DEMONSTRATIO MATHEMATICA
Vol. XLI No 3 2008

S. S. Dragomir, D. Comănescu

ON THE TORRICELLIAN POINT IN INNER PRODUCT SPACES

Abstract. The concept of Torricellian point related to a set of n vectors in normed linear spaces is introduced and the general properties obtained. The existence and uniqueness of the Torricellian point in inner product spaces are established.

1. Introduction

The problem of minimizing the sum of the distances from a variable point to three fixed points in the plane, posed and solved by Torricelli in the 17th century, is well known. He found that the point for which the minimum is realised is either a vertex of the fixed triangle, if the measure of the corresponding angle is greater than $\frac{2\pi}{3}$, or the unique point for which each edge is seen under $\frac{2\pi}{3}$.

In this paper, the concept of *Torricellian point* for the case of normed linear spaces and related with a set of n distinct given vectors $\{a_1, \ldots, a_n\} \subset X$ $(n \geq 1)$ is introduced and some of its general properties obtained. The existence and uniqueness of Torricellian point in inner product spaces and characterisations with a geometrical interpretation are established as well. The obtained results build on the case of three vectors in inner product spaces that has been considered in [4].

2. Preliminary results

We start with the following definition:

DEFINITION 1. Let $(X; \|\cdot\|)$ be a real normed linear space, $n \geq 1$ a natural number and $\{a_1, \ldots, a_n\} \subset X$ a set of distinct vectors in X. We say that the point $x_0 \in X$ is a Torricellian point for the set $\{a_1, \ldots, a_n\}$ if the following

---

2000 *Mathematics Subject Classification*: Primary 46B20, 46B05.
Key words and phrases: Torricellian points, Normed spaces, Inner product spaces.

condition holds:

$$\sum_{i=1}^{n} \|x_0 - a_i\| \leq \sum_{i=1}^{n} \|x - a_i\| \qquad \text{for all } x \in X,$$

i.e., the element x_0 minimizes the (nonlinear) functional $T : X \to [0, \infty)$, called the Torricelli functional, and given by: $T(x) := \sum_{i=1}^{n} \|x - a_i\|$.

The set of Torricellian points associated with the set $\{a_1, \ldots, a_n\}$ will be denoted by $\mathcal{T}_X \{a_1, \ldots, a_n\}$.

REMARK 1. Naturally, the above concepts can be introduced in the more general case of metric spaces. The Torricellian point is also known in the literature as the median point of the finite set $F = \{a_1, \ldots, a_n\}$, [1]-[3] and [5]-[7], however we believe that, taking into account the history of the problem, the name Torricellian point is perhaps more appropriate.

For the sake of completeness, we introduce some notations that will be used in the sequel:

(i) $dr(a,b) := \{\lambda a + (1-\lambda) b | \lambda \in \mathbb{R}\}$ where $a, b \in X$ and $a \neq b$ will be called the *right line* determined by the elements a and b;
(ii) $[a,b] := \{\lambda a + (1-\lambda) b | \lambda \in [0,1]\}$ where a, b are as above, will be the *segment* determined by a and b;
(iii) The points of the set $M \subset H$ are said to be *colinear* iff there exists a right line $dr(a,b)$ such that $M \subseteq dr(a,b)$.
(iv) The normed space $(X, \|\cdot\|)$ will be called *strictly convex* iff for every $x, y \in X$ with $x \neq y$ and such that $\|x+y\| = \|x\| + \|y\|$, there exists a real number t such that $x = ty$.
(v) $Sp[a_1, \ldots, a_n]$ is the linear subspace generated by the set of vectors $\{a_1, \ldots, a_n\}$.
(vi) Let $(X, \|\cdot\|)$ be a normed linear space and $h : D \subseteq X \to \mathbb{R}$ (D is open in X). Suppose that $x_0 \in D$. We will say that h is *Gâteaux differentiable* in x_0 iff there exists the limit:

$$\lim_{t \to 0} \frac{h(x_0 + ty) - h(x_0)}{t} = \frac{\partial h}{\partial y}(x_0)$$

for all $y \in X$.

Some of the fundamental properties of the Torricellian mapping T associated with the set of distinct points $\{a_1, \ldots, a_n\}$ are embodied in the following proposition:

PROPOSITION 1. *With the above assumptions,*

(i) *T is nonlinear;*
(ii) *T is continuous on X in the norm topology;*

(iii) *T is nonnegative and* $\lim_{\|x\|\to\infty} T(x) = \infty$;
(iv) *T is convex on X.*

Proof. (i) and (ii) are obvious.

(iii) We have:

$$T(x) = \sum_{i=1}^{n} \|x - a_i\| \geq \sum_{i=1}^{n} |\|x\| - \|a_i\|| \\ \geq \sum_{i=1}^{n} (\|x\| - \|a_i\|) = n\|x\| - \sum_{i=1}^{n} \|a_i\|$$

which shows that $T(x) \to \infty$ as $\|x\| \to \infty$.

(iv) Utilising the triangle inequality we have:

$$T(\alpha x + \beta y) = \sum_{i=1}^{n} \|\alpha x + \beta y - a_i\| = \sum_{i=1}^{n} \|\alpha(x - a_i) + \beta(y - a_i)\| \\ \leq \alpha \sum_{i=1}^{n} \|x - a_i\| + \beta \sum_{i=1}^{n} \|y - a_i\| = \alpha T(x) + \beta T(y)$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$. ∎

The next proposition also holds.

PROPOSITION 2. *Let $(X, \|\cdot\|)$ be a strictly convex normed linear space. If $\{a_1, \ldots, a_n\}$ $(n \geq 1)$ is a set of non-colinear vectors in X, then T is strictly convex on X.*

Proof. Since T is convex, one has:

$$T(\lambda x_1 + (1-\lambda) x_2) \leq \lambda T(x_1) + (1-\lambda) T(x_2) \tag{2.1}$$

for all $\lambda \in [0,1]$ and $x_1, x_2 \in X$.

Now, let $\lambda \in (0,1)$ and $x_1, x_2 \in X$ with $x_1 \neq x_2$ and assume that the inequality (2.1) becomes an equality, i.e.,

$$\sum_{i=1}^{n} \|\lambda(x_1 - a_i) + (1-\lambda)(x_2 - a_i)\| = \lambda \sum_{i=1}^{n} \|x_1 - a_i\| + (1-\lambda) \sum_{i=1}^{n} \|x_2 - a_i\|$$

which gives us (by the triangle inequality) that:

$$\|\lambda(x_1 - a_i) + (1-\lambda)(x_2 - a_i)\| = \|\lambda(x_1 - a_i)\| + \|(1-\lambda)(x_2 - a_i)\| \tag{2.2}$$

for all $i \in \{1, \ldots, n\}$.

Since $(X, \|\cdot\|)$ is strictly convex, then there exists $t_i \in \mathbb{R}$ such that

$$\lambda(x_1 - a_i) = t_i (1-\lambda)(x_2 - a_i) \qquad \text{for all } i \in \{1, \ldots, n\}.$$

Suppose that $x_r \neq a_i$, $r \in \{1,2\}$ for all $i \in \{1,\dots,n\}$. Then by the above equality we get:

$$a_i\left(\lambda - t_i\left(1-\lambda\right)\right) = -\lambda x_1 + t_i\left(1-\lambda\right)x_2 \quad \text{for all } i \in \{1,\dots,n\}.$$

Now, if $t_i = \frac{\lambda}{1-\lambda}$, then we get $x_1 = x_2$ which contradicts the previous assumption, hence

$$a_i = -\frac{\lambda}{\lambda - t_i\left(1-\lambda\right)}x_1 + \frac{t_i\left(1-\lambda\right)}{\lambda - t_i\left(1-\lambda\right)}x_2 \quad \text{for all } i \in \{1,\dots,n\},$$

which shows that $a_i \in dr\left(x_1,x_2\right)$ for all $i \in \{1,\dots,n\}$, i.e., a contradiction to the fact that $\{a_1,\dots,a_n\}$ are non-colinear.

If there exists $i_0 \in \{1,\dots,n\}$ such that $x_2 = a_{i_0}$, then the argument goes likewise and we omit the details. ■

COROLLARY 1. *If $(H;\langle\cdot,\cdot\rangle)$ is an inner product space and $\{a_1,\dots,a_n\}$ $(n \geq 3)$ are non-colinear, then T is strictly convex on X.*

We also have:

PROPOSITION 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space and $\{a_1,\dots,a_n\}$ a set of n distinct vectors in H. Then T is Gâteaux differentiable on $H\backslash\{a_1,\dots,a_n\}$ and*

$$\frac{\partial T}{\partial y}\left(x\right) = \left\langle y, g\left(x\right)\right\rangle \quad \text{for all } x \in H\backslash\{a_1,\dots,a_n\} \quad \text{and } y \in H,$$

where

$$g : H\backslash\{a_1,\dots,a_n\} \to H, \quad g\left(x\right) := \sum_{i=1}^{n}\frac{x-a_i}{\|x-a_i\|}.$$

Proof. Let $y \in H$ and $t \in \mathbb{R}$. Then for all $x \in H\backslash\{a_1,\dots,a_n\}$ one has:

$$\begin{aligned}\lim_{t\to 0}\frac{T\left(x+ty\right)-T\left(x\right)}{t} &= \lim_{t\to 0}\frac{\sum_{i=1}^{n}\left(\|x+ty-a_i\| - \|x-a_i\|\right)}{t}\\ &= \lim_{t\to 0}\frac{1}{t}\sum_{i=1}^{n}\frac{\|x+ty-a_i\|^2 - \|x-a_i\|^2}{\|x+ty-a_i\| + \|x-a_i\|}\\ &= \lim_{t\to 0}\frac{1}{t}\sum_{i=1}^{n}\frac{2t\left\langle y, x-a_i\right\rangle + t^2\|y\|^2}{\|x+ty-a_i\| + \|x-a_i\|}\\ &= \sum_{i=1}^{n}\frac{\left\langle y, x-a_i\right\rangle}{\|x-a_i\|} = \left\langle y, g\left(x\right)\right\rangle,\end{aligned}$$

which proves the statement. ■

3. The existence and uniqueness of Torricellian point in inner product spaces

We start this section with the following decomposition theorem which holds in inner product spaces (not necessarily Hilbert spaces).

LEMMA 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space and G a finite-dimensional subspace in H. Then for all $x \in H$ there exists a unique element $x_1 \in G$ and a unique element $x_2 \in G^{\perp}$ (the orthogonal complement of G) such that:*

$$x = x_1 + x_2. \tag{3.1}$$

We denote this by $H = G \bigoplus G^{\perp}$.

Proof. Let $x \in H$. If $x \in G$, then $x = x + 0$ with $G \in G^{\perp}$ and the decomposition (3.1) holds.

If $x \in X \backslash G$, then by the well known theorem of the best approximation element from finite-dimensional linear subspaces, there exists $x_1 \in G$ such that $d(x, x_1) = d(x, G)$. Put $x_2 := x - x_1$. Then for all $y \in G$ and $\lambda \in \mathbb{K}$ one has

$$\|x_2 + \lambda y\| = \|x - x_1 + \lambda y\| = \|x - (x_1 - \lambda y)\| \geq \|x - x_1\| = \|x_2\|,$$

which is clearly equivalent with $x_2 \perp y$, i.e., $x_2 \in G^{\perp}$ and the representation (3.1) holds.

Now, suppose that there exists another representation $x = y_1 + y_2$ with $y_1 \in G$ and $y_2 \in G^{\perp}$. Then one gets:

$$G \ni x_1 - y_1 = y_2 - x_2 \in G^{\perp}.$$

Since $G \cap G^{\perp} = \{0\}$, we deduce that $x_1 = y_1$ and $x_2 = y_2$ and the uniqueness in decomposition (3.1) is proved. ∎

THEOREM 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space. If $\{a_1, \ldots, a_n\}$ is a set of $n \geq 3$ non-colinear distinct vectors in H, then $\mathcal{T}_X \{a_1, \ldots, a_n\}$ has a unique element.*

Proof. *The existence.* Consider $H_n := Sp[a_1, \ldots, a_n]$ the finite-dimensional subspace generated by $\{a_1, \ldots, a_n\}$. Then $2 \leq \dim H_n \leq n$. By the above lemma we have:

$$H = H_n \bigoplus H_n^{\perp}.$$

Now, let $x \in H \backslash H_n$. Then there exists a unique $x_1 \in H_n$ and a unique $x_2 \in H_n^{\perp}$ such that $x = x_1 + x_2$ and $x_2 \neq 0$.

For all $a \in H_n$ one has:

$$\|x-a\| = \|x_1 + x_2 - a\| = \|x_2 + (x_1 - a)\| = \left(\|x_2\|^2 + \|x_1 - a\|^2\right)^{\frac{1}{2}} > \|x_1 - a\|$$

because $\|x_2\| > 0$. Thus, for $a = a_i$, $i \in \{1, \dots, n\}$, we get:

$$T(x) = \sum_{i=1}^{n} \|x - a_i\| > \sum_{i=1}^{n} \|x_1 - a_i\| = T(x_1),$$

which shows that the vectors which minimize the functional T on H are in the finite-dimensional subspace H_n.

Let $x_0 \in H_n$. Since $\lim_{\|x\| \to \infty} T(x) = \infty$, there exists $r > 0$ such that $T(y) > T(x_0)$ for all $y \in H_n$ with $\|y\| > r$.

Denote $\bar{B}_n(0,r) = \bar{B}(0,r) \cap H_n$. Then $\bar{B}_n(0,r)$ is compact in H_n and since T is continuous on $\bar{B}_n(0,r)$, it follows that there exists an element $x_0 \in \bar{B}_n(0,r)$ such that:

$$T(x_0) = \inf_{x \in \bar{B}_n(0,r)} T(x) \le T(y) \quad \text{for all} \quad y \in H_n.$$

Now, by the above considerations we can state that x_0 is a point which minimizes the functional T on H.

The uniqueness. Suppose that there exist two vectors $x_1, x_2 \in H$ with $x_1 \neq x_2$ such that:

$$T(x_1) = T(x_2) = \inf_{x \in H} T(x).$$

Consider $x_t := tx_1 + (1-t)x_2$ with $t \in (0,1)$ (i.e., $x_t \neq x_1, x_2$). Since T is strictly convex (see Corollary 1) we have:

$$T(x_t) = T(tx_1 + (1-t)x_2) < tT(x_1) + (1-t)T(x_2) = T(x_1)$$

which contradicts the fact that x_1 minimizes the functional T on H.

The proof of the theorem is thus completed. ∎

4. Sets which are Torricellian degenerate in inner product spaces

We start with the following definition.

DEFINITION 2. Let $\{a_1, \dots, a_n\}$ be a set of n non-colinear distinct vectors. The set $\{a_1, \dots, a_n\}$ is said to be Torricellian degenerate if $\mathcal{T}_H\{a_1, \dots, a_n\} \in \{a_1, \dots, a_n\}$, i.e., there exists $a_j \in \{a_1, \dots, a_n\}$ so that $\mathcal{T}_H\{a_1, \dots, a_n\} = \{a_j\}$, $(j \in \{1, \dots, n\})$.

We have the following lemma which is of interest in itself.

LEMMA 2. *Let $F : X \to \mathbb{R}$ be a convex mapping in the normed linear space $(X, \|\cdot\|)$ and $x_0 \in X$. The following statements are equivalent:*

(i) *x_0 minimizes the functional F on X;*
(ii) *One has the inequality:*

$$\lim_{t\to 0+}\frac{F(x_0+tx)-F(x_0)}{t}\geq 0\geq \lim_{s\to 0-}\frac{F(x_0+sx)-F(x_0)}{s} \tag{4.1}$$

for all $x \in X$.

Proof. Consider the mapping $\Psi_{x_0,x}:\mathbb{R}\to\mathbb{R}$ given by

$$\Psi_{x_0,x}(t) := F(x_0+tx), \quad x\in X.$$

A simple calculation shows that $\Psi_{x_0,x}$ is convex on $\mathbb{R}$ for all $x\in X$, hence there exists the limits:

$$\lim_{t\to 0+}\frac{\Psi_{x_0,x}(t)-\Psi_{x_0,x}(0)}{t}, \quad \lim_{s\to 0-}\frac{\Psi_{x_0,x}(s)-\Psi_{x_0,x}(0)}{s}$$

and

$$\begin{aligned}\frac{\Psi_{x_0,x}(t)-\Psi_{x_0,x}(0)}{t} &\geq \lim_{t\to 0+}\frac{\Psi_{x_0,x}(t)-\Psi_{x_0,x}(0)}{t}\\ &\geq \lim_{s\to 0-}\frac{\Psi_{x_0,x}(s)-\Psi_{x_0,x}(0)}{s}\geq \frac{\Psi_{x_0,x}(s)-\Psi_{x_0,x}(0)}{s}\end{aligned}$$

for all $s<0<t$, i.e., one has

$$\begin{aligned}\frac{F(x_0+tx)-F(x_0)}{t} &\geq \lim_{t\to 0+}\frac{F(x_0+tx)-F(x_0)}{t}\\ &\geq \lim_{s\to 0-}\frac{F(x_0+sx)-F(x_0)}{s}\geq \frac{F(x_0+sx)-F(x_0)}{s}\end{aligned} \tag{4.2}$$

for all $t>0>s$ and $x\in X$.

"(i) $\Longrightarrow$ (ii)". If we assume that x_0 minimizes the functional F, then $F(x_0+tx)-F(x_0)\geq 0$ for all $t\in\mathbb{R}$ which implies, by (4.2), that the inequality (4.1) holds.

"(ii) $\Longrightarrow$ (i)". If (4.1) holds, then for all $t>0>s$, we have:

$$\frac{F(x_0+tx)-F(x_0)}{t}\geq 0\geq \frac{F(x_0+sx)-F(x_0)}{s}$$

which gives :

$$F(x_0+uy)\geq F(x_0) \quad \text{for all } u\in\mathbb{R} \text{ and } y\in X.$$

Choosing $u=1$ and $y=v-x_0$ (v is arbitrary in X), we get

$$F(v)\geq F(x_0), \quad \text{for each } v\in X$$

i.e., x_0 minimizes the functional F. ∎

THEOREM 2. *Let $(H; \langle , \cdot \rangle)$ be an inner product space over the real number field and $\{a_1, \dots, a_n\}$ a set of n non-colinear distinct vectors. The following statements are equivalent:*

(i) $\mathcal{T}_H \{a_1, \dots, a_n\} = \{a_j\}, \; j \in \{1, \dots, n\}$;
(ii) *One has the inequality:*

$$\left\| \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{a_i - a_j}{\|a_i - a_j\|} \right\| \leq 1. \tag{4.3}$$

Proof. We have:

$$\begin{aligned} T(x) - T(a_j) &= \|x - a_j\| + \sum_{\substack{i=1 \\ i \neq j}}^{n} (\|x - a_i\| - \|a_j - a_i\|) \\ &= \|x - a_j\| + \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{\|x - a_j\|^2 + 2\langle x - a_j, a_j - a_i \rangle}{\|x - a_i\| + \|a_j - a_i\|}. \end{aligned}$$

Let $y \in X$ and $t \in \mathbb{R}$. Then we have:

$$T(a_j + ty) - T(a_j) = |t| \, \|y\| + \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{t^2 \|y\|^2 + 2t \langle y, a_j - a_i \rangle}{\|ty + a_j - a_i\| + \|a_j - a_i\|}.$$

A simple calculation shows that:

$$\lim_{t \to 0+} \frac{T(a_j + ty) - T(a_j)}{t} = \|y\| - \left\langle y, \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{a_i - a_j}{\|a_i - a_j\|} \right\rangle$$

and

$$\lim_{s \to 0-} \frac{T(a_j + sy) - T(a_j)}{s} = -\|y\| - \left\langle y, \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{a_i - a_j}{\|a_i - a_j\|} \right\rangle.$$

"(i) $\Longrightarrow$ (ii)". If a_j minimizes the convex functional T, then by the implication "(i) $\Longrightarrow$ (ii)" of the above lemma, we have:

$$\|y\| - \left\langle y, \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{a_i - a_j}{\|a_i - a_j\|} \right\rangle \geq 0 \geq -\|y\| - \left\langle y, \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{a_i - a_j}{\|a_i - a_j\|} \right\rangle$$

for all $y \in X$ which yields that

$$\left| \left\langle y, \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{a_i - a_j}{\|a_i - a_j\|} \right\rangle \right| \leq \|y\| \quad \text{for all} \quad y \in X. \tag{4.4}$$

Put $u := \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} \frac{a_i - a_j}{\|a_i - a_j\|}$ and $f_u : X \to \mathbb{R}$, $f_u(y) = \langle y, u \rangle$. Then by (4.4) one has $|f_u(u)| \leq \|u\|$ which gives that $\|f_u\| \leq 1$.

On the other hand, it is clear that $\|f_u\| = \|u\|$ and the inequality (4.3) is thus proved.

"(ii) $\Longrightarrow$ (i)". Suppose that $\|u\| \leq 1$, then by Schwarz's inequality, we have

$$|\langle y, u \rangle| \leq \|u\| \|y\| \quad \text{for all} \quad y \in X,$$

which is clearly equivalent with (4.4), i.e.,

$$\lim_{t \to 0+} \frac{T(a_j + ty) - T(a_j)}{t} \geq 0 \geq \lim_{s \to 0-} \frac{T(a_j + sy) - T(a_j)}{s}$$

and by the implication "(ii) $\Longrightarrow$ (i)" we conclude that a_j minimizes the functional T, i.e., $\mathcal{T}_H \{a_1, \ldots, a_n\} = \{a_j\}$. ∎

5. Characterisation of Torricellian points for non-degenerate sets

In this section we point out a characterisation result for the Torricellian point associated with a non-degenerate set of n distinct non-colinear vectors in an inner product space.

We can state and prove the following theorem:

THEOREM 3. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $\{a_1, \ldots, a_n\} \subset H$ a non-degenerate set of $n \geq 3$ non-colinear distinct vectors. If $x_0 \in H$, the following statements are equivalent:*

(i) $x_0 \in \mathcal{T}_H \{a_1, \ldots, a_n\}$

(ii) x_0 *is a solution of the equation:*

$$g(x) = \sum_{i=1}^{n} \frac{x - a_i}{\|x - a_i\|} = 0, \qquad x \in H; \tag{5.1}$$

(iii) x_0 *is a solution of the system:*

$$\begin{cases} \sum_{i=1}^{n} \cos \varphi_{i1}(x) = 0, \\ \dots\dots\dots\dots\dots\dots & x \in H; \\ \sum_{i=1}^{n} \cos \varphi_{in}(x) = 0 \end{cases} \tag{5.2}$$

where

$$\cos \varphi_{ij}(x) = \frac{\langle x - a_i, x - a_j \rangle}{\|x - a_i\| \|x - a_j\|}, \qquad i, j \in \{1, \ldots, n\},$$

and in all cases x_0 is unique.

Proof. "(i) $\Longrightarrow$ (ii)". If x_0 minimizes the functional T, then by the implication "(i) $\Longrightarrow$ (ii)" of Lemma 2, we deduce:

$$\lim_{t\to 0+}\frac{T(x_0+tx)-T(x_0)}{t}\geq 0\geq \lim_{s\to 0-}\frac{T(x_0+sx)-T(x_0)}{s}. \tag{5.3}$$

Since the mapping T is Gâteaux differentiable, hence by Proposition 3 we have:

$$\frac{\partial T(x_0)}{\partial x}=\langle x,g(x_0)\rangle \qquad \text{for all } x\in X.$$

By the relation (5.3) we get $\frac{\partial T(x_0)}{\partial x}=0$ for all $x\in X$, i.e., $g(x_0)=0$, which shows that x_0 is a solution of the equation (5.1).

"(ii) $\Longrightarrow$ (iii)". Suppose that x_0 is a solution of (5.1), then

$$\sum_{i=1}^{n}\frac{x_0-a_i}{\|x_0-a_i\|}=0,$$

which yields that:

$$\left\langle\sum_{i=1}^{n}\frac{x_0-a_i}{\|x_0-a_i\|},\frac{x_0-a_j}{\|x_0-a_j\|}\right\rangle=0 \qquad \text{for all } j\in\{1,\dots,n\}$$

i.e.,

$$\sum_{i=1}^{n}\frac{\langle x_0-a_i,x_0-a_j\rangle}{\|x_0-a_i\|\,\|x_0-a_j\|}=0 \qquad \text{for all } j\in\{1,\dots,n\},$$

which means that x_0 is a solution of the system (5.2).

"(iii) $\Longrightarrow$ (ii)". If x_0 is a solution of the system (5.2), then

$$\left\langle\sum_{i=1}^{n}\frac{x_0-a_i}{\|x_0-a_i\|},\frac{x_0-a_j}{\|x_0-a_j\|}\right\rangle=0 \qquad \text{for all } j\in\{1,\dots,n\},$$

i.e.,

$$\left\langle g(x_0),\frac{x_0-a_j}{\|x_0-a_j\|}\right\rangle=0 \qquad \text{for all } j\in\{1,\dots,n\}.$$

Summing over j from 1 to n, we get

$$0=\langle g(x_0),g(x_0)\rangle=\|g(x_0)\|^2$$

which means that x_0 is a solution of the equation (5.1), then

$$\frac{\partial T(x_0)}{\partial x}=\langle x,g(x_0)\rangle=0,$$

which, by Lemma 2, shows that x_0 minimizes the functional T, i.e., $x_0\in \mathcal{T}_H\{a_1,\dots,a_n\}$.

The uniqueness of the solution for the equations (5.1) and (5.2) is obvious by the uniqueness of the Torricellian point associated with a set of $n \geq 3$ non-colinear distinct vectors in H, and the proof is complete. ∎

6. The case of three vectors in inner product spaces

It is natural to consider the case of $n = 3$ vectors in inner product spaces and show that the classical result due to Torricelli can be naturally recaptured from the more general results stated above.

If $\{a_1, a_2, a_3\}$ are colinear and

$$a_i = \lambda_i a + (1 - \lambda_i b) \quad \text{with} \quad i = \{1, 2, 3\}, \; \lambda_1 < \lambda_2 < \lambda_3 \quad \text{and} \quad a, b \in H$$

then one can easily show that $\mathcal{T}_H \{a_1, a_2, a_3\} = \{a_2\}$.

The case of Torricellian degenerated vectors is embodied in the following proposition (see also [4]):

PROPOSITION 4. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $\{a_1, a_2, a_3\}$ a set of three non-colinear vectors in H. The following statements are equivalent:*

(i) $\mathcal{T}_H \{a_1, a_2, a_3\} = \{a_2\}$;
(ii) *The angle θ between $a_1 - a_2$ and $a_1 - a_3$ is greater that $\frac{2\pi}{3}$.*

Proof. By Theorem 2 one has that $a_2 \in \mathcal{T}_H \{a_1, a_2, a_3\}$ if and only if

$$\left\| \frac{a_1 - a_2}{\|a_1 - a_2\|} + \frac{a_3 - a_2}{\|a_3 - a_2\|} \right\| \leq 1,$$

which is equivalent with

$$\frac{\|a_1 - a_2\|^2}{\|a_1 - a_2\|^2} + 2 \left\langle \frac{a_1 - a_2}{\|a_1 - a_2\|}, \frac{a_3 - a_2}{\|a_3 - a_2\|} \right\rangle + \frac{\|a_3 - a_2\|^2}{\|a_3 - a_2\|^2} \leq 1$$

i.e.,

$$\cos\theta = \frac{\langle a_1 - a_2, a_3 - a_2 \rangle}{\|a_1 - a_2\| \, \|a_3 - a_2\|} \leq -\frac{1}{2},$$

which shows that $\theta \in \left[\frac{2\pi}{3}, \pi\right)$. ∎

The case of non-degenerate sets is embodied in the following proposition (see also [4]):

PROPOSITION 5. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $\{a_1, a_2, a_3\}$ a set of three non-colinear non-degenerate vectors in H. The following statements are equivalent:*

(i) $\mathcal{T}_H \{a_1, a_2, a_3\} = \{x_0\}$.
(ii) *We have $\theta_{12} = \theta_{23} = \theta_{31} = \frac{2\pi}{3}$, where θ_{ij} is the angle between $a_i - x_0$, $a_j - x_0$, $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$.*

Proof. By Theorem 3, we have that $\mathcal{T}_H\{a_1,a_2,a_3\} = \{x_0\}$ iff x_0 is the unique solution of the system:

$$\begin{cases} \cos\theta_{11}(x) + \cos\theta_{12}(x) + \cos\theta_{13}(x) = 0 \\ \cos\theta_{21}(x) + \cos\theta_{22}(x) + \cos\theta_{23}(x) = 0 \\ \cos\theta_{31}(x) + \cos\theta_{32}(x) + \cos\theta_{33}(x) = 0 \end{cases}$$

where $\cos\varphi_{ij}(x) = \frac{\langle x-a_i, x-a_j\rangle}{\|x-a_i\|\|x-a_j\|}$.

This system is equivalent with

$$\begin{cases} \cos\theta_{12}(x) + \cos\theta_{31}(x) = -1 \\ \cos\theta_{12}(x) + \cos\theta_{23}(x) = -1 \\ \cos\theta_{31}(x) + \cos\theta_{23}(x) = -1 \end{cases}$$

which gives us $\cos\theta_{12} = \cos\theta_{23} = \cos\theta_{31} = -\frac{1}{2}$ and the proposition is proved. ∎

References

[1] D. Amir, *Characterisation of Inner Product Spaces*, Birkhauser-Verlag, Basel, 1986.

[2] M. Baronti, E. Casini and P. L. Papini, *Equilateral sets and their central points*, Rend. di Mat., Ser VII, 13 (1993), 133–148.

[3] B. Beauzamy and M. Maurey, *Points minimax et ensembles optimeaux dans les espaces de Banach*, J. Funct. Anal. 24 (1977), 107–139.

[4] S. S. Dragomir, D. Comănescu, Y. J. Cho and S. S. Kim, *On Torricelli's problem in inner product spaces*, Austral. Math. Soc. Gaz. 27 (4) (2000), 173–180.

[5] A. Daurat, A. Del Longo and M. Nivat, *Medians of discrete sets according to a linear distance*, Discrete Comput. Geom. 23 (2000), no. 4, 465–483.

[6] J. Gatial and P. Kaprálik, *On median point of the system of elements of A-structure*, Math. Slovaca 51 (2001), no. 3, 275–280.

[7] R. B. Holmes, *A Course on Optimization and Best Approximation*, Lecture Notes, Math. 257, Springer-Verlag, Berlin, 1972.

S. S. Dragomir
SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS
VICTORIA UNIVERSITY
PO BOX 14428, MCMC 8001, VIC, AUSTRALIA
e-mail: Sever.Dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir/

D. Comănescu
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE WEST TIMIŞOARA
B-dul. V. Pârvan, No. 4
RO-1900 TIMIŞOARA, ROMANIA
e-mail: comanescu@hilbert.math.uvt.ro

Received October 11, 2007.