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GLOBAL SOLUTIONS FOR VOLTERRA ORDINARY AND RETARDED INTEGRAL EQUATIONS

Abstract. Using a generalization of Darbo's fixed point theorem, we obtain the existence of global solutions for nonlinear Volterra-type integral equations in Banach spaces. The involved functions are supposed to be continuous only with respect to some variables, integrability or essential boundedness conditions being also imposed. Our result improves the similar result given in [10] (where uniform continuity was required), as well as those referred by the authors of the cited paper. Finally, following the same ideas, the existence of continuous solutions is proved for a Volterra-type retarded integral equation, under less restrictive assumptions than in the others related results in literature.

1. Introduction

The importance of Volterra-type integral equations in solving various nonlinear problems in science determined many authors to study the existence of (continuous or better) solutions (see e.g. [5], [6], [9], [10], [11], [12], [13]). Different fixed point theorems were applied in order to obtain the existence results: Darbo's theorem (in [5]) and a generalization of it (in [10]), Mönch's fixed point theorem (in [9]) and some Mönch-type results (in [12] and [13]). In the present paper, applying a Darbo's fixed point theorem established in [10], we obtain the existence of global continuous solutions for the nonlinear Volterra integral equation

$$u(t)=\int_0^t G(t,s)f\Big(s,u(s),\int_0^s k(s,\tau)u(\tau)d\tau,\int_0^1 h(s,\tau)u(\tau)d\tau\Big)ds.$$

The setting is that of a separable Banach space and the assumptions made on the operators are much weaker than those made by the previously cited

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authors for similar results. Mainly, we require some partial continuity of f and G, along with some integrability and boundedness conditions. Essential is the use of Kuratowski measure of noncompactness.

Applying the same method, retarded nonlinear equations are then studied and the existence of continuous solutions is obtained under similar assumptions. The result is more general than the related results we know (see e.g. on the real line [7] and [3] and in the case of a general Banach space via Bochner integral [4]).

2. Notations and preliminary facts

Through this paper, X is a separable Banach space with the norm $\|\cdot\|$ and T_R is its ball of radius R. The space $C([0,1],X)$ of continuous functions is endowed with the usual (Banach space) norm $\|f\|_C = \sup\limits_{t\in[0,1]} \|f(t)\|$. By $\left(L^1([0,1],X), \|\|_{L^1}\right)$ we denote the space of Bochner integrable X-valued functions and by $\|\cdot\|_{L^\infty}$ the essential supremum of a real function. For the Kuratowski measure of noncompactness α we refer the reader to [8].
In [10] the following generalization of Darbo's fixed point theorem was given:

LEMMA 1. *Let F be a closed convex subset of a Banach space and the operator $A: F \to F$ be continuous with $A(F)$ bounded. Suppose that for the sequence defined for any bounded $B \subset F$ by*

$$\tilde{A}^1(B) = A(B) \quad \text{and} \quad \tilde{A}^n(B) = A\left(\overline{co}\left(\tilde{A}^{n-1}(B)\right)\right), \forall n \geq 2$$

there exist a positive constant $0 \leq k < 1$ and a number n_0 such that for every bounded $B \subset F$, $\alpha(\tilde{A}^{n_0}(B)) \leq k\alpha(B)$. Then A has a fixed point.

Let us make the following

REMARK 2. If the Banach space is separable, then the previously considered operator A has a fixed point if it satisfies the inequality $\alpha(\tilde{A}^{n_0}(B)) \leq k\alpha(B)$ for every bounded countable $B \subset F$.

Indeed, for every positive integer n, $\alpha(\tilde{A}^n(\overline{B})) = \alpha(\tilde{A}^n(B))$:

$$\alpha\left(\tilde{A}^1\left(\overline{B}\right)\right) = \alpha\left(A(\overline{B})\right) \leq \alpha\left(\overline{A(B)}\right) = \alpha(A(B)) = \alpha\left(\tilde{A}^1(B)\right)$$

thanks to the continuity of A and, for every $n \geq 2$, it follows by induction that $\tilde{A}^n(\overline{B}) = \tilde{A}^n(B)$, since

$$\tilde{A}^2(\overline{B}) = A\left(\overline{co}\left(A(\overline{B})\right)\right) \subset A\left(\overline{co}\left(\overline{A(B)}\right)\right) = A\left(\overline{co}(A(B))\right) = \tilde{A}^2(B).$$

THEOREM 3. ([2], see also [1] for the Hausdorff measure of noncompactness) *Let $\mathcal{K} \subset C([0,1],X)$ be bounded and equi-continuous. Then $\alpha(\mathcal{K}) = \sup\limits_{t\in[0,1]} \alpha(\mathcal{K}(t))$.*

We will use a property of sequences of integrable functions which can be found in [8] (see also [12]):

THEOREM 4. *Assume that E is a Banach space and $M \subset L^1([0,1],E)$ is countable with $\|u(t)\| \le h(t)$ for all $u \in M$ a.e. for some $h \in L^1([0,1],\mathbb{R})$. Then $\alpha(M(\cdot)) \in L^1([0,1],\mathbb{R})$ and*

$$\alpha\Big(\int_0^t M(s)ds\Big) \le 2\int_0^t \alpha(M(s))ds, \quad \forall t \in [0,1].$$

In the final part, an existence result will be given for retarded nonlinear Volterra integral equations. In order to do this, we need some basic facts on this type of equations. Let $\overline{r}$ be a positive number and, following [7], we define, for each $s \in [0,1]$, the function which "contains" as information the history of the system $u_s : [-\overline{r}, 0] \to X$ by $u_s(\theta) = u(s+\theta)$.

3. Main results

THEOREM 5. *Let X be a real separable Banach space, $f : [0,1] \times X^3 \to X$ and $h, k, G : [0,1]^2 \to \mathbb{R}$ satisfy the following conditions:*

i) for each $t \in [0,1]$, $h(t,\cdot)$, $k(t,\cdot)$ and $G(t,\cdot)$ are in $L^\infty([0,1],\mathbb{R})$, the mapping $t \mapsto G(t,\cdot)$ is L^∞-continuous and $t \mapsto h(t,\cdot)$, $t \mapsto k(t,\cdot)$ are L^∞-bounded;

ii) f is measurable with respect to the first variable and:

ii1) for each positive R, one can find $\phi_R \in L^1([0,1],\mathbb{R}_+)$ such that

$$\|f(t,x,y,z)\| \le \phi_R(t), \quad \forall t \in [0,1], x, y, z \in T_R;$$

ii2) there are three positive integrable functions satisfying

$$\|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\| \le \sum_{i=1}^{3} \overline{L}_i(t)\|x_i - y_i\|, \ \forall t \in [0,1], x_i, y_i \in X;$$

ii3) denoting by $M(R) = \|\phi_R\|_{L^1}$,

$$\limsup_{R\to\infty} \frac{M(R)}{R} < \frac{1}{\sup\limits_{t\in[0,1]} \|G(t,\cdot)\|_{L^\infty} \max\Big\{1, \sup\limits_{t\in[0,1]} \|k(t,\cdot)\|_{L^\infty}, \sup\limits_{t\in[0,1]} \|h(t,\cdot)\|_{L^\infty}\Big\}};$$

iii) there exist three positive integrable functions $L_i(t)$ such that for any bounded $D_i \subset X$ and any $t \in [0,1]$,

$$\alpha\left(f(t,D_1,D_2,D_3)\right) \le \sum_{i=1}^{3} L_i(t)\alpha(D_i).$$

Then the Volterra-type integral equation

$$u(t) = \int_0^t G(t,s) f\Big(s, u(s), \int_0^s k(s,\tau)u(\tau)d\tau, \int_0^1 h(s,\tau)u(\tau)d\tau\Big) ds$$

has a continuous solution on $[0,1]$.

Proof. We follow the ideas of proof of Theorem 3.1 in [10]. In order to simplify the calculation, denote by $(Tu)(t) = \int_0^t k(t,s)u(s)ds$ and by $(Su)(t) = \int_0^1 h(t,s)u(s)ds$.

By hypothesis i), let $a = \sup_{t\in[0,1]} \|k(t,\cdot)\|_{L^\infty}$, $b = \sup_{t\in[0,1]} \|G(t,\cdot)\|_{L^\infty}$ and $c = \sup_{t\in[0,1]} \|h(t,\cdot)\|_{L^\infty}$. One can find $0 < r < \dfrac{1}{b\max\{1,a,c\}}$ and $R_0 > 0$ such that for any $R \geq R_0 \max\{1,a,c\}$,

$$M(R) < rR.$$

Consider $A : C([0,1],X) \to C([0,1],X)$ defined by

$$Au(t) = \int_0^t G(t,s) f(s, u(s), (Tu)(s), (Su)(s)) ds, \quad \forall u \in C([0,1],X).$$

We claim that A is a continuous operator mapping the closed ball B_{R_0} of $C([0,1],X)$ into itself. Indeed, for any $u \in C([0,1],X)$ with $\|u\|_C \leq R_0$, we have

$$\begin{aligned}\|Au\|_C &\leq \sup_{t\in[0,1]} \|G(t,\cdot)\|_{L^\infty} \int_0^t \|f(s,u(s),(Tu)(s),(Su)(s))\| ds \\ &\leq b\int_0^t \phi_{R_0 \max\{1,a,c\}}(s) ds\end{aligned}$$

since $\|Tu\|_C \leq aR_0$ and $\|Su\|_C \leq cR_0$.

Therefore

$$\|Au\|_C \leq bM(R_0\max\{1,a,c\}) < brR_0 \max\{1,a,c\} < R_0.$$

Concerning the continuity, one can see that

$$\begin{aligned}\|Au_1 - Au_2\|_C &= \sup_{t\in[0,1]} \|Au_1(t) - Au_2(t)\| \\ &= \sup_{t\in[0,1]} \Big\| \int_0^t G(t,s)(f(s,u_1(s),(Tu_1)(s),(Su_1)(s)) \\ &\qquad - f(s,u_2(s),(Tu_2)(s),(Su_2)(s)))ds \Big\|\end{aligned}$$

$$\leq b\int_0^t \|f(s,u_1(s),(Tu_1)(s),(Su_1)(s)) - f(s,u_2(s),(Tu_2)(s),(Su_2)(s))\|\,ds$$

$$\leq b\int_0^t \overline{L}_1(s)\|u_1(s)-u_2(s)\|ds$$

$$+b\int_0^t \overline{L}_2(s)\|(Su_1)(s)-(Su_2)(s)\| + \overline{L}_3(s)\|(Tu_1)(s)-(Tu_2)(s)\|ds$$

$$\leq b\int_0^t \overline{L}_1(s)ds\|u_1-u_2\|_C$$

$$+b\Big(\int_0^t \overline{L}_2(s)ds\|Tu_1-Tu_2\|_C + \int_0^t \overline{L}_3(s)ds\|Su_1-Su_2\|_C\Big)$$

$$\leq b\int_0^t \overline{L}_1(s)ds\|u_1-u_2\|_C + b\int_0^t \overline{L}_2(s)ds\; t\|k(t,\cdot)\|_{L^\infty}\|u_1-u_2\|_C$$

$$+b\int_0^t \overline{L}_3(s)ds\|h(t,\cdot)\|_{L^\infty}\|u_1-u_2\|_C.$$

By hypothesis $i)$,

$$\|Au_1-Au_2\|_C \leq b\left(\|\overline{L}_1\|_{L^1} + a\|\overline{L}_2\|_{L^1} + c\|\overline{L}_3\|_{L^1}\right)\|u_1-u_2\|_C$$

whence the continuity of A follows.

We prove now that $F=\overline{co}A(B_{R_0})$ is equi-continuous. For this, it suffices to show by Lemma 2.1 in [10], that $A(B_{R_0})$ is equi-continuous. But

$$\|Au(t_1)-Au(t_2)\|$$

$$= \Big\|\int_0^{t_1} G(t_1,s)f(s,u(s),(Tu)(s),(Su)(s))ds$$

$$-\int_0^{t_2} G(t_2,s)f(s,u(s),(Tu)(s),(Su)(s))ds\Big\|$$

$$\leq \Big\|\int_0^{t_1}(G(t_1,s)-G(t_2,s))f(s,u(s),(Tu)(s),(Su)(s))ds\Big\|$$

$$+\Big\|\int_{t_1}^{t_2} G(t_2,s)f(s,u(s),(Tu)(s),(Su)(s))ds\Big\|$$

$$\leq \|G(t_1,\cdot)-G(t_2,\cdot)\|_{L^\infty}M(R_0\max\{1,a,c\})$$

$$+b\int_{t_1}^{t_2} \phi_{R_0\max\{1,a,c\}}(s)ds, \forall u\in B_{R_0}$$

and, by hypothesis $i)$ and $ii1)$, this can be made less than some fixed ε for t_1, t_2 with an appropriately small distance between them. So, the equicontinuity follows.

Obviously, $A : F \to F$ is bounded and continuous.

Let us prove, by the method of mathematical induction, that for every $B \subset F$ and any $n \in \mathbb{N}$, $\tilde{A}^n(B) \subset A(B_{R_0})$, so it is bounded and equicontinuous. For $n = 1$, this is valid, since $A(B) \subseteq A(F) \subset A(B_{R_0})$. Suppose now that this is true for $n-1$ and prove it for n: $\tilde{A}^n(B) = A(\overline{co}(\tilde{A}^{n-1}(B))) \subset A(\overline{co}(A(B_{R_0}))) \subset A(\overline{co}(B_{R_0})) = A(B_{R_0})$.

By Theorem 3,

$$\alpha\left(\tilde{A}^n(B)\right) = \sup_{t\in[0,1]} \alpha\left(\tilde{A}^n(B)(t)\right), \quad \forall n \in \mathbb{N}.$$

Similarly to the second part of the proof of Theorem 3.1 in [10], one can show that there exist a constant $0 \le k < 1$ and a positive integer n_0 such that for any $B \subset F$, $\alpha(\tilde{A}^{n_0}(B)) \le k\alpha(B)$.

Let (v_n) be an arbitrary countable subset of $\tilde{A}^1(B) = A(B)$. There exists a sequence $(u_n) \subset B$ such that $v_n = Au_n$. Hypothesis $ii1)$ allows us to use Theorem 4 and to obtain that

$$\begin{aligned}\alpha\left(\{v_n(t), n \in \mathbb{N}\}\right) &= \alpha\left(\{Au_n(t), n \in \mathbb{N}\}\right)\\ &= \alpha\Big(\int_0^t G(t,s) f(s, \{u_n(s), n\}, \{(Tu_n)(s), n\}, \{(Su_n)(s), n\}) ds\Big)\\ &\le 2b \int_0^t \alpha\left(f(s, \{u_n(s), n\}, \{(Tu_n)(s), n\}, \{(Su_n)(s), n\})\right) ds.\end{aligned}$$

By hypothesis $iii)$, it follows that

$$\begin{aligned}&\alpha(\{v_n(t), n \in \mathbb{N}\})\\ &\quad \le 2b \int_0^t L_1(s)\alpha(\{u_n(s)\}) + L_2(s)\alpha(\{(Tu_n)(s)\}) + L_3(s)\alpha(\{(Su_n)(s)\}) ds.\end{aligned}$$

Applying again Theorem 4 gives that

$$\begin{aligned}\alpha(\{v_n(t), n \in \mathbb{N}\}) &\le 2b \int_0^t \left(L_1(s) + 2aL_2(s) + 2cL_3(s)\right) \alpha(\{u_n(s)\}) ds\\ &\le 2b \int_0^t \left(L_1(s) + 2aL_2(s) + 2cL_3(s)\right) ds\, \alpha(B).\end{aligned}$$

Since the Banach space is separable and the Kuratowski measure of noncompactness is preserved when the set under discussion is replaced by its closure, then

$$\alpha\left(\tilde{A}^1(B)(t)\right) \leq 2b\int_0^t (L_1(s) + 2aL_2(s) + 2cL_3(s)))\, ds\, \alpha(B).$$

As $L_1(s) + 2aL_2(s) + 2cL_3(s)) \in L^1([0,1], \mathbb{R})$ and continuous functions are dense in $L^1([0,1], \mathbb{R})$ with respect to the usual norm, one can make an evaluation of the form $\alpha(\tilde{A}^1(B)(t)) \leq (\varepsilon + Mt)\alpha(B)$. It can be shown, by mathematical induction, that

$$\alpha\left(\tilde{A}^m(B)(t)\right) \leq \left(\varepsilon^m + C_m^1 \varepsilon^{m-1} Mt + \cdots + \frac{(Mt)^m}{m!}\right) \alpha(B), \quad \forall t \in [0,1].$$

Suppose the inequality is valid for m and prove it for $m+1$. For any countable subset (v_n) of $\tilde{A}^{m+1}(B) = A\left(\overline{co}\left(\tilde{A}^m(B)\right)\right)$, there exist $(u_n) \subset \overline{co}\left(\tilde{A}^m(B)\right)$ such that $v_n = Au_n$. Then, as before,

$$\alpha(\{v_n(t), n \in \mathbb{N}\}) \leq 2b\int_0^t (L_1(s) + 2aL_2(s) + 2cL_3(s))\, ds\, \alpha\left(\tilde{A}^m(B)\right),$$

whence

$$\alpha\left(\tilde{A}^{m+1}(B)(t)\right) \leq 2b\int_0^t (L_1(s) + 2aL_2(s) + 2cL_3(s))\, ds\, \alpha\left(\tilde{A}^m(B)\right)$$

and so the assertion follows. The rest of the calculus goes as in [10]: for some integer n_0 the evaluation term $\varepsilon^{n_0} + C_{n_0}^1 \varepsilon^{n_0 - 1} Mt + \cdots + \frac{(Mt)^{n_0}}{n_0!}$ can be made less than 1 and so, by Lemma 1, A has a fixed point, which is a global solution to our equation. ∎

As a particular case, when $G(t,s) = 1$ for every $t, s \in [0,1]$, one can deduce an existence result for integral equations already studied in literature (see [10] and the references therein).

THEOREM 6. *Let X be a real separable Banach space, $f : [0,1] \times X^3 \to X$ and $h, k : [0,1]^2 \to \mathbb{R}$ satisfy the assumptions ii1), ii2) and iii) of Theorem 5, together with:*

i′) for each $t \in [0,1]$, $h(t,\cdot)$, $k(t,\cdot)$ are in $L^\infty([0,1], \mathbb{R})$ and the applications $t \mapsto h(t,\cdot)$, $t \mapsto k(t,\cdot)$ are L^∞-bounded;

ii′3) denoting by $M(R) = \|\phi_R\|_{L^1}$,

$$\limsup_{R \to \infty} \frac{M(R)}{R} < \frac{1}{\max\{1, \sup_{t \in [0,1]} \|k(t,\cdot)\|_{L^\infty}, \sup_{t \in [0,1]} \|h(t,\cdot)\|_{L^\infty}\}};$$

Then the integral equation

$$u(t) = \int_0^t f\Big(s, u(s), \int_0^s k(s,\tau)u(\tau)d\tau, \int_0^1 h(s,\tau)u(\tau)d\tau\Big)ds$$

has a continuous solution on $[0,1]$.

Following the same method, with some minor modifications, the existence of global solutions for nonlinear Volterra retarded integral equations can be obtained.

THEOREM 7. *Let* X *be a real separable Banach space,* $f : [0,1] \times X \times C([-\overline{r},0],X) \times X \to X$ *and* $k, G : [0,1]^2 \to \mathbb{R}$ *satisfy the following conditions:*

i) for each $t \in [0,1]$, $k(t,\cdot)$ *and* $G(t,\cdot)$ *are in* $L^\infty([0,1],\mathbb{R})$, *the mapping* $t \mapsto G(t,\cdot)$ *is* L^∞*-continuous and* $t \mapsto k(t,\cdot)$ *is* L^∞*-bounded;*

ii) f *is measurable with respect to the first variable and:*

ii1) for each positive R, *one can find* $\phi_R \in L^1([0,1],\mathbb{R}_+)$ *such that*

$$\|f(t,x,y,z)\| \le \phi_R(t), \quad \forall t \in [0,1], x, z \in T_R, y \in B_R;$$

ii2) there are three positive integrable functions satisfying, for all $t \in [0,1]$, $x_1, x_3, y_1, y_3 \in X$, $x_2, y_2 \in C([-\overline{r},0],X)$,

$$\|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\| \le \sum_{i=1,3} \overline{L}_i(t)\|x_i - y_i\| + \overline{L}_2(t)\|x_2 - y_2\|_C;$$

ii3) denoting by $M(R) = \|\phi_R\|_{L^1}$,

$$\limsup_{R\to\infty} \frac{M(R)}{R} < \frac{1}{\sup\limits_{t\in[0,1]} \|G(t,\cdot)\|_{L^\infty} \max\Big\{1, \sup\limits_{t\in[0,1]} \|k(t,\cdot)\|_{L^\infty}\Big\}};$$

iii) there exist three positive integrable functions such that for any bounded $D_1, D_3 \subset X$, $D_2 \subset C([-\overline{r},0],X)$ *and any* $t \in [0,1]$,

$$\alpha\left(f(t,D_1,D_2,D_3)\right) \le \sum_{i=1}^{3} L_i(t)\alpha(D_i).$$

Then the retarded integral equation

$$u(t) = \int_0^t G(t,s) f\Big(s, u(s), u_s, \int_0^s k(s,\tau)u(\tau)d\tau\Big)ds$$

has a continuous solution on $[0,1]$.

Proof. Denote by $(Tu)(t) = \int_0^t k(t,s)u(s)ds$. There is $0 < r < \frac{1}{b\max\{1,a\}}$ and $R_0 > 0$ such that for any $R \ge R_0 \max\{1,a\}$, $M(R) < rR$.

Let ξ be an arbitrarily chosen constant function on $[-\overline{r}, 0]$ with $\|\xi\| \le R_0$ and make the convention that whenever for a continuous function on $[0,1]$ the function u_s will intervene, u will be considered cintinued to $[-\overline{r}, 0]$ by ξ.

Consider $A : C([0,1], X) \to C([0,1], X)$ defined by

$$Au(t) = \int_0^t G(t,s) f(s, u(s), u_s, (Tu)(s))ds, \quad \forall u \in C([0,1], X).$$

The operator A maps the closed ball B_{R_0} of $C([0,1], X)$ into itself since, for any $u \in C([0,1], X)$ with $\|u\|_C \le R_0$, $\|u.\|_C \le R_0$ and $\|Tu\|_C \le aR_0$ and so,

$$\begin{aligned}\|Au\|_C &\le b\int_0^t \phi_{R_0 \max\{1,a\}}(s)ds \\ &\le bM(R_0 \max\{1,a\}) < brR_0 \max\{1,a\} < R_0.\end{aligned}$$

The continuity of A follows from

$$\begin{aligned}\|Au_1 - Au_2\|_C &\le b\int_0^t \overline{L}_1(s)ds\|u_1 - u_2\|_C \\ &\quad + b\Big(\int_0^t \overline{L}_2(s)ds\|(u_1)_s - (u_2)_s\|_C + \int_0^t \overline{L}_3(s)ds\|Tu_1 - Tu_2\|_C\Big) \\ &\le b\int_0^t\Big(\overline{L}_1(s)ds + \int_0^t \overline{L}_2(s)ds + a\int_0^t \overline{L}_3(s)ds\Big)\|u_1 - u_2\|_C.\end{aligned}$$

The proof of the equi-continuity of $F = \overline{co}A(B_{R_0})$ does not necessitate modifications, neither the proof of the equality

$$\alpha\left(\tilde{A}^n(B)\right) = \sup_{t\in[0,1]} \alpha\left(\tilde{A}^n(B)(t)\right), \quad \forall n \in \mathbb{N}.$$

One can show that there exist a constant $0 \le k < 1$ and a positive integer n_0 such that for any $B \subset F$, $\alpha(\tilde{A}^{n_0}(B)) \le k\alpha(B)$.

Let $v_n = Au_n$ be an arbitrary countable subset of $\tilde{A}^1(B) = A(B)$. From Theorem 4,

$$\begin{aligned}&\alpha\left(\{v_n(t), n \in \mathbb{N}\}\right) = \\ &\le 2b\int_0^t \alpha\left(f(s, \{u_n(s), n\}, \{(u_n)_s, n\}, \{(Tu_n)(s), n\})\right)ds \\ &\le 2b\int_0^t L_1(s)\alpha(\{u_n(s)\}) + L_2(s)\alpha(\{(u_n)_s\}) + L_3(s)\alpha(\{(Tu_n)(s)\})ds \\ &\le 2b\int_0^t \left(L_1(s) + L_2(s) + 2aL_3(s))\right)ds\; \alpha(B)\end{aligned}$$

since $\alpha(\{(u_n)_s\}) = \sup_{\theta\in[-\overline{r},0]} \alpha\left(\{(u_n)_s(\theta), n\}\right) = \sup_{\theta\in[-\overline{r},0]} \alpha\left(\{u_n(s+\theta), n\}\right) \leq \alpha(B)$. This implies that

$$\alpha\left(\tilde{A}^1(B)(t)\right) \leq 2b\int_0^t (L_1(s) + L_2(s) + 2aL_3(s)))\, ds\, \alpha(B)$$

and, similarly,

$$\alpha\left(\tilde{A}^{m+1}(B)(t)\right) \leq 2b\int_0^t (L_1(s) + L_2(s) + 2aL_3(s))\, ds\, \alpha\left(\tilde{A}^m(B)\right)$$

By mathematical induction,

$$\alpha\left(\tilde{A}^m(B)(t)\right) \leq \left(\varepsilon^m + C_m^1\varepsilon^{m-1}Mt + \cdots + \frac{(Mt)^m}{m!}\right)\alpha(B), \quad \forall t \in [0,1].$$

The rest of the calculus goes as in [10]: for some integer n_0 the evaluation term $\varepsilon^{n_0} + C_{n_0}^1\varepsilon^{n_0-1}Mt + \cdots + \frac{(Mt)^{n_0}}{n_0!}$ can be made less than 1 and so, by Lemma 1, A has a fixed point. ∎

REMARK 8. Our Theorems 5 and 6 improve the related results given in [10], as well as those cited therein: [5], [6], [9] and [11]. On the other hand, in the theory of retarded equations in Banach spaces, Theorem 7 is, as far as we know, more general than all results in literature (see e.g. [7], [3] and [4]).

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