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SPECTRA OF THE OPERATOR OF THE FIRST DIFFERENCE IN s_α, s_α^0, $s_\alpha^{(c)}$ AND $l_p(\alpha)$ $(1 \leq p < \infty)$ AND APPLICATION TO MATRIX TRANSFORMATIONS

Abstract. In this paper we deal with the spectrum of the operator of the first difference Δ considered as an operator from E to itself where E is one of the sets s_α, s_α^0, or $s_\alpha^{(c)}$, or $l_p(\alpha)$ $(1 \leq p < \infty)$. We apply these results to characterize matrix transformations mapping in $E((\Delta - \lambda I)^\chi)$ where E is either of the sets s_α^0, or $l_p(r)$, for $1 \leq p < \infty$ and $\chi \in \mathbb{C}$, or $\mathbb{N}$. This paper generalizes some results given in [8] and [3].

1. Introduction

In this paper we are interested in the study of the spectra of the operator of the first difference Δ. In [8] it was shown that the spectrum of Δ considered as an operator mapping from s_r to itself is equal to $\overline{D(1, 1/R)}$. Altay and Başar studied [2] the fine spectra of the difference operator Δ on c_0 and c. Then in [3] Akhmedov and Başar dealt with the fine spectra Δ over the sequence space l_p $(1 \leq p < \infty)$. In [18, 19], Rhoades dealt with the spectra of the weighted mean operator $\overline{N}_q$ in $\mathcal{B}(l_p)$ and in bv_0. In de Malafosse [12] it was shown that under some conditions the spectrum of $\overline{N}_q$ considered as operator from s_α to itself and from s_α^0 to itself is equal to $\{0\} \cup \{q_n/Q_n \ : \ n \geq 1\}$, where $Q_n = \sum_{m=1}^{n} q_m$. Many results were gathered by Malkowsky and Rakočević in [16], among other things they have provided characterizations of operators mapping from $\chi(\Delta^m)$ to χ', where m is an integer, χ and χ' are either of the sets c_0, c, or l_∞. Characterizations of the set $\left(l_\infty\left(\left(\Delta^t\right)^m\right), l_\infty\right)$ for any given integer $m \geq 1$ were given by Kizmaz [6], Çolak and Et [4]. We also have in [8], necessary and sufficient conditions for a matrix map to belong to $(s_r((\Delta - \lambda I)^\mu), s_r)$, for any given complex numbers μ and λ.

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Here we deal with the spectrum of Δ considered as an operator from E to itself where E is one of the sets s_α, s_α^0, $s_\alpha^{(c)}$, or $l_p(\alpha)$ $(1 \leq p < \infty)$. Then we apply these results to matrix transformations mapping in $s_\alpha^0\left((\Delta - \lambda I)^h\right)$ and mapping in $l_p(r)\left((\Delta - \lambda I)^h\right)$ where h is a complex number.

This paper is organized as follows. In Section 2, we recall some definitions and results on *sequence spaces and matrix transformations.* Then in Section 3 we study the spectrum of Δ considered as an operator from E to itself for $E = s_\alpha$, s_α^0, $s_\alpha^{(c)}$, or $l_p(\alpha)$. Finally in Section 4 we characterize matrix transformations between $s_\alpha^0\left((\Delta - \lambda I)^h\right)$ and F and between $l_p(r)\left((\Delta - \lambda I)^h\right)$ and F, where F is either of the sets s_γ, s_γ^0, $s_\gamma^{(c)}$, $1 < p < \infty$ and h is a complex number.

2. Notations and preliminary results

For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$ we define the *operators* A_n for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m \tag{1}$$

where $X = (x_m)_{m\geq 1}$, and the series are assumed to be convergent for all n. So we are led to the study of the operator A defined by $AX = (A_n(X))_{n\geq 1}$ mapping a sequence space into another sequence space. Throughout this paper we will consider X and AX as column vectors.

A *BK space* E *is a Banach sequence space with continuous coordinates* $P_n : E \to \mathbb{C}$ where $P_n(X) = x_n$ for all $X \in E$ and $n = 0, 1, \ldots$. A *BK space* E is said to have AK if every sequence $X = (x_n)_{n\geq 1} \in E$ *has a unique representation* $X = \sum_{m=1}^{\infty} x_m e^{(m)}$ where $e^{(m)}$ denotes the sequence with $e_m^{(m)} = 1$ and $e_j^{(m)} = 0$ for $j \neq m$.

We will denote by s, c_0, c, l_∞ the sets of all sequences, the set of sequences that converge to zero, convergent and bounded respectively. Then for given sequence $a \in s$ we define the infinite diagonal matrix D_a by $[D_a]_{nn} = a_n$ for all n. We will use the set U^+ of all sequences $(u_n)_{n\geq 1} \in s$ such that $u_n > 0$ for all n. Using Wilansky's notations [20], we define for any sequence $\alpha = (\alpha_n)_{n\geq 1} \in U^+$ and for any set of sequences E, the set

$$(1/\alpha)^{-1} * E = \left\{(x_n)_{n\geq 1} \in s \ : \ (x_n/\alpha_n)_{n\geq 1} \in E\right\}.$$

Throughout this paper we will write $D_\alpha E = (1/\alpha)^{-1} * E$ and put $s_\alpha = D_\alpha l_\infty$, $s_\alpha^0 = D_\alpha c_0$ and $s_\alpha^{(c)} = D_\alpha c$, for $\alpha = (\alpha_n)_{n\geq 1} \in U^+$, see [9]. Each of the spaces $D_\alpha E$, where $E \in \{l_\infty, c_0, c\}$, is a BK space normed by $\|X\|_{s_\alpha} = \sup_{n\geq 1}(|x_n|/\alpha_n)$ and s_α^0 has AK, see [13].

Now let $\alpha = (\alpha_n)_{n\geq 1}$, $\mu = (\mu_n)_{n\geq 1} \in U^+$. By $S_{\alpha,\mu}$ we denote the set of infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that

$$\|A\|_{S_{\alpha,\mu}} = \sup_{n\geq 1}\left(\frac{1}{\mu_n}\sum_{m=1}^{\infty}|a_{nm}|\,\alpha_m\right) < \infty.$$

The set $S_{\alpha,\mu}$ *is a Banach space* with the norm $\|A\|_{S_{\alpha,\mu}}$. Let E and F be any subsets of s. When A maps E into F we shall write $A \in (E, F)$, see [7]. So for every $X \in E$, $AX \in F$, ($AX \in F$ will mean that for each $n \geq 1$ the series defined by $A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m$ is convergent and $(A_n(X))_{n\geq 1} \in F$). For any subset E of s, we put

$$AE = \{Y \in s \ : Y = AX \text{ for some } X \in E\}.$$

If F is a subset of s, we shall denote

$$F(A) = F_A = \{X \in s \ : Y = AX \in F\}.$$

In [14] it was shown that $A \in (s_\alpha, s_\mu)$ if and only if $A \in S_{\alpha,\mu}$. So we can write that $(s_\alpha, s_\mu) = S_{\alpha,\mu}$. This result comes from the next elementary lemma we will use throughout this paper.

LEMMA 1. *Let $\alpha, \mu \in U^+$ and let E, $F \subset s$. Then we have $A \in (D_\alpha E, D_\mu F)$ if and only if $D_{1/\mu}AD_\alpha \in (E, F)$.*

When $s_\alpha = s_\mu$ we obtain the *Banach algebra with identity* $S_{\alpha,\mu} = S_\alpha$, (cf. [14]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$. We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$.

For any BK space E we denote by $\mathcal{B}(E)$ the *Banach algebra of all bounded linear operators that map E to itself.* In this paper we will use the set $l_p(\alpha) = (1/\alpha)^{-1} * l_p = D_\alpha l_p$ for $p \geq 1$. It can easily be seen that

$$l_p(\alpha) = \left\{X : \|X\|_{l_p(\alpha)}^p = \sum_{n=1}^{\infty}\left(\frac{|x_n|}{\alpha_n}\right)^p < \infty\right\}.$$

The set $l_p(\alpha) = l_p(D_{1/\alpha})$ *is a BK space normed by* $\|X\|_{l_p(\alpha)}$ and has AK. We deduce that the set $\mathcal{B}(l_p(\alpha))$ of all bounded operators mapping $l_p(\alpha)$ to itself is a *Banach algebra* (cf. [13]). Since $l_p(\alpha)$ *has AK,* by [13, Lemma 4, pp. 44] we have

$$\mathcal{B}(l_p(\alpha)) = (l_p(\alpha), l_p(\alpha)).$$

If $\alpha = (r^n)_{n\geq 1}$, the matrix D_α and the sets S_α, s_α, s_α^0, $s_\alpha^{(c)}$ and $l_p(\alpha)$ are denoted by D_r, S_r, s_r, $\overset{0}{s_r}$, $s_r^{(c)}$ and $l_p(r)$ respectively (see [8, 9]). When $r = 1$, we obtain $s_1 = l_\infty$, $\overset{0}{s_1} = c_0$ and $s_1^{(c)} = c$, and putting $e = (1, 1, \dots)$ we have $S_1 = S_e$. It is well known, see [7] that $(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1$. In the following we will use the next result.

LEMMA 2. *i)* $A \in (c_0, c_0)$ *if and only if* $A \in S_1$ *and*

$$\lim_{n\to\infty} a_{nm} = 0 \text{ for all } m \geq 1.$$

ii) $A \in (c_0, c)$ *if and only if* $A \in S_1$ *and*

$$\lim_{n\to\infty} a_{nm} = l_m \text{ for some } l_m \in \mathbb{C} \text{ and for all } m \geq 1. \tag{2}$$

iii) $A \in (c, c)$ *if and only if* $A \in S_1$, *(2) holds and*

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{nm} = l \text{ for some } l \in \mathbb{C}.$$

3. The spectra of the operator Δ in the sets s_α, s_α^0, $s_\alpha^{(c)}$, or $l_p(\alpha)$ $(1 \leq p < \infty)$

In this section we study the spectrum of the operator of the first difference mapping in s_α and we explictly calculate the spectra $\sigma(\Delta, s_R)$, $\sigma(\Delta, s_R^0)$, $\sigma\left(\Delta, s_R^{(c)}\right)$ and $\sigma\left(\Delta, s_{(n)_n}\right)$. Then we deal with the sets $\sigma(\Delta, l_1(\alpha))$, $\sigma(\Delta^+, l_1(\alpha))$, $\sigma(\Delta, l_p(r))$ and $\sigma(\Delta^+, l_p(r))$ for $1 \leq p < \infty$.

3.1. On the spectrum of Δ considered as operator in s_α, s_α^0, or $s_\alpha^{(c)}$

Recall that B. Altay and F. Başar [2] dealt with the fine spectra of the operator Δ considered as operator in c_0, c and l_∞ respectively. Let E be a set of sequences and A be an operator mapping E to itself. We denote by $\sigma(A, E)$ the set of all complex numbers λ such that $\lambda I - A$ considered as an operator from E to itself is not invertible. We have the next result where we use the notation $\overline{D(\lambda_0, r)} = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\}$ for $\lambda_0 \in \mathbb{C}$ and $r > 0$.

THEOREM 3. *[2, Theorems 2.1-2.12], [3, Theorem 2.8]. Let* $1 \leq p \leq \infty$

$$\sigma(\Delta, c_0) = \sigma(\Delta, c) = \sigma(\Delta, l_p) = \overline{D(1, 1)}.$$

To state the next results we need to recall some properties of the sequence $C(\alpha)\alpha$.

3.1.1. Properties of the sequence $C(\alpha)\alpha$

Here we shall deal with the operators represented by $C(\xi)$ and $\Delta(\xi)$, see for instance [8]. The infinite matrix $T = (t_{nm})_{n,m\geq 1}$ is said to be a triangle if $t_{nm} = 0$ for $m > n$ and $t_{nn} \neq 0$ for all n. Now let U be the set of all sequences $(u_n)_{n\geq 1} \in s$ with $u_n \neq 0$ for all n. The infinite matrix $C(\xi) = (c_{nm})_{n,m\geq 1}$ for $\xi = (\xi_n)_{n\geq 1} \in U$, is defined by

$$c_{nm} = \begin{cases} \dfrac{1}{\xi_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It can be proved that the matrix $\Delta(\xi) = (c'_{nm})_{n,m\geq 1}$ with

$$c'_{nm} = \begin{cases} \xi_n & \text{if } m = n, \\ -\xi_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

is the inverse of $C(\xi)$, that is $C(\xi)(\Delta(\xi)X) = \Delta(\xi)(C(\xi)X)$ for all $X \in s$. If $\xi = e$ we get the well-known operator of the first difference represented by $\Delta(e) = \Delta$ and usually denoted by $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and Δ and Σ belong to any given space S_R with $R > 1$.

Consider the following sets

$$\widehat{C_1} = \left\{\alpha \in U^+ : C(\alpha)\alpha = \left(\frac{1}{\alpha_n}\left(\sum_{m=1}^{n} \alpha_m\right)\right)_{n\geq 1} \in s_1 = l_\infty\right\},$$

and

$$\Gamma = \left\{\alpha \in U^+ : \limsup_{n\to\infty}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\}.$$

From [9, Proposition 2.1, pp. 1786] we get

LEMMA 4. *Let $\alpha \in U^+$. Then*

i) $\Gamma \subset \widehat{C_1}$,

ii) If $\alpha \in \widehat{C_1}$ there are $K > 0$ and $\gamma > 1$ such that

$$\alpha_n \geq K\gamma^n \text{ for all } n.$$

3.1.2. On the spectrum of Δ considered as an operator from E to itself where $E = s_\alpha$, s_α^0, or $s_\alpha^{(c)}$

Now we can state the following result where $\rho(A,E) = \sigma(A,E)^c$ is the resolvent set of $A \in (E,E)$.

THEOREM 5. *Let $\alpha \in U^+$ and assume $\sup_n(\alpha_{n-1}/\alpha_n) < \infty$. Then*

i) $\sigma(\Delta, s_\alpha) = \sigma(\Delta, s_\alpha^0)$ *and* $\lambda \in \sigma(\Delta, s_\alpha)$ *if and only if*

$$\lambda = 1 \text{ or } (|\lambda - 1|^n \alpha_n)_{n\geq 1} \notin \widehat{C}_1.$$

ii) a) We have

$$\sigma(\Delta, s_\alpha) = \sigma(\Delta, s_\alpha^0) \subset \overline{D\left(1, \limsup_{n\to\infty}\frac{\alpha_{n-1}}{\alpha_n}\right)}, \tag{3}$$

and the inclusion is strict.

b) If $(\alpha_{n-1}/\alpha_n)_{n\geq 2} \in c$ *then*

$$\sigma\left(\Delta, s_\alpha^{(c)}\right) \subset \overline{D\left(1, \lim_{n\to\infty} \frac{\alpha_{n-1}}{\alpha_n}\right)}.$$

iii) For any $R > 0$ *we have*

$$\sigma\left(\Delta, s_R\right) = \sigma\left(\Delta, s_R^0\right) = \sigma\left(\Delta, s_R^{(c)}\right) = \overline{D\left(1, 1/R\right)}.$$

Proof. i) First show $\lambda \in \sigma(\Delta, s_\alpha)$ if and only if $\lambda = 1$ or $(|\lambda - 1|^n \alpha_n)_{n\geq 1} \notin \widehat{C}_1$. Let $\lambda \in \rho(\Delta, s_\alpha)$. Then $\Delta - \lambda I : s_\alpha \to s_\alpha$ is invertible and $(\Delta - \lambda I)^{-1} \in (s_\alpha, s_\alpha)$. First by Lemma 1 we have $\Delta - \lambda I \in (s_\alpha, s_\alpha)$ if and only if

$$\Delta'_\alpha = D_{1/\alpha}(\lambda I - \Delta) D_\alpha \in S_1,$$

that is for any sequence α satisfying $\sup_n (\alpha_{n-1}/\alpha_n) < \infty$. For $\lambda \neq 1$ we get $(\lambda I - \Delta)^{-1} = (\xi_{nm})_{n,m\geq 1}$, where

$$\xi_{nm} = \begin{cases} \dfrac{(-1)^{n-m}}{(\lambda - 1)^{n-m+1}} & \text{if} \quad m \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Since $S_\alpha = (s_\alpha, s_\alpha)$ the condition $(\lambda I - \Delta)^{-1} \in S_\alpha$ is then equivalent to $\lambda \neq 1$ and

$$\chi = \sup_n \left(\frac{\sum_{m=1}^{n} |\lambda - 1|^m \alpha_m}{|\lambda - 1|^n \alpha_n}\right) < \infty, \tag{5}$$

that is $(|\lambda - 1|^n \alpha_n)_{n\geq 1} \in \widehat{C}_1$. We conclude that $\lambda \in \sigma(\Delta, s_\alpha)$ if and only if $\lambda = 1$ or $(|\lambda - 1|^n \alpha_n)_{n\geq 1} \notin \widehat{C}_1$.

Now show $\sigma(\Delta, s_\alpha) = \sigma\left(\Delta, s_\alpha^0\right)$. First show $\sigma(\Delta, s_\alpha) \subset \sigma\left(\Delta, s_\alpha^0\right)$. For this take $\lambda \in \rho\left(\Delta, s_\alpha^0\right)$. Then $\lambda I - \Delta$ considered as operator from s_α^0 to itself is invertible and $(\lambda I - \Delta)^{-1} \in \left(s_\alpha^0, s_\alpha^0\right)$. Since

$$\left(s_\alpha^0, s_\alpha^0\right) \subset \left(s_\alpha^0, s_\alpha\right) = (s_\alpha, s_\alpha)$$

we deduce $\lambda I - \Delta$ maps s_α to itself, $(\lambda I - \Delta)^{-1} \in (s_\alpha, s_\alpha)$ and $\lambda \in \rho(\Delta, s_\alpha)$. We conclude $\rho\left(\Delta, s_\alpha^0\right) \subset \rho(\Delta, s_\alpha)$ and $\sigma(\Delta, s_\alpha) \subset \sigma\left(\Delta, s_\alpha^0\right)$. Now show $\sigma\left(\Delta, s_\alpha^0\right) \subset \sigma(\Delta, s_\alpha)$. For this take $\lambda \in \rho(\Delta, s_\alpha)$. Then $\lambda I - \Delta$ considered as operator from s_α to itself is invertible and $(\lambda I - \Delta)^{-1} \in (s_\alpha, s_\alpha)$. From the characterization of $\left(s_\alpha^0, s_\alpha^0\right)$ we only need to show that

$$\left|\left[\Delta'_\alpha\right]_{nm}\right| \to 0 \ (n \to \infty) \text{ for all } m.$$

As we have seen above $(\lambda I - \Delta)^{-1} \in S_\alpha$ implies $(|\lambda - 1|^n \alpha_n)_{n\geq 1} \in \widehat{C}_1$ and by Lemma 4 ii) $|\lambda - 1|^n \alpha_n \to \infty$ $(n \to \infty)$ and so

$$\left|[\Delta'_\alpha]_{nm}\right| = \left|\frac{\alpha_m}{(\lambda - 1)^{n-m+1}\alpha_n}\right|$$
$$= |(\lambda - 1)^m| \alpha_m \frac{1}{|(\lambda - 1)^n| \alpha_n} = 0(1) \quad (n \to \infty)$$

for each $m = 1, 2, \dots$. We deduce $\lambda \in \rho\left(\Delta, s_\alpha^0\right)$. So we have shown $\sigma\left(\Delta, s_\alpha^0\right) \subset \sigma\left(\Delta, s_\alpha\right)$ and since $\sigma\left(\Delta, s_\alpha\right) \subset \sigma\left(\Delta, s_\alpha^0\right)$ we conclude $\sigma\left(\Delta, s_\alpha\right) = \sigma\left(\Delta, s_\alpha^0\right)$.

ii) a) Let $\lambda \in \sigma\left(\Delta, s_\alpha\right)$. By Lemma 4 i) we easily deduce that $\left(|\lambda - 1|^n \alpha_n\right)_{n \geq 1} \notin \widehat{C}_1$ implies $\left(|\lambda - 1|^n \alpha_n\right)_{n \geq 1} \notin \Gamma$ and

$$\frac{1}{|\lambda - 1|} \limsup_{n \to \infty} \left(\frac{\alpha_{n-1}}{\alpha_n}\right) \geq 1$$

that is $|\lambda - 1| \leq \limsup_{n \to \infty}\left(\alpha_{n-1}/\alpha_n\right)$. This shows that (3) holds. Furthermore since $\widehat{C}_1 \neq \Gamma$, by [10, Remark 6, p. 255] the inclusion is strict and we have shown ii) a).

ii) b) Let λ be such that $|\lambda - 1| > \lim_{n \to \infty}\left(\alpha_{n-1}/\alpha_n\right) = \limsup_{n \to \infty}\left(\alpha_{n-1}/\alpha_n\right)$. Then there is an integer q_0 such that

$$\sup_{n \geq q_0 + 1}\left(\frac{1}{|\lambda - 1|}\frac{\alpha_{n-1}}{\alpha_n}\right) < 1.$$

Now define by $\Sigma^{q_0} = \left(\sigma_{nm}\right)_{nm \geq 1}$ the triangle whose entries of the q_0 first rows and of the q_0 first columns are equal to those of $\Delta_\lambda = \left(1/(\lambda - 1)\right)(\lambda I - \Delta)$, $\sigma_{nn} = 1$ for all n and $\sigma_{nm} = 0$ for all n, m with $n \geq q_0 + 1$ and $m < n$. We easily see that $\lambda I - \Delta \in \left(s_\alpha^{(c)}, s_\alpha^{(c)}\right)$ and

$$\|I - \Delta_\lambda \Sigma^{q_0}\|_{S_\alpha} = \sup_{n \geq q_0 + 1}\left(\frac{1}{|\lambda - 1|}\frac{\alpha_{n-1}}{\alpha_n}\right) < 1,$$

from [13, Corollary 9, p. 47] we deduce $\Lambda = \left(\Delta_\lambda \Sigma^{q_0}\right)^{-1} \in \left(s_\alpha^{(c)}, s_\alpha^{(c)}\right)$ and

$$(\lambda I - \Delta)^{-1} = (\lambda - 1)^{-1} \Sigma^{q_0} \Lambda \in \left(s_\alpha^{(c)}, s_\alpha^{(c)}\right).$$

So we have shown that $\lambda \notin \overline{D\left(1, \lim_{n \to \infty}\left(\alpha_{n-1}/\alpha_n\right)\right)}$ implies $\lambda \in \rho\left(\Delta, s_\alpha^{(c)}\right)$ and we conclude that ii) b) holds.

iii) The identity $\sigma\left(\Delta, s_R\right) = \overline{D(1, 1/R)}$ was shown in [8], so by i) we also have $\sigma\left(\Delta, s_R^0\right) = \overline{D(1, 1/R)}$.

Now we show $\sigma\left(\Delta, s_R^{(c)}\right) = \overline{D(1, 1/R)}$. From the characterization of (c, c) we deduce the characterization of $\left(s_R^{(c)}, s_R^{(c)}\right)$. For the convenience of the reader recall that $A \in \left(s_R^{(c)}, s_R^{(c)}\right)$ if and only if $D_{1/R}AD_R \in (c,c)$, that is $D_{1/R}AD_R \in S_1$, $\lim_{n\to\infty}\left[D_{1/R}AD_R\right]_{nm} = l_m$ for all m, and $\lim_{n\to\infty}\sum_{m=1}^{\infty}\left[D_{1/R}AD_R\right]_{nm} = l$. Now take $\lambda \in \rho\left(\Delta, s_R^{(c)}\right)$. Putting $\Delta'_R = D_{1/R}(\lambda I - \Delta)^{-1} D_R$, we easily see that

$$\left\|\Delta'_R\right\|_{S_1} = \frac{1}{R} + |\lambda - 1| < \infty.$$

So we have $\Delta'_R \in S_1$ and since Δ'_R is a band matrix we have $(\lambda I - \Delta) \in \left(s_R^{(c)}, s_R^{(c)}\right)$ for all $\lambda \in \mathbb{C}$. Furthermore we have $(\lambda I - \Delta)^{-1} \in \left(s_R^{(c)}, s_R^{(c)}\right)$. Then $(\lambda I - \Delta)^{-1} \in S_R$ and by (4) we obtain

$$\chi = R\sup_n\left(\sum_{m=1}^{n}\frac{1}{(|\lambda-1|\,R)^{n-m+1}}\right) \tag{6}$$

$$= R\sup_n\left\{\frac{\left(\frac{1}{|\lambda-1|R}\right)^{n+1} - \frac{1}{|\lambda-1|R}}{\frac{1}{|\lambda-1|R} - 1}\right\} < \infty.$$

Then $1/(|\lambda - 1|\,R) < 1$ and we conclude $\lambda \notin \overline{D(1,1/R)}$ and $\overline{D(1,1/R)} \subset \sigma\left(\Delta, s_R^{(c)}\right)$.

The inclusion $\sigma\left(\Delta, s_R^{(c)}\right) \subset \overline{D(1,1/R)}$ is a direct consequence of ii) b) and we conclude $\sigma\left(\Delta, s_R^{(c)}\right) = \overline{D(1,1/R)}$. ■

As a direct consequence of Theorem 5 ii) we can state the following corollary.

COROLLARY 6. *i)* $\sigma\left(\Delta, s_{(n)_n}\right) = \sigma\left(\Delta, s^0_{(n)_n}\right) = \overline{D(1,1)}$.

ii) If $\limsup_{n\to\infty}(\alpha_{n-1}/\alpha_n) = 0$ *then* $\sigma(\Delta, s_\alpha) = \{1\}$.

iii) $\sigma(\Delta, s_\alpha) = \{1\}$ *where* $\alpha = (n!)_n$, *or* $(n^n)_n$.

Proof. i) By Theorem 5 i) $(|\lambda - 1|^n\, n)_{n\geq 1} \in \widehat{C}_1$ if and only if

$$c_n = \frac{|\lambda-1| + 2|\lambda-1|^2 + \cdots + n|\lambda-1|^n}{n|\lambda-1|^n} = O(1) \quad (n \to \infty).$$

It is well known that for $x \neq 1$ real we have

$$x + 2x^2 + \cdots + nx^n = x\left(1 + x + x^2 + \cdots + x^n\right)' = x\frac{x^n(nx - n - 1) + 1}{(1-x)^2}.$$

Hence we obtain

$$c_n = |\lambda - 1| \frac{n|\lambda - 1| - n - 1}{n\left(1 - |\lambda - 1|^2\right)} + \frac{1}{n} \frac{1}{\left(1 - |\lambda - 1|^2\right)|\lambda - 1|^{n-1}}$$

and $c_n = O(1)$ $(n \to \infty)$ if and only if $|\lambda - 1| > 1$. We conclude that $\left(|\lambda - 1|^n n\right)_{n\geq 1} \notin \widehat{C}_1$ if and only if $|\lambda - 1| \leq 1$ and by i) we conclude that $\sigma\left(\Delta, s_{(n)_n}\right) = \overline{D(1,1)}$.

ii) is a direct consequence of Theorem 5 ii) a).

iii) Trivially we have $\limsup_{n\to\infty}(\alpha_{n-1}/\alpha_n) = 0$ for $\alpha = (n^n)_n$, or $(n!)_n$. ∎

3.2. Spectrum of Δ considered as an operator in $l_p(\alpha)$

In this subsection we will characterize the sets $\sigma(\Delta, l_1(\alpha))$ and $\sigma(\Delta^+, l_1(\alpha))$ and explicitly calculate $\sigma(\Delta, l_p(\alpha))$ and $\sigma(\Delta^+, l_p(\alpha))$.

In the following we will use two lemmas where we consider the set $E^* = \mathcal{B}(E, \mathbb{C})$ of all continuous linear functionals on E where E is a Banach space and the β-dual of E is denoted by E^β. For $L \in \mathcal{B}(E)$ the operator L^* is defined by $L^*(\varphi)(u) = \varphi(L(u))$ for all $\varphi \in E^*$ and for all $u \in E$. We have the following well known result.

LEMMA 7 [5, pp. 71]. *Let E be a normed space. Then $L \in \mathcal{B}(E)$ implies $\sigma(L, E) = \sigma(L^*, E^*)$.*

When E is a BK space with AK there is a relation between E^* and E^β. Recall that E^β is the set of all sequences $a = (a_n)_{n\geq 1}$ such that $\sum_{n=1}^\infty a_n x_n$ is convergent for all $X \in E$. From [20, Theorem 7.2.9, pp. 107] we deduce that the map $\widehat{\ }: E^\beta \to E^*$ defined by $\widehat{\ }(a) = \widehat{a} : E \to \mathbb{C}$ $\left(a \in E^\beta\right)$ where $\widehat{a}(X) = \sum_{n=1}^\infty a_n x_n$ for all $X \in E$ is an isomorphism onto E^*. This means that E^* is isomorphic to E^β which is denoted by $E^* \equiv E^\beta$. From [20, Theorem 4.4.2, pp. 66] we deduce $(D_\alpha E)^* \equiv (D_\alpha E)^\beta$. As a direct consequence of the preceding we get the next lemma.

LEMMA 8. *Let $\alpha \in U^+$ and assume that E is a BK space with AK. Then $(D_\alpha E)^* \equiv D_{1/\alpha} E^\beta$.*

We will also use the sets Γ^+ and $\widehat{C_1^+}$ defined by

$$\Gamma^+ = \left\{\alpha \in U^+ : \limsup_{n\to\infty}\left(\frac{\alpha_{n+1}}{\alpha_n}\right) < 1\right\}.$$

and

$$\widehat{C_1^+} = \left\{\alpha \in U^+ \bigcap cs \ : \ \frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty} \alpha_k\right) = O(1) \ \ (n \to \infty)\right\},$$

where cs is the set of all convergent series. From [11, Proposition 6.1, pp. 3199] we have the following result.

LEMMA 9. *We have* $\Gamma^{+} \subset \widehat{C_1^{+}}$.

Concerning the spectrum of the operator Δ from $l_p(\alpha)$ to itself, we have the next result where we put $\Delta^{+} = \Delta^{t}$.

THEOREM 10. *Let* $\alpha \in U^{+}$. *Then*

i) a) $\lambda \in \sigma(\Delta, l_1(\alpha))$ *if and only if*

$$\lambda = 1 \text{ or } \left(\frac{1}{|\lambda - 1|^{n} \alpha_n}\right)_{n\geq 1} \notin \widehat{C_1^{+}}. \tag{7}$$

b) $\lambda \in \sigma(\Delta^{+}, l_1(\alpha))$ *if and only if*

$$\lambda = 1 \text{ or } \left(\frac{|\lambda - 1|^{n}}{\alpha_n}\right)_{n\geq 1} \notin \widehat{C_1}. \tag{8}$$

ii) We have

$$\sigma(\Delta, l_1(\alpha)) \subset \overline{D\left(1, \limsup_{n\to\infty}(\alpha_n/\alpha_{n+1})\right)} \text{ and}$$

$$\sigma\left(\Delta^{+}, l_1(\alpha)\right) \subset \overline{D\left(1, \limsup_{n\to\infty}(\alpha_{n+1}/\alpha_n)\right)}.$$

iii) Let $r > 0$ *and* $1 \leq p < \infty$. *Then*

$$\sigma(\Delta, l_p(r)) = \overline{D(1, 1/r)} \text{ and } \sigma\left(\Delta^{+}, l_p(r)\right) = \overline{D(1, r)}.$$

Proof. i) a) By Lemma 7 we have $\sigma(\Delta, l_1(\alpha)) = \sigma(\Delta^{+}, (l_1(\alpha))^{*})$ and since by Lemma 8 we have

$$(l_1(\alpha))^{*} = (D_\alpha l_1)^{*} \equiv D_{1/\alpha} l_1^{\beta} = D_{1/\alpha} l_\infty = s_{1/\alpha}$$

we deduce $\sigma(\Delta, l_1(\alpha)) = \sigma\left(\Delta^{+}, s_{1/\alpha}\right)$. Then $\lambda \in \rho\left(\Delta^{+}, s_{1/\alpha}\right)$ means that $\left(\lambda I - \Delta^{+}\right)^{-1} \in S_{1/\alpha}$, that is

$$\sup_n \left\{ \sum_{m=n}^{\infty} \frac{1}{|\lambda - 1|^{m-n+1}} \frac{\alpha_n}{\alpha_m} \right\} < \infty.$$

Reasoning as in Theorem 5 we conclude $\lambda \in \sigma(\Delta, l_1(\alpha))$ if and only if $\lambda = 1$ or

$$\left(\frac{1}{|\lambda - 1|^{n} \alpha_n}\right)_{n\geq 1} \notin \widehat{C_1^{+}}. \tag{9}$$

i) b) We have $\sigma(\Delta^{+}, l_1(\alpha)) = \sigma(\Delta, (l_1(\alpha))^{*})$ and as we have just seen $(l_1(\alpha))^{*} \equiv s_{1/\alpha}$. So $\sigma(\Delta^{+}, l_1(\alpha)) = \sigma\left(\Delta, s_{1/\alpha}\right)$ and we conclude by Theorem 5 i).

ii) As we have just seen $\lambda \in \sigma(\Delta, l_1(\alpha))$ implies that either $\lambda = 1$ or (7) holds. By Lemma 9 condition (7) implies $\left(|\lambda - 1|^{-n}/\alpha_n\right)_{n\geq 1} \notin \Gamma^+$ and

$$|\lambda - 1| \leq \limsup_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}}.$$

So we have shown the first inclusion. The second inclusion comes from the identity $\sigma(\Delta^+, l_1(\alpha)) = \sigma\left(\Delta, s_{1/\alpha}\right)$ and Theorem 5 ii).

iii) To show $\sigma(\Delta, l_p(r)) \subset \overline{D(1, 1/r)}$ we will show $\overline{D(1,1/r)}^c \subset \rho(\Delta, l_p(r))$. Take $\lambda \notin \overline{D(1,1/r)}$. We have

$$\|(I - \Delta_\lambda)X\|^p_{l_p(r)} = \frac{1}{|\lambda - 1|^p} \sum_{n=2}^{\infty} \left(\frac{|x_{n-1}|^p}{r^{np}}\right) = \frac{1}{(|\lambda - 1|\, r)^p} \sum_{n=2}^{\infty} \left(\frac{|x_{n-1}|^p}{r^{(n-1)p}}\right)$$

and

$$\|(I - \Delta_\lambda)X\|_{l_p(r)} = \frac{1}{|\lambda - 1|\, r} \|X\|_{l_p(r)}.$$

Hence $\|I - \Delta_\lambda\|_{\mathcal{B}(l_p(r))} < 1$ whenever $|\lambda - 1|\, r > 1$, and since $\mathcal{B}(l_p(r)) = (l_p(r), l_p(r))$ is a Banach algebra, (cf.[13]) we have $(\Delta_\lambda)^{-1} \in (l_p(r), l_p(r))$ and $\lambda \in \rho(\Delta, l_p(r))$. So we have shown that $\sigma(\Delta, l_p(r)) \subset \overline{D(1, 1/r)}$.

Conversely we show $\overline{D(1,1/r)} \subset \sigma(\Delta, l_p(r))$. For this take $\lambda \in \rho(\Delta, l_p(r))$. Then $(\lambda I - \Delta)^{-1} \in (l_p(r), l_p(r))$ and since $e^{(1)} \in l_p(r)$ we deduce

$$(\lambda I - \Delta)^{-1}\left(e^{(1)}\right)^t = \left(\frac{(-1)^{n-1}}{(\lambda - 1)^n}\right)_{n\geq 1} \in l_p(r).$$

So

$$\sum_{n=1}^{\infty} \left[\frac{1}{(r\,|\lambda - 1|)^n}\right]^p < \infty$$

and $1/(r\,|\lambda - 1|) < 1$. We have shown $\lambda \in \rho(\Delta, l_p(r))$ implies $|\lambda - 1| > 1/r$ that is $\overline{D(1,1/r)} \subset \sigma(\Delta, l_p(r))$. We conclude $\sigma(\Delta, l_p(r)) = \overline{D(1, 1/r)}$.

We show $\sigma(\Delta^+, l_p(r)) = \overline{D(1, r)}$. First consider the case $p > 1$. Let $q > 1$ such that $1/p + 1/q = 1$. Then $l_q^* \equiv l_p$ and by Lemma 8 we have $\left(D_{1/r} l_q\right)^* \equiv D_r l_q^\beta$ and $(l_q(1/r))^* = \left(D_{1/r} l_q\right)^* \equiv D_r l_q^\beta = D_r l_p = l_p(r)$. Then we show $\Delta \in (l_q(1/r), l_q(1/r))$. Put $\Delta_r = D_r \Delta D_{1/r}$, from Minkowski's inequality we get

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |[\Delta_r X]_n|^q\right)^{1/q} &= \left(\sum_{n=1}^{\infty} \left|x_n - \frac{1}{r}x_{n-1}\right|^q\right)^{1/q} \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^q\right)^{1/q} + \frac{1}{r}\left(\sum_{n=1}^{\infty} |x_{n-1}|^q\right)^{1/q} \leq \left(1 + \frac{1}{r}\right) \|X\|_{l_q}.\end{aligned}$$

So $\Delta_r \in (l_q, l_q)$ and $\Delta \in (l_q(1/r), l_q(1/r))$. By Lemma 7 we conclude $\sigma(\Delta^+, l_p(r)) = \sigma(\Delta, l_q(1/r)) = \overline{D(1,r)}$.

Case when $p = 1$. As we have seen in i) a) we have $(l_1(r))^* \equiv s_{1/r}$. We conclude

$$\sigma\left(\Delta^+, l_1(r)\right) = \sigma\left(\Delta, (l_1(r))^*\right) = \sigma\left(\Delta, s_{1/r}\right) = \overline{D(1,r)}. \qquad \blacksquare$$

4. Application to matrix transformations mapping in $s^0_\alpha\left((\Delta - \lambda I)^h\right)$ and in $l_p(r)\left((\Delta - \lambda I)^N\right)$, $r > 0$, $h \in \mathbb{C}$ and N integer

4.1. On the set $(E(T), F)$

Here we will reformulate a theorem due to Malkowsky and Rakočević [17]. For this we consider the triangle T and put $T^{-1} = (s_{nm})_{n,m \geq 1}$ and $R = \left(T^{-1}\right)^t$. We can state the next result, where $\widehat{A}$ is the matrix with rows $\widehat{A_n} = \left(RA^t_n\right)^t$ for all n and $A_n = (a_{n1}, \dots, a_{nm}, \dots)$.

Lemma 11. *[17] Let E be a BK space with AK and F be an arbitrary subset of s. Then we have $A \in (E(T), F)$ if and only if*

$$\widehat{A} \in (E, F)$$

and

$$V^{(i)} \in (E, c) \text{ for all } i,$$

where $V^{(i)} = \left(w^i_{nm}\right)_{n,m \geq 1}$ is defined by

$$w^i_{nm} = \begin{cases} \sum_{j=m}^n s_{jm} a_{ij} & 1 \leq m \leq n, \\ 0 & \text{for } m > n. \end{cases}$$

Lemma 12. *Let E be a BK space with AK and F be any set of sequences. Then $A \in (E(T), F)$ if and only if*

$$AT^{-1} \in (E, F)$$

and

$$\Sigma D_{(a_{in})_n} T^{-1} \in (E, c) \text{ for all } i = 1, 2, \dots.$$

Proof. We show $\widehat{A} = AT^{-1}$ and $V^{(i)} = \Sigma D_{(a_{in})_n} T^{-1}$. We have

$$\left(\widehat{A_n}\right)^t = RA^t_n = \left(T^{-1}\right)^t A^t_n = \left(A_n T^{-1}\right)^t.$$

So

$$\widehat{A_n} = A_n T^{-1} = \left(\sum_{j=m}^{\infty} a_{nj} s_{jm}\right)_{m \geq 1}$$

and $\widehat{A} = AT^{-1}$. Now for each integer i we have

$$\Sigma D_{(a_{in})_n} = \begin{pmatrix} a_{i1} & & & & \\ a_{i1} & a_{i2} & & \mathbf{0} & \\ . & . & . & & \\ a_{i1} & a_{i2} & . & a_{in} & \\ . & . & . & . & . \end{pmatrix}$$

and trivially $V^{(i)} = \Sigma D_{(a_{in})_n} T^{-1}$. This concludes the proof. ■

4.2. Application to matrix transformations from $s_\alpha^0\left((\Delta - \lambda I)^h\right)$ to s_μ

In the following we will use the operator Δ^h, where $h \in \mathbb{C}$. For this we need the next notation

$$\binom{-h+k-1}{k} = \begin{cases} \dfrac{-h(-h+1)\dots(-h+k-1)}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0, \end{cases}$$

(cf. [1]). To simplify we will write $[-h, k] = \binom{-h+k-1}{k}$. Then if $\Delta^h = (\tau_{nm})_{n,m\geq 1}$, we have

$$\tau_{nm} = \begin{cases} [-h, n-m] & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases} \tag{10}$$

We can state the following result.

THEOREM 13. *Let α, $\mu \in U^+$. Then*

i) Let $h \in \mathbb{C}$ and $\lambda \neq 1$. We have

a) $A \in \left(s_\alpha^0\left((\Delta - \lambda I)^h\right), s_\mu\right)$ if and only if

$$\sup_n \left\{ \frac{1}{\mu_n} \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} [h, m-k] \frac{a_{nm}}{(1-\lambda)^{h+m-k}} \right| \alpha_k \right\} < \infty, \tag{11}$$

$$\sup_n \left\{ \sum_{k=1}^{n} \left| \sum_{m=k}^{n} [h, m-k] \frac{a_{im}}{(1-\lambda)^{h+m-k}} \right| \alpha_k \right\} < \infty \textit{ for } i = 1, 2, \dots, \tag{12}$$

$$\lim_{n\to\infty} \left\{ \sum_{m=k}^{n} [h, m-k] \frac{a_{im}}{(1-\lambda)^{h+m-k}} \right\} = l_k \textit{ for some } l_k,\ k = 1, 2, \dots; \tag{13}$$

b) $A \in \left(s_\alpha^0\left((\Delta-\lambda I)^h\right), s_\mu^0\right)$ *if and only if (11), (12), (13) hold and*

$$\lim_{n\to\infty}\left\{\frac{1}{\mu_n}\sum_{m=k}^{n}[h,m-k]\frac{a_{im}}{(1-\lambda)^{h+m-k}}\right\}=0 \text{ for } k=1,2,\dots;$$

c) $A \in \left(s_\alpha^0\left((\Delta-\lambda I)^h\right), s_\mu^{(c)}\right)$ *if and only if (11), (12), (13) hold and*

$$\lim_{n\to\infty}\left\{\frac{1}{\mu_n}\sum_{m=k}^{n}[h,m-k]\frac{a_{im}}{(1-\lambda)^{h+m-k}}\right\}=l_k \text{ for some } l_k,\ k,i=1,2,\dots,$$

ii) Let h *be an integer and assume* $|\lambda-1| > \limsup_{n\to\infty}(\alpha_{n-1}/\alpha_n)$. *Then*

a) $A \in \left(s_\alpha^0\left((\Delta-\lambda I)^h\right), s_\mu\right)$ *if and only if*

$$\sup_n\left(\frac{1}{\mu_n}\sum_{m=1}^{\infty}|a_{nm}|\,\alpha_m\right)<\infty; \tag{14}$$

b) $A \in \left(s_\alpha^0\left((\Delta-\lambda I)^h\right), s_\mu^0\right)$ *if and only if (14) holds and* $a_{nm}/\mu_n \to 0$ $(n\to\infty)$ *for* $m=1,2,\dots;$

c) $A \in \left(s_\alpha^0\left((\Delta-\lambda I)^h\right), s_\mu^{(c)}\right)$ *if and only if (14) holds and* $a_{nm}/\mu_n \to l'_m$ $(n\to\infty)$ *for some* l'_m, $m=1,2,\dots.$

P r o o f. i) Since s_α^0 is a BK space with AK, it is enough to apply Lemma 12 with $T=(\Delta-\lambda I)^h$. We easily see that $T^{-1}=(\Delta-\lambda I)^{-h}$ is the triangle defined by

$$\left[(\Delta-\lambda I)^{-h}\right]_{nm}=\frac{[h,n-m]}{(1-\lambda)^{h+n-m}} \text{ for } m\le n.$$

Then

$$\left[A(\Delta-\lambda I)^{-h}\right]_{nk}=\sum_{m=k}^{\infty}[h,m-k]\frac{a_{nm}}{(1-\lambda)^{h+m-k}} \text{ for all } n,k\ge 1.$$

Since $\left(s_\alpha^0, s_\mu\right)=S_{\alpha,\mu}$ the condition $A(\Delta-\lambda I)^{-h}\in\left(s_\alpha^0, s_\mu\right)$ is equivalent to (11). For each integer i we obtain

$$\left[\Sigma D_{(a_{in})_n}(\Delta-\lambda I)^{-h}\right]_{nk}=\sum_{m=k}^{n}a_{im}\frac{[h,m-k]}{(1-\lambda)^{h+m-k}} \text{ for } k\le n.$$

We conclude from the characterization of $\left(s_\alpha^0, c\right)$ which can be deduced from Lemma 2.

ii) a) If $|\lambda - 1| > \limsup_{n\to\infty} (\alpha_{n-1}/\alpha_n)$, then $\lambda \in \rho\left(\Delta, s_\alpha^0\right)$ and $\Delta - \lambda I$ is bijective from s_α^0 to itself and $s_\alpha^0\left((\Delta - \lambda I)^h\right) = s_\alpha^0$. We then have

$$\left(s_\alpha^0\left((\Delta - \lambda I)^h\right), s_\mu\right) = S_{\alpha,\mu},$$

this gives the conclusion of ii) a). The cases ii) b) and ii) c) can be shown in a similar way using Lemma 2. ∎

Reasoning as above and using Theorem 10 iii) and the characterization of (l_p, l_∞), (cf. [16]) we get the next result whose the proof is elementary and left to the reader.

THEOREM 14. *Let N be an integer, let r be a real > 0 and $\mu \in U^+$. Assume $1 < p < \infty$ and put $q = p/(p-1)$. Then*

i) $A \in \left(l_p(r)\left((\Delta - \lambda I)^N\right), s_\mu\right)$ if and only if

$$\sup_n \left\{ \frac{1}{\mu_n} \sum_{k=1}^{n} \left| \sum_{m=k}^{\infty} [N, m-k] \frac{a_{nm}}{(1-\lambda)^{N+m-k}} \right|^q r^{kq} \right\} < \infty, \tag{15}$$

$$\sup_n \left\{ \sum_{k=1}^{n} \left| \sum_{m=k}^{n} [N, m-k] \frac{a_{im}}{(1-\lambda)^{N+m-k}} \right|^q r^{kq} \right\} < \infty \text{ for } i = 1, 2, \dots, \tag{16}$$

$$\lim_{n\to\infty} r^k \sum_{m=k}^{n} [N, m-k] \frac{a_{im}}{(1-\lambda)^{N+m-k}} = l_k \text{ for some } l_k \in \mathbb{C}\ k, i = 1, 2, \dots.$$

ii) If $|\lambda - 1| > 1/r$ then $A \in \left(l_p(r)\left((\Delta - \lambda I)^N\right), s_\mu\right)$ if and only if

$$\sup_n \left\{ \frac{1}{\gamma_n} \sum_{m=1}^{\infty} \left(|a_{nm}| r^m\right)^{p/(p-1)} \right\} < \infty.$$

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References

[1] A. F. Andersen, *Summation of nonintegral order*, Mat. Tidsskr. B. (1946), 33–52.
[2] B. Altay, F. Başar, *On the fine spectrum of the difference operator on c_0 and c*, Inform. Sci. 168 (2004), 217–224.
[3] A. M. Akhmedov, F. Başar, *The fine spectra of the difference operator Δ over the sequence space l_p*, $(1 \le p < \infty)$, Inform. Sci. (2004).
[4] R. Colak, M. Et, *On some generalized difference spaces and related matrix transformations*, Hokkaido Math. J. 2 (1997), 483-492.
[5] S. Goldberg, *Unbounded Linear Operators*, Dover Publications Inc. New York 1985.

[6] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull. 24 (2) (1981), 19–176.
[7] I. J. Maddox, *Infinite Matrices of Operators*, Springer-Verlag, Berlin, Heidelberg and New York, 1980.
[8] B. de Malafosse, *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ*, Hokkaido Math. J. 31 (2002), 283–299.
[9] B. de Malafosse, *On some BK space*, Internat. J. Math. and Math. Sc. 28 (2003), 1783–1801.
[10] B. de Malafosse, *Sum and product of certain BK spaces and matrix transformations between these spaces*, Acta Math. Hungarica 104 (3), (2004) 241–263.
[11] B. de Malafosse, *On the Banach algebra $B(l_p(\alpha))$*, Internat. J. Math. Sci. and Math. Sc. 60 (2004) 3187–3203.
[12] B. de Malafosse, *Linear operators mapping in new sequence spaces*, Soochow J. Math. 31 N°2 (2005), 403–427.
[13] B. de Malafosse, *The Banach algebra $B(X)$, where X is a BK space and applications*, Vesnik Math. J. 57 (2005), 41–60.
[14] B. de Malafosse, E. Malkowsky, *Sequence spaces and inverse of an infinite matrix*, Rend. Circ. Mat. Palermo Serie II, 51 (2002), 277–294.
[15] E. Malkowsky, *Recent results in the theory of matrix transformations in sequence spaces*, Vesnik Math. J. 49 (1997), 187–196.
[16] E. Malkowsky, V. Rakočević, *An introduction into the theory of sequence spaces and measure of noncompactness*, Zb. Rad. Mat. Inst. SANU 9 (17) (2000), 143–243.
[17] E. Malkowsky, V. Rakočević, *On matrix domains of triangles*, Under review. To appear in Applied Math. and Computation (2007).
[18] B. E. Rhoades, *The fine spectra of some weighted mean operators in $B(l_p)$*, Integral Equat. Operator Theory 12 (1989), 82–98.
[19] B. E. Rhoades, *The spectra of weighted mean operators on bv_0*, J. Austral. Math. Soc. (Series A) 52 (1992), 242–250.
[20] A. Wilansky, *Summability through Functional Analysis*, North-Holland Mathematics Studies 85, 1984.

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