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APPROXIMATION OF COMMON FIXED POINTS OF A FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

Abstract. In this paper, we study the convergence of the sequence of Ishikawa iteration of rank-r to common fixed points of a finite family of asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces. Our results extend and improve some known recent results.

1. Introduction

Let C be a subset of normed space X and $T : C \to C$ be a mapping. Then T is said to be an asymptotically quasi-nonexpansive mapping, if $F(T) \neq \emptyset$ and there is a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\| \text{ for all } x \in C \text{ and } p \in F(T)$$

($F(T)$ denotes the set of fixed points of T). T is an asymptotically nonexpansive mapping [2], if there is a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C.$$

If for each $n \in \mathbb{N}$, there are constants $L > 0$ and $\alpha > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|^\alpha \text{ for all } x, y \in C,$$

then T is called uniformly $(L - \alpha)$-Lipschitz. Every asymptotically nonexpansive mapping is uniformly $(L - 1)$-Lipschitz mapping.

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In [3], Ishikawa introduced a new iteration process as follows:

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_n T y_n, \\ y_n \quad = (1-b_n)x_n + b_n T x_n, n = 1, 2, \cdots, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$ satisfying certain restrictions.

In 1973, Petryshyn and Williamson [5] gave necessary and sufficient conditions for Mann iterative sequence (cf.[4]) to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [1] extended the results of Petryshyn and Williamson [5] and gave necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Qihou [6] extended results of [1, 5] and gave the necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed point of asymptotically quasi-nonexpansive mappings.

Recently, first author [8] introduced Ishikawa iteration process of rank-r which is similar to the following:

$$(1.1) \quad \begin{cases} \quad x_1 \in C; \\ x_{n+1} = (1-a_{n,i})x_n + a_{n,i}T_i y_{n,i}; \\ y_{n,i} \ = (1-a_{n,i+1})x_n + a_{n,i+1}T_i y_{n,i+1}; i = 1, 2, 3, \cdots, r-1; \\ y_{n,r} = x_n. \end{cases}$$

The modified Ishikawa iteration process of rank r is the following:

$$(1.2) \quad \begin{cases} \quad x_1 \in C; \\ x_{n+1} = (1-a_{n,i})x_n + a_{n,i}T_i^n y_{n,i}; \\ y_{n,i} \ = (1-a_{n,i+1})x_n + a_{n,i+1}T_i^n y_{n,i+1}; i = 1, 2, 3, \cdots, r-1; \\ y_{n,r} = x_n. \end{cases}$$

It is very useful in computing to common fixed points of nonlinear mappings.

In this paper, we study the convergence of Ishikawa iteration of rank 3 for three uniformly $(L-\alpha)$-Lipschitz type asymptotically quasi-nonexpansive mappings on a compact convex subset of a uniform convex Banach space. Our scheme is given as follows:

Let C be a nonempty compact convex subset of a uniformly convex Banach space X and for $i = 1, 2, 3$, let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$.

Define a sequence $\{x_n\}$ in C as follows:

$$\text{(1.3)}\qquad \begin{cases} x_1 \in C, \\ x_{n+1} = (1-a_n)x_n + a_n T_1^n y_n, \\ y_n = (1-b_n)x_n + b_n T_2^n z_n, \\ z_n = (1-c_n)x_n + c_n T_3^n x_n \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. Our results generalize and improve the results of [6, 7, 10].

2. Preliminaries

The following lemmas will be used to prove the main theorems.

LEMMA 2.1 ([6]). *Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty}$ be three sequences of non-negative numbers satisfying $\alpha_{n+1} \le (1+\beta_n)\alpha_n + \gamma_n \ \forall\, n \in N$ and $\sum_{n=1}^{\infty} \beta_n < +\infty, \sum_{n=1}^{\infty} \gamma_n < +\infty$. Then $\lim_{n\to\infty} \alpha_n$ exists.*

LEMMA 2.2 ([9]). *Let X be a uniformly convex Banach space, $0 < \alpha \le t_n \le \beta < 1$, $x_n, y_n \in X$, $\limsup_{n\to\infty} \|x_n\| \le a, \limsup_{n\to\infty} \|y_n\| \le a$, $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a, a \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

LEMMA 2.3. *Let C be a nonempty convex subset of a uniformly convex Banach space X and for $i = 1, 2, 3$, let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-a_n)x_n + a_n T_1^n y_n, \\ y_n = (1-b_n)x_n + b_n T_2^n z_n, \\ z_n = (1-c_n)x_n + c_n T_3^n x_n \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2) \cap F(T_3)$.

Proof. Let $p \in \bigcap_{i=1}^{3} F(T_i)$. Then

$$\text{(2.1)}\qquad \begin{aligned} \|z_n - p\| &= \|(1-c_n)x_n + c_n T_3^n x_n - p\| \\ &\le (1-c_n)\|x_n - p\| + c_n\|T_3^n x_n - p\| \\ &\le k_n^{(3)}\|x_n - p\|, \end{aligned}$$

and

$$\begin{aligned}\|y_n - p\| &= \|(1-b_n)x_n + b_nT_2^nz_n - p\| \\ &\le (1-b_n)\|x_n - p\| + b_n\|T_2^nz_n - p\| \\ &\le k_n^{(2)}k_n^{(3)}\|x_n - p\|.\end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-a_n)\|x_n - p\| + a_n\|T_1^ny_n - p\| \\ &\le k_n^{(1)}k_n^{(2)}k_n^{(3)}\|x_n - p\|.\end{aligned} \tag{2.3}$$

Observe that

$$\begin{aligned}\sum_{n=1}^{\infty}(k_n^{(1)}k_n^{(2)}k_n^{(3)} - 1) &= \sum_{n=1}^{\infty}[k_n^{(1)}k_n^{(2)}(k_n^{(3)} - 1) + k_n^{(1)}(k_n^{(2)} - 1) + k_n^{(1)} - 1] \\ &\le K_1\sum_{n=1}^{\infty}(k_n^{(3)} - 1) + K_2\sum_{n=1}^{\infty}(k_n^{(2)} - 1) \\ &\quad + \sum_{n=1}^{\infty}(k_n^{(1)} - 1) < \infty\end{aligned}$$

for some constants $K_1, K_2 > 0$. Using Lemma 2.1, we obtain that $\lim_{n\to\infty}\|x_n - p\|$ exists. ∎

3. Main results

THEOREM 3.1. *Let C be a nonempty compact convex subset of a uniformly convex Banach space X and for $i = 1, 2, 3$, let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-a_n)x_n + a_nT_1^ny_n, \\ y_n = (1-b_n)x_n + b_nT_2^nz_n, \\ z_n = (1-c_n)x_n + c_nT_3^nx_n \text{ for all } n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ such that $0 < \underline{a} < a_n \le \overline{a} < 1, 0 < \underline{b} \le b_n \le \overline{b} < 1$ and $0 < \underline{c} \le c_n \le \overline{c} < 1$.

If $F(T_1) \cap F(T_2) \cap F(T_3) \ne \emptyset$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 2.3, we have $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. Set $\lim_{n\to\infty}\|x_n - p\| = d$ for some $d > 0$. Then, from (2.1) and

(2.2), we have

$$\limsup_{n\to\infty} \|z_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = d$$

and

$$\limsup_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = d,$$

respectively.

Note that

$$\limsup_{n\to\infty} \|T_1^n y_n - p\| \le \limsup_{n\to\infty} (k_n^{(1)} \|y_n - p\|) \le d$$

and

$$\lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|(1-a_n)(x_n - p) + a_n(T_1^n y_n - p)\| = d.$$

Thus, from Lemma 2.2, we get

$$\lim_{n\to\infty} \|x_n - T_1^n y_n\| = 0. \tag{3.2}$$

Next,

$$\begin{aligned} \|x_n - p\| &\le \|x_n - T_1^n y_n\| + \|T_1^n y_n - p\| \\ &\le \|x_n - T_1^n y_n\| + k_n^{(1)} \|y_n - p\|, \end{aligned}$$

which gives that

$$d \le \liminf_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|y_n - p\| \le d$$

and hence

$$\lim_{n\to\infty} \|y_n - p\| = d.$$

Note that

$$\limsup_{n\to\infty} \|T_2^n z_n - p\| \le \limsup_{n\to\infty} (k_n^{(2)} \|z_n - p\|) \le d$$

and

$$d = \lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|(1-b_n)(x_n - p) + b_n(T_2^n z_n - p)\|.$$

Thus, from Lemma 2.2, we get

$$\lim_{n\to\infty} \|x_n - T_2^n z_n\| = 0. \tag{3.3}$$

Note

$$\begin{aligned} \|x_n - p\| &\le \|x_n - T_2^n z_n\| + \|T_2^n z_n - p\| \\ &\le \|x_n - T_2^n z_n\| + k_n^{(2)} \|z_n - p\|, \end{aligned}$$

which gives that

$$d \le \liminf_{n\to\infty} \|z_n - p\| \le \limsup_{n\to\infty} \|z_n - p\| \le d,$$

and hence $\lim_{n\to\infty} \|z_n - p\| = d$.

Since

$$\limsup_{n\to\infty} \|T_3^n x_n - p\| \le \limsup_{n\to\infty} (k_n^{(3)} \|x_n - p\|) \le d,$$

and

$$d = \lim_{n\to\infty} \|z_n - p\| = \lim_{n\to\infty} \|(1 - c_n)(x_n - p) + c_n(T_3^n x_n - p)\|.$$

Thus, from Lemma 2.2, we get

$$\lim_{n\to\infty} \|x_n - T_3^n x_n\| = 0. \tag{3.4}$$

Since C is compact, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Let

$$\lim_{k\to\infty} x_{n_k} = p. \tag{3.5}$$

Then from (3.1), (3.2) and (3.3), we have

$$\|x_{n_k+1} - x_{n_k}\| \le a_{n_k} \|T_1^{n_k} y_{n_k} - x_{n_k}\| \to 0 \tag{3.6}$$

and

$$\|y_n - x_n\| \le b_n \|T_2^n z_n - x_n\| \to 0. \tag{3.7}$$

Again, from (3.2) and (3.5), we have

$$\lim_{k\to\infty} T_1^{n_k} y_{n_k} = p. \tag{3.8}$$

Since $\lim_{k\to\infty} x_{n_k+1} = p$, we have

$$\lim_{k\to\infty} T^{n_k+1} y_{n_k+1} = p. \tag{3.9}$$

From (3.6), (3.7), (3.8) and (3.9) we have

$$\begin{aligned}
0 \le \|p - T_1 p\| &= \|p - T_1^{n_k+1} y_{n_k+1} + T_1^{n_k+1} y_{n_k+1} - T_1^{n_k+1} x_{n_k+1} \\
&\quad + T_1^{n_k+1} x_{n_k+1} - T_1^{n_k+1} x_{n_k} + T_1^{n_k+1} x_{n_k} - T_1^{n_k+1} y_{n_k} \\
&\quad + T_1^{n_k+1} y_{n_k} - T_1 p\| \\
&\le \|p - T_1^{n_k+1} y_{n_k+1}\| + \|T_1^{n_k+1} y_{n_k+1} - T_1^{n_k+1} x_{n_k+1}\| \\
&\quad + \|T_1^{n_k+1} x_{n_k+1} - T_1^{n_k+1} x_{n_k}\| + \|T_1^{n_k+1} x_{n_k} - T_1^{n_k+1} y_{n_k}\| \\
&\quad + \|T_1^{n_k+1} y_{n_k} - T_1 p\| \\
&\le \|p - T_1^{n_k+1} y_{n_k+1}\| + L_1 \|y_{n_k+1} - x_{n_k+1}\|^{\alpha_1} \\
&\quad + L_1 \|x_{n_k+1} - x_{n_k}\|^{\alpha_1} + L_1 \|x_{n_k} - y_{n_k}\|^{\alpha_1} \\
&\quad + L_1 \|T_1^{n_k} y_{n_k} - p\|^{\alpha_1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Next,

$$\|z_n - x_n\| \le c_n \|T_3^n x_n - x_n\| \to 0. \tag{3.10}$$

From (3.3) and (3.5), we have

$$\lim_{k\to\infty} T_2^{n_k} z_{n_k} = p. \tag{3.11}$$

Since $\lim_{k\to\infty} x_{n_k+1} = p$, then

$$\lim_{k\to\infty} T_2^{n_k+1} z_{n_k+1} = p. \tag{3.12}$$

From (3.6), (3.11) and (3.12), we have

$$\begin{aligned}
0 \le \|p - T_2 p\| &= \|p - T_2^{n_k+1} z_{n_k+1} + T_2^{n_k+1} z_{n_k+1} - T_2^{n_k+1} x_{n_k+1} \\
&\quad + T_2^{n_k+1} x_{n_k+1} - T_2^{n_k+1} x_{n_k} + T_2^{n_k+1} x_{n_k} - T_2^{n_k+1} z_{n_k} \\
&\quad + T_2^{n_k+1} z_{n_k} - T_2 p\| \\
&\le \|p - T_2^{n_k+1} z_{n_k+1}\| + \|T_2^{n_k+1} z_{n_k+1} - T_2^{n_k+1} x_{n_k+1}\| \\
&\quad + \|T_2^{n_k+1} x_{n_k+1} - T_2^{n_k+1} x_{n_k}\| + \|T_2^{n_k+1} x_{n_k} - T_2^{n_k+1} z_{n_k}\| \\
&\quad + \|T_2^{n_k+1} z_{n_k} - T_2 p\| \\
&\le \|p - T_2^{n_k+1} z_{n_k+1}\| + L_2 \|z_{n_k+1} - x_{n_k+1}\|^{\alpha_2} \\
&\quad + L_2 \|x_{n_k+1} - x_{n_k}\|^{\alpha_2} + L_2 \|x_{n_k} - z_{n_k}\|^{\alpha_2} \\
&\quad + L_2 \|T_2^{n_k} z_{n_k} - p\|^{\alpha_2} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Now from (3.4) and (3.5), we have

$$\lim_{k\to\infty} T_3^{n_k} x_{n_k} = p. \tag{3.13}$$

Since $\lim_{k\to\infty} x_{n_k+1} = p$, it follows from (3.4) that

$$\lim_{k\to\infty} T_3^{n_k+1} x_{n_k+1} = p. \tag{3.14}$$

From (3.6) and (3.14), we obtain

$$\begin{aligned}
0 \le \|p - T_3 p\| &= \|p - T_3^{n_k+1} x_{n_k+1} + T_3^{n_k+1} x_{n_k+1} - T_3^{n_k+1} x_{n_k} \\
&\quad + T_3^{n_k+1} x_{n_k} - T_3 p\| \\
&\le \|p - T_3^{n_k+1} x_{n_k+1}\| + L_3 \|x_{n_k+1} - x_{n_k}\|^{\alpha_3} \\
&\quad + L_3 \|T_3^{n_k} x_{n_k} - p\|^{\alpha_3} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, p is a common fixed point of T_1, T_2 and T_3. Since the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to p and $\lim_{n\to\infty} \|x_n - p\|$ exists, we conclude that $\lim_{n\to\infty} x_n = p$. ∎

COROLLARY 3.2. *Let C be a nonempty compact convex subset of a uniformly convex Banach space and for $i = 1, 2$, let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence*

$\{k_n^{(i)}\}$ such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1) < \infty$. Define a sequence $\{x_n\}$ in C as follows:

$$(3.15)\qquad \begin{cases} x_1 \in C, \\ x_{n+1} = (1-a_n)x_n + a_n T_1^n y_n, \\ y_n = (1-b_n)x_n + b_n T_2^n x_n \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$ such that $0 < \underline{a} < a_n \leq \overline{a} < 1$ and $0 < \underline{b} \leq b_n \leq \overline{b} < 1$.

If $F(T_1) \cap F(T_2) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

REMARK 3.3. Corollary 3.2 is an improvement of the results of Qihou [7] when c_n and $c_n' = 0$.

In the same manner we have the following theorem.

THEOREM 3.4. *Let C be a nonempty compact convex subset of a uniformly convex Banach space and for $i = 1, 2, \cdots, r$; let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$. Let $\{x_n\}$ be an iterative sequence of the modified Ishikawa iteration process of rank r defined in C by (1.2), where $\{a_{n,i}\}$ (i=1, 2, $\cdots, r$) be sequences of real numbers in $[0,1]$ such that $0 \leq \underline{a}_i \leq a_{n,i} \leq \overline{a}_i < 1$ for all $i \in 1, 2, \cdots, r$ and $n \in \mathbb{N}$. If $F(T_1) \cap F(T_2) \cap \cdots \cap F(T_r) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \cdots, T_r$.*

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