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OSCILLATION CRITERIA FOR SECOND-ORDER FUNCTIONAL DIFFERENCE EQUATION WITH NEUTRAL TERMS

Abstract. In this manuscript, two type of new oscillation criteria are obtained respect to coefficient a_k in the following Eq. (1.1). In the subsection 2.1 considered as $a_k \geq 0$. In the subsection 2.2 allowed it to be an oscillating sequence. There are no results for the oscillation of second order difference equations with oscillating coefficients up to now.

1. Introduction

Recently, the oscillation and nonoscillation problems of second order difference equations have recieved a great amount of attention. This is probably due to the closeness of such phenomenon to those of the analogous differential equations. In addition, these equations have many applications in physics and in other fields (see [1-5]). Particularly, including neutral and delay terms equations find numerous applications in natural science and technology [29-34]. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. In this paper we consider a class of general second order nonlinear difference equation with general nonlinear neutral terms of the form

$$(1.1) \qquad \Delta(p_k F(\Delta(y_k + a_k y_{k-\tau}))) + q_k G(\Delta(y_k + a_k y_{k-\tau})) + H(k, y_k, y_{k-r_1}, \ldots, y_{k-r_n}) = 0$$

where $k \in \mathbb{N}$, and obtain two type new oscillaion criteria respect to sequence $a_k \geq 0$ in the subsection 2.1 and it is even an oscillating sequence in the subsection 2.2. There are no results for the oscillation of second order difference equations with oscillating coefficients up to now.

The following conditions are always assumed to hold:

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i) $p_k > 0$ and $q_k > 0$ for every $k \in \mathbb{N}(k_0)$ where $\mathbb{N}(k_0) = \{k_0, k_0 + 1, \ldots, \}$ and $k_0 \in \mathbb{N}$,

ii) $F : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function such that $uF(u) > 0$ for $u \neq 0$,

iii) $G : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $0 < m_1 \leq G(u) \leq m_2$ where m_1 and m_2 are constants,

iv) $H : \mathbb{N}(k_0) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous increasing function with respect to $v_0, v_1, \ldots, v_n$, further $H(k, v_0, v_1, v_2, \ldots, v_n)$ has same sign with respect to $v_0, v_1, v_2, \ldots, v_n$,

v) τ, $r_1, r_2, \ldots, r_n \in \mathbb{N}(1)$ and $(k - \tau) \to +\infty$, $(k - r_i) \to +\infty$ as $k \to \infty$ for every $i = 1, 2, \ldots, n$.

Choosing Eq. (1.1) for this study is motivated by the numerous research on the oscillatory properties of several particular cases of Eq. (1.1). For example, the linear difference equation

$$\Delta(r_n \Delta x_{n-1}) + q_n x_n = 0$$

has been studied by [9-11, 15, 26 and the references cited therein] which is a special case of Eq. (1.1). The discrete Emden-Fowler equation

$$\Delta^2 x_{n-1} + q_n \left|x_n\right|^{\upsilon-1} x_n = 0\,, \quad \upsilon > 1$$

and its generalizations

$\Delta(r_n \Delta x_{n-1}) + q_n \phi(x_n) = 0$, where ϕ has the same properties as H and F,

$\Delta(r_n \Delta x_{n-1}) + g(n, x_n) = 0$, where g has the same properties as H and F,

have been investigated by [12, 13, 20-24 and the references cited therein]. Another very important special case of Eq. (1.1), which arises in the theory of radial solutions for p-Laplacian equation on an annular domain (see [8] and the references cited therein), is the half- linear equation

$$\Delta(r_n \left|\Delta x_{n-1}\right|^{p-2} \Delta x_{n-1}) + q_n \left|x_n\right|^{p-2} x_n = 0, \quad p > 1,$$

and its more genaral form

$$\Delta(r_n \left|\Delta x_{n-1}\right|^{p-2} \Delta x_{n-1}) + g(n, x_n) = 0, \qquad p > 1,$$

have been studied by [14, 16-19, 27 and the references cited therein) which are also special cases of Eq. (1.1). The delayed or advanced versions of the above equations have been study by [25 and the references cited therein]. Finally, the particular cases of Eq. (1.1), the equations

$$\Delta(r_n f(\Delta x_{n-1})) + g(n, x_n) = 0$$

and

$$\Delta(r_n f(\Delta x_{n-1})) + g(n, x_{\tau_n}) = 0$$

have been investigated by [28].

Recall that Δ is a forward difference operator which is defined by $\Delta y(k) = y(k+1) - y(k)$. Throughout this work we imply $y(k) = y_k$.

Let $\sigma = \max\{\tau, r_i\}$, $i = 1, 2, \ldots, n$, and $\mathbb{N}_0$ be a fixed nonnegative integer. By a solution of Eq. (1.1), we mean a real sequence $\{y_k\}$ which is defined for all $k \geq \mathbb{N}_0 - \sigma$ and satisfies Eq. (1.1) for $k \geq \mathbb{N}_0$. A solution $\{y_k\}$ of Eq. (1.1) is said to be nonoscillatory if all the terms y_k are eventually of fixed sign. Otherwise, the solution $\{y_k\}$ is called oscillatory. In this paper, we shall be concerned only with the nontrivial solutions of Eq. (1.1).

2. Main results

2.1. Oscillation criteria for the case of $0 \leq a_k < 1$. We consider the coefficient a_k as $0 \leq a_k < 1$ in the following Lemma 1, Theorem 1, Theorem 2 and Theorem 3.

LEMMA 1. *Assume that y_k is nonoscillatory solution of Eq. (1.1). If the condition*

$$(C_1) \qquad \sum^{\infty} F^{-1}\left(\frac{-c}{p_k}\right) = -\infty \quad (c > 0)$$

is satisfied, then there exists $k_1 \in \mathbb{N}(k_0)$ such that

$$(y_k + a_k y_{k-\tau})\Delta(y_k + a_k y_{k-\tau}) > 0$$

for all $k \in \mathbb{N}(k_1)$.

Proof. Suppose that there exists a $k_1 \in \mathbb{N}(k_0)$ such that $y_k > 0$ for all $k \in \mathbb{N}(k_1)$. Since $(k - \tau) \to \infty$ and $(k - r_i) \to \infty$ as $k \to \infty$ for $i = 1, 2, \ldots, n$, one can find $k_2 \in \mathbb{N}(k_1)$ such that $y_{k-\tau} > 0$ and $y_{k-r_i} > 0$ for every $i = 1, 2, \ldots, n$ and all $k \in \mathbb{N}(k_2)$. Define

$$z_k = y_k + a_k y_{k-\tau}.$$

Since $0 \leq a_k < 1$, there exist a $k_3 \geq k_2$ and a constant λ $(0 < \lambda < 1)$ such that

$$y_k = z_k - a_k y_{k-\tau} \geq \lambda z_k > 0 \quad \text{for } k \in \mathbb{N}(k_3).$$

Therefore we can find a $k_4 \geq k_3$ such that

$$y_{k-r_1} \geq \lambda z_{k-r_1}, y_{k-r_2} \geq \lambda z_{k-r_2}, \ldots, y_{k-r_n} \geq \lambda z_{k-r_n} > 0 \text{ for all } k \in \mathbb{N}(k_4).$$

Then from Eq. (1.1) we have

$$(2.1) \qquad \Delta(p_k F(\Delta z_k)) \leq -q_k G(\Delta z_k) - H(k, \lambda z_k, \lambda z_{k-r_1}, \lambda z_{k-r_2}, \ldots, \lambda z_{k-r_n}) < 0$$

for all $k \in \mathbb{N}(k_4)$. From (2.1) it is clear that $p_k F(\Delta z_k)$ is decreasing. Therefore there are two cases. Either $\Delta z_k < 0$ or $\Delta z_k > 0$.

Assume that $\Delta z_k < 0$ for all $k \in N(k_4)$. Summing up (2.1) from k_4 to $k-1$ we get

$$p_k F(\Delta z_k) \le p_{k_4} F(\Delta z_{k_4}) = -c < 0 \quad (c > 0) \text{ for all } k \in \mathbb{N}(k_4). \tag{2.2}$$

Then from (2.2) we have

$$F(\Delta z_k) \le \frac{-c}{p_k} \text{ for all } k \in \mathbb{N}(k_4). \tag{2.3}$$

Then from (2.3) we obtain

$$\Delta z_k \le F^{-1}\left(\frac{-c}{p_k}\right) \text{ for all } k \in \mathbb{N}(k_4). \tag{2.4}$$

Summing up (2.4) from k_4 to $k-1$, we obtain

$$z_k \le z_{k_4} + \sum_{j=k_4}^{k-1} F^{-1}\left(\frac{-c}{p_j}\right). \tag{2.5}$$

But, according to (C_1), inequality (2.5) implies that $z_k < 0$ as $k \to \infty$, which contradicts to $z_k = y_k + a_k y_{k-\tau} > 0$. Hence $\Delta z_k = \Delta(y_k + a_k y_{k-\tau}) > 0$.

If $y_k < 0$ for all $k \in \mathbb{N}(k_1)$, then similar reasoning implies a contradiction. We omit the details to avoid repetition. □

Theorem 1. *Let (C_1) hold and the condition*

$$\sum^{\infty} F^{-1}\left(\frac{\phi_j}{p_j}\right) = -\infty \tag{C_2}$$

is satisfied, where $\phi_j = p_{k_5} F(\Delta z_{k_5}) - \sum_{s=j_5}^{j-1} H(s, c, c, c, \ldots, c)$ with a positive constant c. Then every solution of Eq. (1.1) is oscillatory.

Proof. Suppose that there exists a $k_1 \in \mathbb{N}(k_0)$ such that $y_k > 0$ for all $k \in \mathbb{N}(k_1)$. Since $(k-\tau) \to \infty$ and $(k - r_i) \to \infty$ as $k \to \infty$ for $i = 1, 2, \ldots, n$, one can find $k_2 \in \mathbb{N}(k_1)$ such that $y_{k-\tau} > 0$ and $y_{k-r_i} > 0$ for every $i = 1, 2, \ldots, n$ and all $k \in \mathbb{N}(k_2)$. Define

$$z_k = y_k + a_k y_{k-\tau}.$$

Since $0 \le a_k < 1$, there exist a $k_3 \ge k_2$ and a constant λ $(0 < \lambda < 1)$ such that

$$y_k = z_k - a_k y_{k-\tau} \ge \lambda z_k > 0 \quad \text{for } k \in \mathbb{N}(k_3).$$

Therefore we can find a $k_4 \ge k_3$ such that

$$y_{k-r_1} \ge \lambda z_{k-r_1}, y_{k-r_2} \ge \lambda z_{k-r_2}, \ldots, y_{k-r_n} \ge \lambda z_{k-r_n} > 0 \text{ for } k \in \mathbb{N}(k_4).$$

Then from Eq. (1.1) we have

$$\Delta(p_k F(\Delta z_k)) \le -q_k G(\Delta z_k) - H(k, \lambda z_k, \lambda z_{k-r_1}, \lambda z_{k-r_2}, \ldots, \lambda z_{k-r_n}) < 0$$

for all $k \in \mathbb{N}(k_4)$. By Lemma 1 since $\Delta z_k > 0$, z_k is increasing. Therefore there exists a constant $c > 0$ and a $k_5 \geq k_4$ such that $z_k > c > 0$, $z_{k-r_1} > c > 0$, $z_{k-r_2} > c > 0, \ldots$, $z_{k-r_n} > c > 0$ for all $k \in \mathbb{N}(k_5)$. Hence we have from last inequality

$$\Delta(p_k F(\Delta z_k)) \leq -H(k, c, c, c, \ldots, c) \tag{2.6}$$

for all $k \in \mathbb{N}(k_5)$. Summing up (2.6) from k_5 to $k-1$, we obtain

$$p_k F(\Delta(y_k + a_k y_{k-\tau})) \leq p_{k_5} F(\Delta z_{k_5}) - \sum_{s=k_5}^{k-1} H(s, c, c, c, \ldots, c). \tag{2.7}$$

Let us take $\phi_k = p_{k_5} F(\Delta z_{k_5}) - \sum_{s=k_5}^{k-1} H(s, c, c, c, \ldots, c)$. Then from (2.7) we have

$$\Delta z_k < F^{-1}\left(\frac{\phi_k}{p_k}\right). \tag{2.8}$$

Summing up (2.8) from k_5 to $k-1$, we have

$$z_k < z_{k_5} + \sum_{j=k_5}^{k-1} F^{-1}\left(\frac{\phi_j}{p_j}\right). \tag{2.9}$$

But, according to the condition (C_2), inequality (2.9) implies that $z_k = y_k + a_k y_{k-\tau} = -\infty$ as $k \to \infty$, which contradicts to the fact that $y_k > 0$ and $y_k + a_k y_{k-\tau} > 0$. If $y_k < 0$ for all $k \in N(k_1)$, then similar reasoning implies a contradiction. This completes the proof. □

THEOREM 2. *Let (C_1) hold. Moreover, suppose that following conditions are satisfied:*

$$p_k \Delta s_k \leq -M, \tag{C_3}$$

where s_k is a positive sequence with $\Delta s_k \leq 0$, and M is a nonnegative number, and

$$\sum^{\infty} F^{-1}\left(\frac{\varphi_\sigma}{s_\sigma p_\sigma}\right) = -\infty \tag{C_4}$$

where φ_k is any negative squence for all sufficiently large k. Then every solutions of Eq. (1.1) is oscillatory.

Proof. Whithout repeating the same assumption, let us consider in here the part of the proof of Theorem 1 untill (2.6). Let s_k be a positive sequence which satisfies condition (C_3). If we multiply the inequality (2.6) with s_k

and later take its sum from k_5 to $k-1$, we obtain

$$(2.10)\quad s_kp_kF(\Delta z_k) - s_{k_5}p_{k_5}F(\Delta z_{k_5}) - \sum_{j=k_5}^{k-1} F(\Delta z_j)p_j\Delta s_j \le -\sum_{j=k_5}^{k-1} s_jH(j,c,c,c,\dots,c).$$

Applying the condition (C_3) to (2.10) we have

$$(2.11)\quad \begin{aligned} s_kp_kF(\Delta z_k) - s_{k_5}p_{k_5}F(\Delta z_{k_5}) &\le -M\sum_{j=k_5}^{k-1} F(\Delta z_j) - \sum_{j=k_5}^{k-1} s_jH(j,c,c,c,\dots,c) \\ &\le -\sum_{j=k_5}^{k-1} s_jH(j,c,c,c,\dots,c). \end{aligned}$$

Let us take $\varphi_k = s_{k_5}p_{k_5}F(\Delta z_{k_5}) - \sum_{j=k_5}^{k-1} s_jH(j,c,c,c,\dots,c)$ in the inequality (2.11). Then from (2.11) we have

$$(2.12)\quad \Delta z_k \le F^{-1}\left(\frac{\varphi_k}{s_kp_k}\right).$$

Summing up (2.12) from k_5 to $k-1$, we obtain

$$(2.13)\quad y_k + a_ky_{k-\tau} \le y_{k_5} + a_{k_5}y_{k_5-\tau} + \sum_{\sigma=k_5}^{k-1} F^{-1}\left(\frac{\varphi_\sigma}{s_\sigma p_\sigma}\right).$$

By condition (C_4) inequality (2.13) implies that $\lim_{k\to\infty}(y_k + a_ky_{k-\tau}) = -\infty$. This is a contradiction. If $y_k < 0$ for all $k \in N(k_1)$, then similar reasoning implies a contradiction. Hence the proof is complete. □

THEOREM 3. *Let conditions (C_1) and (C_3) hold. In addition, assume that following conditions are satisfied:*

$$(C_5)\quad \frac{H(k,v_0,v_1,\dots,v_n)}{H_1(v_0,v_1,\dots,v_n)} \ge \beta_k > 0,$$

where the function $H_1(v_0,v_1,\dots,v_n)$ is a continuous function and has the same sign with respect to $v_0,v_1,\dots,v_n$, and

$$(C_6)\quad \sum^{\infty} F^{-1}\left(\frac{\psi_u}{s_up_u}\right) = -\infty,$$

where $\psi_k = s_{k_5}p_{k_5}F(\Delta z_{k_5}) - \sum_{j=k_3}^{k-1}\mu\beta_js_j$, where s_k is a positive sequence with $\Delta s_k \le 0$ and μ is a positive constant. Then every solution of Eq. (1.1) is oscillatory.

Proof. Suppose that there exists a $k_1 \in \mathbb{N}(k_0)$ such that $y_k > 0$ for all $k \in \mathbb{N}(k_1)$. Since $(k-\tau) \to \infty$ and $(k-r_i) \to \infty$ as $k \to \infty$ for $i = 1, 2, \ldots, n$, one can find $k_2 \in \mathbb{N}(k_1)$ such that $y_{k-\tau} > 0$ and $y_{k-r_i} > 0$ for every $i = 1, 2, \ldots, n$ and all $k \in \mathbb{N}(k_2)$. Hence, we can find any function $H_1(y_k, y_{k-r_1}, y_{k-r_2}, \ldots, y_{k-r_n})$ such that

$$(2.14) \qquad H(k, y_k, y_{k-r_1}, y_{k-r_2}, \ldots, y_{k-r_n}) \geq \beta_k H_1(y_k, y_{k-r_1}, y_{k-r_2}, \ldots, y_{k-r_n}) > 0$$

for all $k \in \mathbb{N}(k_2)$. Define $z_k = y_k + a_k y_{k-\tau}$. Since $0 \leq a_k < 1$, there exist a $k_3 \geq k_2$ and a constant λ $(0 < \lambda < 1)$ such that

$$y_k = z_k - a_k y_{k-\tau} \geq \lambda z_k > 0 \quad \text{for } k \in \mathbb{N}(k_3).$$

Therefore we can find a $k_4 \geq k_3$ such that

$$y_{k-r_1} \geq \lambda z_{k-r_1},\ y_{k-r_2} \geq \lambda z_{k-r_2}, \ldots,\ y_{k-r_n} \geq \lambda z_{k-r_n} > 0 \text{ for } k \in \mathbb{N}(k_4).$$

Then we can rewrite (2.14) in the form

$$H(k, \lambda z_k, \lambda z_{k-r_1}, \lambda z_{k-r_2}, \ldots, \lambda z_{k-r_n}) \geq \beta_k H_1(\lambda z_k, \lambda z_{k-r_1}, \lambda z_{k-r_2}, \ldots, \lambda z_{k-r_n}) > 0.$$

Therefore considering continuity of $H_1(v_0, v_1, \ldots, v_2)$ and since $z_k > 0$ is increasing, we have

$$\liminf_{k \to \infty} H_1(\lambda z_k, \lambda z_{k-r_1}, \lambda z_{k-r_2}, \ldots, \lambda z_{k-r_n}) = H_1(\delta, \delta, \ldots, \delta).$$

Thus, we obtain $0 < H_1(\delta, \delta, \ldots, \delta) < +\infty$. Choose μ such that $0 < \mu < H_1(\delta, \delta, \ldots, \delta) < +\infty$. Then there exists $k_5 \geq k_4$ such that

$$(2.15) \qquad H_1(\lambda z_k, \lambda z_{k-r_1}, \lambda z_{k-r_2}, \ldots, \lambda z_{k-r_n}) > \mu$$

for all $k \in \mathbb{N}(k_5)$. Therefore from Eq. (1.1), (2.14) and (2.15) we obtain

$$(2.16) \qquad \Delta(p_k F(\Delta z_k) + \mu \beta_k \leq 0.$$

If we treat (2.16) as we treat (2.6) in the proof of Theorem 3,

$$(2.17) \qquad s_k p_k F(\Delta z_k) - s_{k_5} p_{k_5} F(\Delta z_{k_5}) \leq \sum_{j=k_5}^{k-1} F(\Delta z_j) p_j \Delta s_j - \sum_{j=k_5}^{k-1} \mu \beta_j s_j.$$

Applying the condition (C_3) to (2.17) we have

$$(2.18) \qquad s_k p_k F(\Delta z_k) - s_{k_5} p_{k_5} F(\Delta z_{k_5}) \leq -M \sum_{j=k_5}^{k-1} F(\Delta z_j) \quad - \sum_{j=k_5}^{k-1} \mu \beta_j s_j \leq - \sum_{j=k_5}^{k-1} \mu \beta_j s_j.$$

Let us take $\psi_k = s_{k_5} p_{k_5} F(\Delta z_{k_5}) - \sum_{j=k_3}^{k-1} \mu \beta_j s_j$ in (2.18). Then summing up (2.18) from k_5 to $k-1$ and considering condition (C_6) from (2.18) we have

$$y_k + a_k y_{k-\tau} < y_{k_5} + a_{k_5} y_{k_5-\tau} + \sum_{u=k_5}^{k-1} F^{-1}\left(\frac{\psi_u}{s_u p_u}\right) \to -\infty$$

as $k \to \infty$. This is a contradiction. If $y_k < 0$ for all $k \in \mathbb{N}(k_1)$, then similar reasoning implies a contradiction. Hence the proof is complete. □

2.2. Oscillation criteria for the case of oscillating coefficient a_k

We consider coefficient a_k as an oscillating sequence in the following Lemma 2, Theorem 4, Theorem 5 and Theorem 6.

LEMMA 2. *Assume that y_k is nonoscillatory bounded solution of Eq. (1.1) and it does not tend to zero as $k \to \infty$. If the condition (C_1) and*

$$(C_7) \qquad \liminf_{k\to\infty} a_k = 0$$

is satisfied, then there exists a $k_* \in \mathbb{N}(k_0)$ such that

$$(y_k + a_k y_{k-\tau})\Delta(y_k + a_k y_{k-\tau}) > 0$$

for all $k \in \mathbb{N}(k_*)$.

Proof. Suppose that y_k is nonoscillatory bounded solution of Eq. (1.1) and without generality it is positive. Then Since $(k-\tau) \to \infty$ and $(k-r_i) \to \infty$ as $k \to \infty$ for $i = 1, 2, \dots, n$, one can find $k_1 \in \mathbb{N}(k_0)$ such that $y_{k-\tau} > c > 0$ and $y_{k-r_i} > c > 0$ for every $i = 1, 2, \dots, n$ and all $k \in \mathbb{N}(k_1)$. Define

$$z_k = y_k + a_k y_{k-\tau}.$$

Since y_k does not tend to zero as $k \to \infty$ and it is bounded, $\liminf_{k\to\infty} a_k y_{k-\tau} = 0$. Therefore $z_k \downarrow c > 0$ as $k \to \infty$. Hence we find a sufficiently large $k_2 \ge k_1$ and a constant λ $(0 < \lambda < 1)$ such that

$$y_k = z_k - a_k y_{k-\tau} \ge \lambda z_k > 0 \quad \text{for } k \in \mathbb{N}(k_2).$$

Therefore we can find a $k_3 \ge k_2$ such that

$$y_{k-r_1} \ge \lambda z_{k-r_1}, y_{k-r_2} \ge \lambda z_{k-r_2}, \dots, y_{k-r_n} \ge \lambda z_{k-r_n} > 0$$

for $k \in \mathbb{N}(k_3)$. Rest of the proof is similar to the proof of Lemma 1 in the section 2.1. □

THEOREM 4. *Assume that conditions (C_1), (C_2) and (C_7) are satisfied. Then every bounded solution of Eq. (1.1) is either oscillatory or tends to zero as $k \to \infty$.*

Proof. Suppose that y_k is nonoscillatory bounded solutions of Eq. (1.1) and without generality it is positive. Furthermore assume that y_k does not tend to zero if $k \to \infty$. Then, since $(k-\tau) \to \infty$ and $(k-r_i) \to \infty$ as

$k \to \infty$ for $i = 1, 2, \ldots, n$, one can find $k_1 \in \mathbb{N}(k_0)$ such that $y_{k-\tau} > c > 0$ and $y_{k-r_i} > c > 0$ for every $i = 1, 2, \ldots, n$ and all $k \in \mathbb{N}(k_1)$. Define

$$z_k = y_k + a_k y_{k-\tau}.$$

By Lemma 2 $z_k = y_k + a_k y_{k-\tau} > 0$ and $\Delta z_k > 0$. Since y_k does not tend to zero as $k \to \infty$ and it is bounded, $\liminf_{k\to\infty} a_k y_{k-\tau} = 0$. Therefore there exist a $k_2 \geq k_1$ and a constant λ $(0 < \lambda < 1)$ such that

$$y_k = z_k - a_k y_{k-\tau} \geq \lambda z_k > 0 \text{ for sufficiently large all } k \in \mathbb{N}(k_2).$$

Therefore we can find a $k_3 \geq k_2$ such that

$$y_{k-r_1} \geq \lambda z_{k-r_1}, y_{k-r_2} \geq \lambda z_{k-r_2}, \ldots, y_{k-r_n} \geq \lambda z_{k-r_n} > 0 \text{ for all } k \in \mathbb{N}(k_3).$$

Rest of the proof is all similar to the proof of Theorem 1 in the section 2.1. □

THEOREM 5. *Assume that conditions* (C_1), (C_3), (C_6) *and* (C_7) *are satisfied. Then every bounded solution of Eq. (1.1) is either oscillatory or tends to zero as* $k \to \infty$.

Proof. The proof is similar to proofs of Theorem 2 in Section 2.1, so we omit it. □

THEOREM 6. *Assume that conditions* (C_1), (C_3), (C_5), (C_6)*and* (C_7) *are satisfied. Then every bounded solution of Eq. (1.1) is either oscillatory or tends to zero as* $k \to \infty$.

Proof. The proof is all similar to the proofs of Theorem 3 in the section 2.1. Therefore we omit the details to avoid repetition. □

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