + d\beta(\phi X)Y - d\beta(\phi Y)X \tag{4.1}$$

or,

$$\beta^2(\eta(Y)X - \eta(X)Y) = \eta(Y)QX - \eta(X)QY + \left(2\beta^2 - \frac{r}{2}\right)(\eta(Y)X - \eta(X)Y). \tag{4.2}$$

The formula (4.2) yields

$$\left(\beta^2 - \frac{r}{2}\right)(\eta(Y)X - \eta(X)Y) = \eta(X)QY - \eta(Y)QX. \tag{4.3}$$

Putting $Y = \xi$ in (4.3) we have

$$QX = \left(\frac{r}{2} - \beta^2\right)(X - \eta(X)\xi) + \eta(X)Q\xi. \tag{4.4}$$

Now from (2.10), $S(\xi,\xi) = 2\beta^2$. Hence $g(Q\xi,\xi) = 2\beta^2 g(\xi,\xi)$. So we have

$Q\xi = 2\beta^2\xi$. From (4.4) and (4.5), we have

$$QX = \left(\frac{r}{2} - \beta^2\right)(X - \eta(X)\xi) + 2\beta^2\eta(X)\xi \tag{4.6}$$
$$= \left(\frac{r}{2} - \beta^2\right)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi.$$

Thus, in view of (4.6) we note that

$$S(X,Y) = (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y). \tag{4.7}$$

From (2.8), (4.6), and (4.7) we obtain

$$R(X,Y)Z = \left(\frac{r}{2} - 2\beta^2\right)[g(Y,Z)X - g(X,Z)Y] \tag{4.8}$$
$$+ \left(3\beta^2 - \frac{r}{2}\right)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi$$
$$+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].$$

Differentiating (4.8) covariently with respect to W we get

$$(\nabla_W R)(X,Y)Z = \left(\frac{dr(W)}{2} - 4\beta d\beta(W)\right)[g(Y,Z)X - g(X,Z)Y]$$
$$+ \left(6\beta d\beta(W) - \frac{dr(W)}{2}\right)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi$$
$$+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]$$
$$+ \left(3\beta^2 - \frac{r}{2}\right)[g(Y,Z)(\nabla_W\eta)(X)\xi + g(Y,Z)\eta(X)(\nabla_W\xi)$$
$$- g(X,Z)(\nabla_W\eta)(Y)\xi - g(X,Z)\eta(Y)(\nabla_W\xi)$$
$$+ \eta(Y)(\nabla_W\eta)(Z)X + (\nabla_W\eta)(Y)\eta(Z)X$$
$$- (\nabla_W\eta)(X)\eta(Z)Y - \eta(X)(\nabla_W\eta)(Z)Y].$$

Here we take X, Y, Z, W orthogonal to ξ. Now we obtain from the above

$$(\nabla_W R)(X,Y)Z = \left(\frac{dr(W)}{2} - 4\beta d\beta(W)\right)[g(Y,Z)X - g(X,Z)Y] \tag{4.9}$$
$$+ \left(3\beta^2 - \tfrac{r}{2}\right)[g(Y,Z)(\nabla_W\eta)(X)\xi$$
$$- g(X,Z)(\nabla_W\eta)(Y)\xi].$$

Applying ϕ^2 on both sides of (4.9) we have

$$\phi^2(\nabla_W R)(X,Y)Z = \left(\frac{dr(W)}{2} - 4\beta d\beta(W)\right)[g(X,Z)Y - g(Y,Z)X].$$

Assuming the manifold as locally ϕ-recurrent we get from above

$$(4.10)\quad A(W)R(X,Y)Z = \left(\frac{dr(W)}{2} - 4\beta d\beta(W)\right)[g(X,Z)Y - g(Y,Z)X].$$

Putting $W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i, i = 1,2,3$, we have

$$R(X,Y)Z = \lambda[g(X,Z)Y - g(Y,Z)X],$$

where $\lambda = \frac{(\frac{1}{2}dr(e_i) - 4\beta d\beta(e_i))}{A(e_i)}$.

Now β is a scalar function and A is a non zero 1-form. Hence λ is a constant by Schurs' theorem. Hence we conclude the following:

THEOREM 4.1. *A three dimensional locally ϕ-recurrent quasi-Sasakian manifold is of constant curvature.*

5. Example

In this section we give an example of locally ϕ-recurrent quasi-Sasakian manifold of dimension three and which is of constant curvature. We take the 3-dimensional manifold $M = \{(x,y,z) \in \mathbf{R}^3 : x \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbf{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M given by

$$E_1 = \frac{2}{x}\frac{\partial}{\partial y},\quad E_2 = 2\frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z},\quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_3) = g(E_2,E_3) = g(E_1,E_2) = 0,$$
$$g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1.$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$.Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$ for any $U, W \in \chi(M)$.

Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a contact metric structure on M.

Hence we have $[E_1, E_2] = 2E_3 + \frac{2}{x}E_1$, $[E_1, E_3] = 0$, $[E_2, E_3] = 2E_1$.

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z])$$
$$- g(Y,[X,Z]) + g(Z,[X,Y]).$$

Taking $E_3 = \xi$ and using the above formula for Riemannian metric g, we can show that

$$\nabla_{E_1}E_3 = -2E_2,\quad \nabla_{E_2}E_3 = 2E_1,\quad \nabla_{E_3}E_3 = 0,\quad \nabla_{E_3}E_1 = 2E_2,$$

$$\nabla_{E_1}E_2 = \tfrac{2}{x}E_1,\quad \nabla_{E_2}E_1 = 0,\quad \nabla_{E_2}E_2 = 0,\quad \nabla_{E_3}E_2 = 0,\quad \nabla_{E_1}E_1 = -2E_2.$$

From the above it follows that the manifold under consideration is a quasi-Sasakian manifold of dimension three. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_2, E_1)E_1 = -\frac{4}{x}E_2, \qquad R(E_1, E_2)E_2 = -\frac{4}{x}E_1, \tag{5.2}$$

and the components which can be obtained from these by symmetric properties. Since $\{E_1, E_2, E_3\}$ form a basis of M^3, any vector field W can be taken as

$$W = a_1E_1 + a_2E_2 + a_3E_3$$

where $a_i \in R^+(\textit{the set of all positive real numbers}), i = 1, 2, 3$. Thus the covariant derivatives of the components of the curvature tensor are given by

$$(\nabla_W R)(E_2, E_1)E_1 = -8\frac{a_1}{x^2}E_1, \qquad (\nabla_W R)(E_1, E_2)E_2 = -8\frac{a_1}{x^2}E_2.$$

Now from the properties of g, ϕ, and $R(X, Y)Z$ it follows that the manifold satisfies

$$\phi^2(\nabla_W R)(X, Y)Z = A(W)R(X, Y)Z, \tag{5.3}$$

for the non-vanishing 1-form $A(W) = \frac{2a_1}{x}$. In view of (5.2) and (5.3) we conclude that the manifold under consideration is locally ϕ-recurrent and is of constant curvature.

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