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POLYNOMIALS IN ADDITIVE FUNCTIONS AND GENERALIZED POLYNOMIALS

Abstract. We consider polynomials P in additive functions $g_1, \dots, g_m$ and present two approaches for a characterization of those generalized polynomials p, which may be represented as $p = P \circ (g_1, \dots, g_m)$. Under rather general assumptions on the domains of the g_i and of P, one of the approaches is based on certain properties of subspaces generated by translates of p. The other approach utilizes the fact, that every p is the diagonalization of an associated multi-Jensen function.

1. Introduction

In [PS] the authors discussed the connections between generalized polynomials p of degree $\leq n$ defined on a vector space V over $\mathbb{Q}$ taking values in a vector space W over $\mathbb{Q}$ and (symmetric) functions $f\colon V^n \to W$ which have the Jensen property in each variable. It has been shown that there is a one-to-one correspondence between the space $\mathcal{P}_n(V, W)$ of all generalized polynomials $p\colon V \to W$ of degree $\leq n$ and the space of all symmetric functions $f\colon V^n \to W$ which are Jensen in each variable.

There is a subclass $\mathcal{P}_n^*(V, W)$ of $\mathcal{P}_n(V, W)$ which is of special importance (see, for example [Sz]). This class, the *space of all polynomials in additive functions (of degree $\leq n$)* is defined by

$$\mathcal{P}_n^*(V, W) := \mathcal{L}_{\mathbb{Q}}\Big(\Big\{\prod_{i=1}^{m} g_i \cdot a \,\Big|\, a \in W,\, n \geq m \in \mathbb{N}_0,\, g_1, g_2, \dots g_m\colon V \to \mathbb{Q} \text{ additive}\Big\}\Big).$$

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Thus (with $\prod_{i=1}^{0} g_i = 1$, as usual) $\mathcal{P}_n^*(V,W)$ is the space of all functions

$$p = \sum_{i=1}^{N} \prod_{j=1}^{m_i} g_j^{(i)} \cdot a_i,$$

where $N \in \mathbb{N}$, $m_1, m_2, \ldots, m_N \in \mathbb{N}_0$, $m_i \leq n$ for $1 \leq i \leq N$, $a_1, a_2, \ldots, a_N \in W$ and where all $g_j^{(i)} : V \to \mathbb{Q}$ are additive. In general $\mathcal{P}_n^*(V,W) \subsetneq \mathcal{P}_n(V,W)$ ([RS]).

The starting point of our considerations has been a question asked by Ludwig Reich [R] about a characterization of polynomials in additive functions by their corresponding multi-Jensen functions. In the course of our treatment it turned out, that it is helpful to direct one's attention in the first place to certain problems concerning general properties of polynomials in additive functions. Assuming that G is an abelian group, K a field with characteristic zero and V a vector space over K, we consider mappings $p : G \to V$ admitting a representation $p = P \circ (g_1, \ldots, g_m)$, where $P : K^m \to V$ is a polynomial in m variables and $g_1, \ldots, g_m : G \to K$ are additive functions.

Beginning with the equation $P \circ (g_1, \ldots, g_m) = 0$, in Section 2 the relationship between different representations $p = P \circ (g_1, \ldots, g_m) = Q \circ (h_1, \ldots, h_n)$ and the problem of minimality of m are investigated. Moreover, a characterization of functions p being a polynomial in additive functions in terms of the dimension of $\mathcal{L}_K(p(G))$ and compositions $\varphi \circ p$ with $\varphi \in V^*$ is given.

Section 3 contains a characterization of generalized polynomials both $p : G \to K$ as well as $p : G \to V$, K not necessarily algebraically closed, admitting a representation $p = P \circ (g_1, \ldots, g_m)$, by translation invariant subspaces of the vector space of all generalized polynomials with values in K or V, respectively. Characterizing in this way, the translation equation is involved crucially. For that reason a former result on diagonalization of homomorphisms $G \to \mathrm{Gl}_n(K)$ and on the form of their entries as exponential polynomials has to be generalized to the considered case of arbitrary fields K of characteristic zero.

In Section 4 we give characterizations of generalized polynomials $p : V \to W$, V, W being $\mathbb{Q}$-vector spaces, admitting a representation $p = P \circ (g_1, \ldots, g_m)$, by specifying the structure of their corresponding multi-Jensen functions. Moreover, an example of a generalized polynomial which is not a polynomial in additive functions is analyzed in detail.

In our approach generalized polynomials are considered as polynomial functions. An algebraic view on generalized polynomials as elements of a certain polynomial ring may be found in [H].

2. Polynomials in additive functions defined on abelian groups

In [RS] mainly such polynomials in additive functions have been considered which are defined on $\mathbb{C}$ and which take values in $\mathbb{C}$. Here we want to generalize this concept. Moreover we will investigate the question to what extent the representations of polynomials in additive functions are unique.

We start with a field K of characteristic 0 and an abelian group G. (In some situations it would be enough to suppose that the number of elements of K is not too small.) Homomorphisms $g\colon G \to K$ will be called *additive.* If V is a vector space over K and $m \in \mathbb{N}$, a function $P\colon K^m \to V$ is called a *polynomial (function)* if there is a family $(a_\nu)_{\nu \in \mathbb{N}_0^m}$ of elements $a_\nu \in V$ such that all $a_\nu = 0$ with at most finitely many exceptions and such that

$$P(x) = \sum_{\nu \in \mathbb{N}_0^m} x^\nu a_\nu$$

for all $x = (x_1, x_2, \ldots, x_m) \in K^m$. (As usual $x^\nu := \prod_{i=1}^m x_i^{\nu_i}$.)

REMARK 1. Well-known arguments ([La, chap. V, p. 121]) show that the family (a_ν) is uniquely determined by P:
If $P(x_1, x_2, \ldots, x_m) = 0$ for all $(x_1, x_2, \ldots, x_m) \in \prod_{i=1}^m X_i$ and if all X_i contain infinitely many elements, then $a_\nu = 0$ for all ν.

We denote the set of all these polynomial functions by $\mathcal{Q}(K^m, V)$. Obviously this set is a vector space over K. Moreover $Q \cdot P \in \mathcal{Q}(K^m, V)$ if $Q \in \mathcal{Q}(K^m, K)$ and $P \in \mathcal{Q}(K^m, V)$. One also can easily verify that for given $Q_1, Q_2, \ldots, Q_m \in \mathcal{Q}(K^p, K)$ the function $P^* := P \circ (Q_1, Q_2, \ldots, Q_m)$ defined by $P^*(y) = P^*(y_1, \ldots, y_p) := \sum_{\nu \in \mathbb{N}_0^m} \prod_{i=1}^m Q_i(y)^{\nu_i} \cdot a_\nu$ is contained in $\mathcal{Q}(K^p, V)$.

We call $p\colon G \to V$ a *polynomial in additive functions* if there are some additive functions $g_1, g_2, \ldots, g_m\colon G \to K$ and if there is some $P \in \mathcal{Q}(K^m, V)$ such that $p = P \circ (g_1, g_2, \ldots, g_m)$ $(p(x) = P(g_1(x), g_2(x), \ldots, g_m(x))$ for all $x \in G)$.

The following theorem generalizes Theorem 1 of [RS].

THEOREM 1. *Let G be an abelian group, let K be a field of characteristic 0. Suppose that V is a vector space over K, that $P \in \mathcal{Q}(K^m, V)$ and that $g_1, \ldots, g_m\colon G \to K$ are m linearly independent additive functions. Then $P \circ (g_1, \ldots, g_m) = 0$ implies $P = 0$.*

Proof. Since $g_1, g_2, \ldots, g_m$ are linearly independent we may find $x_1, x_2, \ldots, x_m \in G$ such that the vectors $(g_1(x_i), g_2(x_i), \ldots, g_m(x_i))$, $i = 1, 2, \ldots, m$ are linearly independent in K^m ([AD, p. 229]). Let $Q_i \in \mathcal{Q}(K^m, K)$ be defined by $Q_i(t_1, t_2, \ldots, t_m) := \sum_{j=1}^m g_i(x_j)t_j$. $P \circ (g_1, \ldots, g_m) = 0$ implies with $x := \sum_{j=1}^{m} n_j x_j$ (and $n_1, n_2, \dots, n_m \in \mathbb{N}_0$) that

$$0 = P(g_1(x), \dots, g_m(x)) = P(Q_1(n_1, \dots, n_m), \dots, Q_m(n_1, \dots, n_m))$$

for all $(n_1, \dots, n_m) \in \mathbb{N}_0^m$. Thus, by Remark 1, $\mathcal{Q}(K^m, V) \ni P^* := P \circ (Q_1, \dots, Q_m) = 0$.

By construction the matrix $(g_i(x_j))$ is regular. Let $\Delta = (\Delta_{ij})$ be its inverse, i. e., $\sum_j g_i(x_j)\Delta_{jk} = \delta_{ik}$. Then, with $l_i(t) = l_i(t_1, \dots, t_m) := \sum_k \Delta_{ik} t_k$, we get

$$Q_i(l_1(t), \dots, l_m(t)) = \sum_j g_i(x_j) \sum_k \Delta_{jk} t_k = \sum_k \delta_{ik} t_k = t_i.$$

So

$$\begin{aligned} 0 = P^*(l_1(t), \dots, l_m(t)) &= P(Q_1(l_1(t), \dots, l_m(t)), \dots, Q_m(l_1(t), \dots, l_m(t))) \\ &= P(t_1, \dots, t_m) \end{aligned}$$

for all $t_1, t_2, \dots, t_m \in K$. Thus $P = 0$. ■

Now we want to describe those polynomial functions P with

$$P \circ (g_1, \dots, g_m) = 0$$

for not necessarily linearly independent additive functions $g_1, \dots, g_m$. To this aim we first need some auxiliary results.

LEMMA 1. *Let $n \in \mathbb{N}$, $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}_0^n$. Then there are polynomials $Q_1, \dots, Q_n$ contained in $\mathcal{Q}(K^{2n}, K)$ such that*

$$x^\mu = a^\mu + \sum_{l=1}^{n} Q_l(x_1, \dots, x_n, a_1, \dots, a_n) \cdot (x_l - a_l)$$

for all $x = (x_1, \dots, x_n), a = (a_1, \dots, a_n) \in K^n$.

Proof. The assertion obviously holds true when $|\mu| := \sum \mu_i = 0$. If $|\mu| = m + 1$ and if the assertion holds for μ' with $|\mu'| = m$, we also may assume that, say, $\mu_n > 0$. Then $x^\mu = x^{\mu'} x_n$, where $\mu' := (\mu_1, \mu_2, \dots, \mu_n - 1)$ (and $|\mu'| = m$). Accordingly

$$\begin{aligned} x^\mu &= x^{\mu'} x_n \\ &= \Big(a^{\mu'} + \sum_{l=1}^{n} Q'_l(x_1, \dots, x_n, a_1, \dots, a_n) \cdot (x_l - a_l)\Big)((x_n - a_n) + a_n) \\ &= a^{\mu'} a_n + \sum_{l=1}^{n} Q_l(x_1, \dots, x_n, a_1, \dots, a_n) \cdot (x_l - a_l), \end{aligned}$$

where $Q_l(x_1, \dots, x_n, a_1, \dots, a_n) := Q'_l(x_1, \dots, x_n, a_1, \dots, a_n) \cdot x_n$ for $l < n$ and where $Q_n(x_1, \dots, x_n, a_1, \dots, a_n) = a^{\mu'} + Q'_n(x_1, \dots, x_n, a_1, \dots, a_n) \cdot x_n$. ■

LEMMA 2. *For every $P \in \mathcal{Q}(K^n, V)$ and every $m \leq n$ there are polynomials $Q_0^* \in \mathcal{Q}(K^n, V)$, $Q_l^* \in \mathcal{Q}(K^{2n-m}, V)$, $l = m+1, m+2, \ldots, n$, such that*

$$(1)\quad P(x_1, \ldots, x_n) = Q_0^*(x_1, \ldots, x_m, a_{m+1}, \ldots, a_n) + \sum_{l=m+1}^{n} Q_l^*(x_1, \ldots, x_n, a_{m+1}, \ldots, a_n) \cdot (x_l - a_l)$$

for all $x_1, \ldots, x_n, a_{m+1}, \ldots, a_n \in K$.

Proof. Given P we may write

$$(2)\qquad P(x_1, \ldots, x_n) = \sum_{\mu' \in M} x^{\mu'} P_{\mu'}(x_1, \ldots, x_m),$$

where M is a finite subset of $\{0\}^m \times \mathbb{N}_0^{n-m}$, i.e., $\mu'_i = 0$ for all $i \leq m$ (and $x^{\mu'} = x_{m+1}^{\mu'_{m+1}} \cdot \ldots \cdot x_n^{\mu'_n}$), and where $P_{\mu'} \in \mathcal{Q}(K^m, V)$ for all $\mu' \in M$. We may assume that $\mu_0 := (0, 0, \ldots, 0) \in M$. According to Lemma 1 we have, with certain $Q_l^{(\mu')} \in \mathcal{Q}(K^{2(n-m)}, K)$,

$$(3)\qquad x^{\mu'} = a^{\mu'} + \sum_{l=m+1}^{n} Q_l^{(\mu')}(x_{m+1}, \ldots, x_n, a_{m+1}, \ldots, a_n) \cdot (x_l - a_l)$$

for all $\mu' \in M$. Inserting (3) into (2) we get (1) if we use

$$Q_0^*(x_1, \ldots, x_m, a_{m+1}, \ldots, a_n) := P_{\mu_0}(x_1, \ldots, x_m) + \sum_{\substack{\mu' \in M \\ \mu' \neq \mu_0}} a^{\mu'} \cdot P_{\mu'}(x_1, \ldots, x_m)$$

and, for $l = m+1, \ldots, n$,

$$Q_l^*(x_1, \ldots, x_n, a_{m+1}, \ldots, a_n) := \sum_{\substack{\mu' \in M \\ \mu' \neq \mu_0}} Q_l^{(\mu')}(x_{m+1}, \ldots, x_n, a_{m+1}, \ldots, a_n) \cdot P_{\mu'}(x_1, \ldots, x_m). \quad \blacksquare$$

COROLLARY 1. *Given P and m, n as in Lemma 2 and given $u_{lj} \in K$, $1 \leq j \leq m$, $m+1 \leq l \leq n$, there is some $P_0^* \in \mathcal{Q}(K^m, V)$ and there are $P_l^* \in \mathcal{Q}(K^n, V)$ such that*

$$(4)\qquad P(x) = P_0^*(x_1, \ldots, x_m) + \sum_{l=m+1}^{n} \Big(x_l - \sum_{j=1}^{m} u_{lj} x_j\Big) P_l^*(x)$$

for all $x = (x_1, \ldots, x_m, \ldots, x_n) \in K^n$.

Proof. This immediately follows from Lemma 2 for $a_l := \sum_{j=1}^{m} u_{lj} x_j$. $\blacksquare$

THEOREM 2. *Let $g_1, g_2, \ldots, g_n \colon G \to V$ be additive, let $g_1, \ldots, g_m$ be linearly independent and let $g_l = \sum_{j=1}^{m} u_{lj} g_j$ for $l > m$. Then $P \in \mathcal{Q}(K^m, V)$*

satisfies

$$P\,(g_1(y),\ldots,g_n(y)) = 0,\; y \in G \tag{5}$$

if and only if there are $P^*_{m+1},\ldots,P^*_n \in \mathcal{Q}(K^n,V)$ *such that*

$$P(x) = \sum_{l=m+1}^{n} (x_l - \sum_{j=1}^{m} u_{lj}x_j)P^*_l(x) \tag{6}$$

for all $x = (x_1,\ldots,x_n) \in K^n$.

Proof. Put, for arbitrary $y \in G$, $x_i := g_i(y)$. Then (6) implies (5) since in this case $x_l = \sum_{j=1}^m u_{lj}x_j$ for all $l > m$.

If (5) is satisfied we use (4) and thus, with x_i as before, we get $0 = P^*_0(g_1(y),\ldots,g_m(y)) + 0 + \ldots + 0$ for all $y \in G$. So, by Theorem 1 and since $g_1,\ldots,g_m$ are linearly independent, $P^*_0 = 0$. This implies (6). ∎

REMARK 2. This theorem shows that a polynomial p in additive functions may have different representations of the form $p = P \circ (g_1,\ldots,g_n)$ when $g_1,\ldots,g_n$ are linearly dependent. But this may even happen when $g_1,\ldots,g_n$ are linearly independent. Take, for example, two linearly independent additive functions g_1, g_2, put $P(x_1,x_2) := x_1 + x_2$, $Q(x_1) := x_1$ and $h := g_1 + g_2$. Then $P \circ (g_1,g_2) = Q \circ h$.

We also note that any polynomial $p = P \circ (g_1,g_2,\ldots,g_n)$ in additive functions $g_1,g_2,\ldots,g_n$ may be written as a polynomial in m ($\leq n$) linearly independent additive functions $h_1,h_2,\ldots,h_m$ (which may be chosen to be a base of the space generated by the g_i).

If the number n of additive functions in a representation $p = P\circ(g_1,\ldots,g_n)$ is *minimal*, we can describe all representations as a polynomial in n additive functions.

THEOREM 3. *Let* $p\colon G \to V$ *be a non constant polynomial in additive functions and let* $n \in \mathbb{N}$ *be minimal with respect to the property that there are additive functions* $g_1,\ldots,g_n\colon K \to V$ *such that* $p = P \circ (g_1,\ldots,g_n)$ *for some* $P \in \mathcal{Q}(K^n,V)$.

Then $g_1,\ldots,g_n$ *are linearly independent. Moreover* $p = Q \circ (h_1,\ldots,h_n)$ *with additive functions* $h_1,\ldots,h_n$ *and suitable* $Q \in \mathcal{Q}(K^n,V)$ *if and only if* $h_1,\ldots,h_n$ *constitute a basis for the space generated by* $g_1,\ldots,g_n$.

Proof. Since p is not constant, the number n must be at least 1. Obviously also $g_1 \neq 0$ when $n = 1$. Suppose $n > 1$. If, say, g_n were a linear combination of $g_1,\ldots,g_{n-1}$, $g_n = \sum_{l=1}^{n-1} u_l g_l$, then $p = Q^* \circ (g_1,\ldots,g_{n-1})$ with $Q^* \in \mathcal{Q}(K^{n-1},V)$ defined by $Q^*(x_1,\ldots,x_{n-1}) := Q(x_1,x_2,\ldots,x_{n-1},\sum_{l=1}^{n-1} u_l x_l)$. This contradicts the minimality of n.

Let, now, $p = P \circ (g_1, \dots, g_n) = Q \circ (h_1, \dots, h_n)$, where $P, Q \in \mathcal{Q}(K^n, V)$ and $g_1, \dots, g_n$, $h_1, \dots, h_n$ are additive functions from G to K (and n minimal). Hence, by the first part of the proof, both $g_1, \dots, g_n$ and $h_1, \dots, h_n$ are linearly independent. Thus it is enough to show that $\{h_1, \dots, h_n\} \subseteq \mathcal{L}_K(\{g_1, \dots, g_n\})$, the vector space generated by $g_1, \dots, g_n$. Assume that this is not true. Then the dimension of $W := \mathcal{L}_K(\{g_1, \dots, g_n, h_1, \dots, h_n\})$ must be greater than n. By standard arguments of Linear Algebra (and by renumbering the h_i if necessary) we may assume that the vectors $g_1, \dots, g_n$, $h_1, \dots, h_k$ form a basis for W for some $1 \le k \le n$. Let $h_j = \sum_{l=1}^n u_{jl} g_l + \sum_{i=1}^k v_{ji} h_i$ for $j = k+1, \dots, n$. We also may assume that all v_{ji} vanish. This always can be achieved by using $h_j^* := h_j - \sum_{i=1}^k v_{ji} h_i$, $k+1 \le j \le n$, and $h_j^* := h_j$, $1 \le j \le k$, instead of $h_1, \dots, h_n$. (Of course $Q \circ (h_1, \dots, h_n) = Q^* \circ (h_1^*, \dots, h_n^*)$ for some $Q^* \in \mathcal{Q}(K^n, V)$.)

Consider $T \in \mathcal{Q}(K^{2n}, V)$ defined by

$$T(x_1, \dots, x_n, y_1, \dots, y_n) := P(x_1, \dots, x_n) - Q(y_1, \dots, y_n).$$

Then $P \circ (g_1, \dots, g_n) = Q \circ (h_1, \dots, h_n)$ implies $T \circ (g_1, \dots, g_n, h_1, \dots, h_n) = 0$.

But $g_1, \dots, g_n, h_1, \dots, h_k$ are linearly independent and $h_j = \sum_{l=1}^n u_{jl} g_l$ for $k+1 \le j \le n$. Theorem 2 then implies the existence of polynomials $R_{k+1}, \dots, R_n \in \mathcal{Q}(K^{2n}, V)$ such that

$$\text{(7)} \quad T(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{j=k+1}^{n} \Big(y_j - \sum_{l=1}^{n} u_{jl} x_l\Big) R_j(x_1, \dots, x_n, y_1, \dots, y_n)$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in K$.

If we let $x_1, \dots, x_n, y_1, \dots, y_k$ be arbitrary and if we put $y_j := \sum_{l=1}^n u_{jl} x_l$ for $j > k$ equation (7) implies that

$$T(x_1, \dots, x_n, y_1, \dots, y_k, \sum_{l=1}^{n} u_{k+1,l} x_l, \dots, \sum_{l=1}^{n} u_{nl} x_l) = 0$$

for all $x_1, \dots, x_n, y_1, \dots, y_k \in K$. The definition of T then gives

$$\text{(8)} \qquad P(x_1, \dots, x_n) = Q(y_1, \dots, y_k, \sum_{l=1}^{n} u_{k+1,l} x_l, \dots, \sum_{l=1}^{n} u_{nl} x_l)$$

for all $x_1, \dots, x_n, y_1, \dots, y_k \in K$.

Let $\widehat{Q} \in \mathcal{Q}(K^{n-k}, V)$ be defined by $\widehat{Q}(y_{k+1}, \dots, y_n) := Q(0, \dots, 0, y_{k+1}, \dots, y_n)$. Then (8) with $y_1 = y_2 = \dots = y_k = 0$ leads to

$$P(x_1, \dots, x_n) = \widehat{Q}\Big(\sum_{l=1}^{n} u_{k+1,l} x_l, \dots, \sum_{l=1}^{n} u_{nl} x_l\Big), \quad x_1, \dots, x_n \in K.$$

Since $h_j = \sum_{l=1}^{n} u_{jl} g_l$ for $k+1 \le j \le n$ we conclude

$$P(g_1(t), \dots, g_n(t)) = \widehat{Q}(h_{k+1}(t), \dots, h_n(t)),\ t \in G.$$

But then $p = \widehat{Q} \circ (h_{k+1}, \dots, h_n)$. Since $k > 0$ this contradicts the minimality of n. ∎

THEOREM 4. *Assume that $g_1, \dots, g_n$ are n linear independent additive functions from G to K. Then there always exist polynomials from G to V which are polynomials in n additive functions and which are not polynomials in less than n additive functions. For $0 \neq a \in V$ the function $p\colon G \to V$ defined by $p(t) := g_1(t) g_2(t) \dots g_n(t) a$ has the desired property.*

Proof. Suppose $p = P \circ (h_1, \dots, h_m)$ with m additive functions h_i, $P \in \mathcal{Q}(K^m, V)$ and suppose also that m is minimal. Assume $m < n$. m must be greater than 0 since p cannot be constant (Theorem 1). $h_1, \dots, h_m$ are linearly independent by Theorem 3. Let $h_1, \dots, h_m, g_1, \dots, g_k$ be a basis of $W := \mathcal{L}_K(\{h_1, \dots, h_m, g_1, \dots, g_n\})$. Then $k > 0$ since $m < n$. Let $g_l = \sum_{j=1}^{m} u_{lj} h_j + \sum_{i=1}^{k} v_{li} g_i$ for $l = k+1, \dots, n$. Theorem 1 applied to $h_1, \dots, h_m, g_1, \dots, g_k$ and to the polynomial Q defined by

$$Q(y_1, \dots, y_m, x_1, \dots, x_k) := \prod_{l=1}^{k} x_l \prod_{l=k+1}^{n} \Big(\sum_{j=1}^{m} u_{lj} y_j + \sum_{i=1}^{k} v_{li} x_i \Big) a - P(y_1, \dots, y_m)$$

implies $Q = 0$. Since $k \ge 1$ we get for $x_1 = 0$ that $0 = Q(y_1, \dots, y_m, 0, x_2, \dots, x_n) = 0 - P(y_1, \dots, y_m)$. But then $P = 0$ which implies the contradiction $p = 0$. ∎

One may ask about a method to find, for a given polynomial p in additive functions, a representation as a polynomial in a minimal number of additive functions. A partial result related to this question is contained in the following theorem.

THEOREM 5. *Let $p\colon G \to V$ be a non constant polynomial in additive functions, $p = P \circ (g_1, \dots, g_n)$ with $P \in \mathcal{Q}(K^n, V)$ and $g_1, \dots, g_n\colon G \to K$ additive. Assume that we also have $p = Q \circ (h_1, \dots, h_m)$, $Q \in \mathcal{Q}(K^m, V)$ and $h_1, \dots, h_m\colon G \to K$ additive, where m is the minimal number of additive functions necessary to represent p as a polynomial in additive functions. Then all h_i are linear combinations of $g_1, g_2, \dots, g_n$.*

Proof. Let $g_1, \dots, g_n, h_1, \dots, h_k$ be a basis of $\mathcal{L}_K(\{g_1, \dots, g_n, h_1, \dots, h_m\})$. (Renumbering the h_l might be necessary.) We must show that $k = 0$. If k were greater than 0, we would be able to write $h_l = \sum_{j=1}^{k} u_{lj} h_j + \sum_{i=1}^{n} v_{li} g_i$, $l > k$. As in the proof of Theorem 3 we may even assume that all u_{lj} vanish.

Then $P \circ (g_1, \dots, g_n) = Q \circ (h_1, \dots, h_m)$ reads as

$$P \circ (g_1, \dots, g_n) - Q \circ \Big(h_1, \dots, h_k, \sum_{i=1}^{n} v_{k+1,i} g_i, \dots, \sum_{i=1}^{n} v_{mi} g_i\Big) = 0.$$

Theorem 1 implies

$$P(x_1, \dots, x_n) - Q\Big(y_1, \dots, y_k, \sum_{i=1}^{n} v_{k+1,i} x_i, \dots, \sum_{i=1}^{n} v_{mi} x_i\Big) = 0$$

for all $x_1, \dots, x_n, y_1, \dots, y_k \in K$. Thus

$$P(x_1, \dots, x_n) = Q^*\Big(\sum_{i=1}^{n} v_{k+1,i} x_i, \dots, \sum_{i=1}^{n} v_{mi} x_i\Big)$$

for all $x_1, \dots, x_n \in K$, where $Q^*(y_{k+1}, \dots, y_m) := Q(0, \dots, 0, y_{k+1}, \dots, y_m)$. With $x_i := g_i(t)$, $t \in G$, this results in $P \circ (g_1, \dots, g_n) = Q^* \circ (h_{k+1}, \dots, h_m)$ which contradicts the minimality of m. ∎

We may characterize polynomials in additive function in the following way.

THEOREM 6. *A mapping $p\colon G \to V$ is a polynomial in additive functions if and only if the subspace $\mathcal{L}_K(p(G))$ of V generated by the image of p is of finite dimension and if $\varphi \circ p\colon G \to K$ is a polynomial in additive functions for all $\varphi \in V^* := \operatorname{Hom}_K(V, K)$.*

Proof. Assume that $p\colon G \to V$ is a polynomial in additive functions: $p = P \circ (g_1, \dots, g_n)$, $g_1, \dots, g_n\colon G \to K$ additive, $P \in \mathcal{Q}(K^n, V)$, $P(x_1, \dots, x_n) = \sum_{\nu \in \mathbb{N}_0^n} x^\nu a_\nu$. Then $p(G)$ is contained in $\mathcal{L}_K\left(\{a_\nu \mid \nu \in \mathbb{N}_0^n, a_\nu \neq 0\}\right)$. Thus $\mathcal{L}_K(p(G))$ is of finite dimension. Moreover, for any $\varphi \in V^*$, $\varphi \circ p = (\varphi \circ P) \circ (g_1, \dots, g_n)$, where $\varphi \circ P \in \mathcal{Q}(K^n, K)$.

On the other hand, let $\{b_1, \dots, b_m\}$ be a basis of $\mathcal{L}_K(p(G))$ and let B be a basis of V containing the m linearly independent vectors $b_1, \dots, b_m$. Then $p = \sum_{i=1}^{m} p_i b_i$ with $p_i\colon G \to K$. Let $\varphi_i \in V^*$ be the projection from V to K determined by $b_i \in B$, i.e., $p_i = \varphi_i \circ p$. By assumption all p_i are polynomials in additive functions. This implies that $p = \sum_i p_i b_i$ is a polynomial in additive functions, too. ∎

Given a polynomial in additive functions, the question of finding representations as a polynomial in a minimal number of additive functions has a purely algebraic component. It has been pointed out in [Le] that such representations may be calculated explicitely as in the proof of the next theorem.

THEOREM 7. *Let K be a field with characteristic zero, $Q \in K[y_1, \dots, y_n]$ be a polynomial of degree $\leq d$,*

$$Q(y_1,\dots,y_n)=\sum_{k=0}^{d}\sum_{i_1+\cdots+i_n=k} a_{i_1\dots i_n}y_1^{i_1}\cdots y_n^{i_n},$$

and suppose that the equation

$$\sum_{j=1}^{n}\lambda_j\frac{\partial Q}{\partial y_j}=0 \tag{9}$$

has exactly $n-m$ linear independent solutions $\ell_i=(\lambda_1^{(i)},\dots,\lambda_n^{(i)})^\top\in K^n$, $i=1,\dots,n-m$. Then there are m linear forms $\varphi_i\in K[y_1,\dots,y_n]$, $i=1,\dots,m$, and a polynomial $P\in K[x_1,\dots,x_m]$ of degree $\le d$, such that

$$P\circ(\varphi_1,\dots,\varphi_m)=Q. \tag{10}$$

Moreover, m is the smallest number, such that a representation (10) *of Q is possible.*

Proof. There are $b_{i_1\dots i_n j}\in K$, such that

$$0=\sum_{j=1}^{n}\lambda_j\frac{\partial Q}{\partial y_j}=\sum_{k=0}^{d-1}\sum_{i_1+\cdots+i_n=k} y_1^{i_1}\cdots y_n^{i_n}\sum_{j=1}^{n}b_{i_1\dots i_n j}\lambda_j,$$

and linear independence of the monomials $y_1^{i_1}\cdots y_n^{i_n}$ imply that the linear system

$$\sum_{j=1}^{n}b_{i_1\dots i_n j}\lambda_j=0,\quad i_1+\cdots+i_n=0,\dots,d-1, \tag{11}$$

has $n-m$ linear independent solutions $\ell_i, i=1,\dots,n-m$. Let $A=(a_{ij})\in K^{m\times n}$ be a Gaussian row-eschelon form of the system matrix of (11), such that (11) may be written as $\sum_{j=1}^{n}a_{ij}\lambda_j=0, i=1,\dots,m$. A is the matrix of a vector $\varphi\in K^m[y_1,\dots,y_n]$ of linear forms, $\varphi(y)=Ay$, with component functions $\varphi_i(y_1,\dots,y_n)=\sum_{j=1}^{n}a_{ij}y_j$, $i=1,\dots,m$. Since $\operatorname{rank}(A)=m$, A is equivalent to $(E_m\ 0)\in K^{m\times n}$, where E_m is the $m\times m$-identity matrix. Hence there is a regular matrix $C\in K^{n\times n}$ satisfying $AC=(E_m\,0)$ and whose columns $m+1,\dots,n$ have to be linear independent solutions of (9), such that it may be written as $C=(b_1\ \cdots\ b_m\ \ell_1\ \cdots\ \ell_{n-m})$. Then $B:=(b_{ij}):=(b_1\ \cdots\ b_m)\in K^{n\times m}$ is the matrix of a vector $\psi\in K^n[x_1,\dots,x_m]$ of linear forms, $\psi(x)=Bx$, with component functions $\psi_i(x_1,\dots,x_m)=\sum_{j=1}^{m}b_{ij}x_j$, $i=1,\dots,n$. Let $P\in K[x_1,\dots,x_m]$ be defined by

$$P:=Q\circ(\psi_1,\dots,\psi_n).$$

Then P is a polynomial of degree $\le d$ and it remains to show that P is satisfying (10). $P=Q\circ\psi$ implies $P\circ\varphi=Q\circ\psi\circ\varphi$, such that we have to show $Q\circ\psi\circ\varphi=Q$. Using the notation $\widetilde{Q}:=Q\circ\psi\circ\varphi$, $q^\top:=(\frac{\partial Q}{\partial y_1},\dots,\frac{\partial Q}{\partial y_n})$

and $\widetilde{q}^\top := (\frac{\partial\widetilde{Q}}{\partial y_1}, \dots, \frac{\partial\widetilde{Q}}{\partial y_n})$, the chain rule renders $\widetilde{q}^\top = q^\top BA$ which leads to $\widetilde{q}^\top B = q^\top B$. Moreover, by (9) we have $q^\top \ell_i = 0$, $i = 1, \dots, n-m$, as well as $\widetilde{q}^\top \ell_i = q^\top BA\ell_i = 0$. It follows $\widetilde{q}^\top C = q^\top C$ and therefore $\widetilde{q} = q$, which together with $\widetilde{Q}(0) = Q(0)$ implies $\widetilde{Q} = Q$.

Suppose there are an m', a polynomial $P' \in K[x_1, \dots, x_{m'}]$ and m' linear mappings $\varphi'_i : K^n \to K$, such that $P' \circ (\varphi'_1, \dots, \varphi'_{m'}) = Q$. Using the notation $U_q := \mathcal{L}_K(\{\frac{\partial Q}{\partial y_i} \mid i = 1, \dots, n\})$, $U_p := \mathcal{L}_K(\{\frac{\partial P'}{\partial x_j} \mid j = 1, \dots, m'\})$, one obtains from $\frac{\partial Q}{\partial y_i} = \sum_{j=1}^{m'} \frac{\partial P'}{\partial x_j}\frac{\partial \varphi'_j}{\partial y_i}$, $i = 1, \dots, n$, that $U_q \subseteq U_p$ and therefore $m' \geq \dim_K U_p \geq \dim_K U_q = m$. ∎

EXAMPLE 1. Let $K = \mathbb{R}$ and consider the polynomial

$$Q(y_1, y_2, y_3, y_4) = 2y_1^2y_3 + 3y_1y_3^2 + y_3^3 - 4y_1y_2^2 - 4y_2^2y_3 - y_1y_4^2 - y_3y_4^2 + 4y_1^2y_2 + 4y_1y_2y_3 + 2y_1^2y_4 + 2y_1y_3y_4 - 4y_1y_2y_4 - 4y_2y_3y_4 - y_1 + 2y_2 + y_4.$$

Then

$$\begin{aligned}\sum_{j=1}^{4} \lambda_j \frac{\partial Q}{\partial y_j} &= y_1^2(4\lambda_2 + 2\lambda_3 + 2\lambda_4) + y_1y_2(8\lambda_1 - 8\lambda_2 + 4\lambda_3 - 4\lambda_4)\\ &\quad + y_1y_3(4\lambda_1 + 4\lambda_2 + 6\lambda_3 + 2\lambda_4) + y_1y_4(4\lambda_1 - 4\lambda_2 + 2\lambda_3 - 2\lambda_4)\\ &\quad + y_2^2(-4\lambda_1 - 4\lambda_3) + y_2y_3(4\lambda_1 - 8\lambda_2 - 4\lambda_4) + y_2y_4(-4\lambda_1 - 4\lambda_3)\\ &\quad + y_3^2(3\lambda_1 + 3\lambda_3) + y_3y_4(2\lambda_1 - 4\lambda_2 - 2\lambda_4)\\ &\quad + y_4^2(-\lambda_1 - \lambda_3) + 1 \cdot (-\lambda_1 + 2\lambda_2 + \lambda_4) = 0\end{aligned}$$

renders system (11) for the λ_i, whose system matrix in a Gaussian row-eschelon form is

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Observing that the number of rows of A equals $\operatorname{rank}(A)$ we can take in this case because of $K = \mathbb{R}$

$$B = A^+ = A^\top (AA^\top)^{-1} = \frac{1}{11}\begin{pmatrix} 6 & -2 \\ -2 & 8 \\ 5 & 2 \\ -1 & 4 \end{pmatrix}$$

to obtain

$$\begin{aligned}P(x_1, x_2) &= Q(\tfrac{1}{11}(6x_1 - 2x_2), \tfrac{1}{11}(-2x_1 + 8x_2), \tfrac{1}{11}(5x_1 + 2x_2), \tfrac{1}{11}(-x_1 + 4x_2))\\ &= 4x_1^2x_2 - 4x_1x_2^2 - x_1 + 2x_2.\end{aligned}$$

Indeed, this polynomial is satisfying

$$P(y_1 + y_3, y_2 + \tfrac{1}{2}y_3 + \tfrac{1}{2}y_4) = Q(y_1, y_2, y_3, y_4)$$

for all $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$.

3. Polynomials in additive functions and their characterization by translation invariant subspaces

The aim of this section is to generalize and to sharpen the main result, Theorem 12, of [RS, section 7]. Since there are many misprints in this paper we allow ourselves also to repeat some arguments from there.

Given an abelian group G and a vector space V over K, K a field of characteristic 0, a mapping $p\colon G \to V$ is called a generalized polynomial of degree $\leq n$, if it may be written in the form $p = \sum_{i=0}^{n} p_i$, where $p_i(x) = \widehat{p_i}(x, x, \ldots, x)$ with some $\widehat{p_i}\colon G^i \to V$ which is (symmetric and) additive in each component for all $i = 0, 1, \ldots, n$. We write $\mathcal{P}_n(G, V)$ for the space of all these mappings. The space $\mathcal{P}(G, V) := \bigcup_{n \in \mathbb{N}_0} \mathcal{P}_n(G, V)$ is called the space of (all) generalized polynomials defined on G with values in V. Since, for any given additive functions $g_1, \ldots, g_m\colon G \to K$ and any $a \in V$, the mapping $G^m \ni (x_1, x_2, \ldots, x_m) \mapsto g_1(x_1) \cdot \ldots \cdot g_m(x_m)a \in V$ is m-additive, the space

$$\mathcal{P}^*(G, V) := \{P \circ (g_1, \ldots, g_n) \mid n \in \mathbb{N},\, P \in \mathcal{Q}(K^n, V),\, g_1, g_2, \ldots, g_n\colon G \to K \text{ additive}\}$$

of polynomials in additive functions is a subspace of $\mathcal{P}(G, V)$.

If $\mathcal{P}(G, V) \ni p = p_0 + p_1 + \ldots + p_n$ with $p_0, \ldots, p_n$ as above this *homogeneous* components p_i of degree i in p are uniquely determined by p. Thus for $0 \neq p \in \mathcal{P}(G, V)$ the *degree* of p, $\partial p := \max\{i \mid p_i \neq 0\}$, is well-defined.

For functions $p\colon G \to V$ and $y \in G$ the *translate* $T_y p\colon G \to V$ is defined by translation in the argument: $(T_y p)(x) := p(x + y)$. As in [RS] it is seen easily that $\mathcal{P}_n(G, V)$ and $\mathcal{P}_n^*(G, V) = \mathcal{P}^*(G, V) \cap \mathcal{P}_n(G, V)$ are invariant under T_y for all y. If $0 \neq p \in \mathcal{P}(G, V)$ we also have $\partial T_y p = \partial p$. More exactly, given $p = p_n + q \in \mathcal{P}(G, V)$ with $0 \neq p_n$ homogeneous of degree n and $q \in \mathcal{P}_{n-1}(G, V)$ the translate $T_y p$ is of the form $T_y p = p_n + q_y$ with some $q_y \in \mathcal{P}_{n-1}(G, V)$.

Theorem 12 of [RS] is the special case with an algebraically closed field K of the following result.

THEOREM 8. *Let K be a field of characteristic 0, not necessarily algebraically closed. Let $p \in \mathcal{P}(G, K)$. Then $p \in \mathcal{P}^*(G, K)$ if and only if the subspace $\mathcal{T}(p) := \mathcal{L}_K\left(\{T_y p \mid y \in G\}\right)$ of $\mathcal{P}(G, K)$ generated by all translates $T_y p$ of p is of finite dimension.*

To prove this theorem we first provide certain preparatory results.

A function $\pi\colon G \to K$ is called *exponential function* if $\pi(x+y) = \pi(x) \cdot \pi(y)$ for all $x, y \in G$. It is well-known ([AD, p. 28]) that then either $\pi = 0$ or $\pi(x) \neq 0$ for all $x \in G$. A function $q\colon G \to V$, V a vector space over K, is called *exponential polynomial* if it is of the form $q = \sum_{i=1}^{n} \pi_i p_i$ with $n \in \mathbb{N}$, exponential functions π_i ($\neq 0$) and generalized polynomials $p_i \in \mathcal{P}(G, V)$.

LEMMA 3. *Let $1 \neq \pi\colon G \to K$ be an exponential function and let $p \in \mathcal{P}(G, V)$. Then $\pi(y)T_y p - p = 0$ for all $y \in G$ is possible only for $p = 0$.*

Proof. Suppose $p \neq 0$ and let $n := \partial p$. Then $p = p_n + q$ where $0 \neq p_n \in \mathcal{P}_n(G, V)$ is homogeneous of degree n and where $q \in \mathcal{P}_{n-1}(G, V)$. Then $T_y p = p_n + q_y$ with $q_y \in \mathcal{P}_{n-1}(G, V)$ for all $y \in G$. Let y' be such that $a := \pi(y') \neq 1$. Then the hypothesis applied for $y = y'$ implies $a(p_n + q_{y'}) = p_n + q$. Hence $ap_n = p_n$ which is impossible when $p_n \neq 0$ since $a \neq 1$. ∎

LEMMA 4. *Let $\pi_1, \pi_1, \ldots, \pi_n\colon G \to V$ be n distinct exponential functions, all different from 0 and let $p_1, p_2, \ldots, p_n \in \mathcal{P}(G, V)$. Then $\sum_{i=0}^{n} \pi_i p_i = 0$ only is possible if $p_1 = p_2 = \ldots = p_n = 0$.*

Proof. (Compare [RS, Theorem 9].) We proceed by induction. For $n = 1$ the equation $\pi_1 p_1 = 0$ implies $p_1 = 0$ since $\pi_1(x) \neq 0$ for all x. Suppose, now, that Lemma 4 holds true for n and consider a relation $\sum_{i=1}^{n+1} \pi_i p_i = 0$ with $n+1$ distinct exponential functions π_i, all $\neq 0$, and $p_1, \ldots, p_{n+1} \in \mathcal{P}(G, V)$. Dividing by π_{n+1} we get

$$q = \sum_{i=1}^{n} \pi_i^* q_i, \tag{12}$$

where $q = p_{n+1}$, $q_i = -p_i$ for $1 \le i \le n$, and $\pi_i^* := \pi_i / \pi_{n+1}$.

Note that $\pi_1^*, \ldots, \pi_n^*$ are n distinct exponential functions, all $\neq 0, 1$. To apply the induction hypothesis it is enough to show that $q = 0$. If $q \neq 0$ we may choose q such that (12) holds and that $m := \partial q$ is minimal. But (12) implies

$$T_y q = \sum_{i=1}^{n} \pi_i^*(y) \pi_i^* T_y q_i,\ y \in G.$$

Thus $T_y q - q = \sum_{i=1}^{n} \pi_i^* \cdot (\pi_i^*(y) T_y q_i - q_i)$. But $T_y q - q = 0$ if $m = 0$, and $T_y q - q \in \mathcal{P}_{m-1}(G, V)$ if $m \ge 1$. Since $\pi_i^*(y) T_y q_i - q_i \in \mathcal{P}(G, V)$ the minimality of m implies $T_y q - q = 0$ also for $m \ge 1$. But then, by induction hypothesis, $\pi_i^*(y) T_y q_i - q_i = 0$ for all i and all y. Hence by Lemma 3 all q_i must vanish. But then $q = 0$ contradicting our assumption that $q \neq 0$. ∎

The following lemma generalizes Theorem 10 of [RS].

LEMMA 5. *Let $A\colon G \to \mathrm{Gl}_n(K)$, K a field of characteristic 0, $\mathrm{Gl}_n(K)$ the group of regular $n \times n$-matrices with entries in K, be a homomorphism,* *i. e., $A(x+y) = A(x)A(y)$ for all $x \in G$. Assume that the characteristic polynomial $\chi_{A(x)}$ splits over K for all $x \in G$.*

Then there is some $S \in \mathrm{Gl}_n(K)$ such that $B\colon G \to \mathrm{Gl}_n(K)$ defined by $B(x) := SA(x)S^{-1}$ is a block diagonal matrix of the form $B(x) = \mathrm{diag}(B_1(x), B_2(x), \ldots, B_r(x))$, $B_i\colon G \to \mathrm{Gl}_{n_i}(K)$, $1 \le i \le r$, $n_1, \ldots, n_r \ge 1$, $n_1 + \ldots + n_r = n$. Moreover the B_i are given by $B_i(x) = \pi_i(x)\exp(C_i(x))$ where all π_i are non zero exponential functions and where the $C_i\colon G \to \mathrm{M}_{n_i}(K)$, $\mathrm{M}_{n_i}(K)$ the ring of all $n_i \times n_i$-matrices with entries in K, are such that all $C_i(x)$ are nilpotent lower triangular matrices. All entries of all C_i are additive functions from G to K and $C_i(x)$ commutes with $C_i(y)$ for all i and all $x, y \in G$.

Proof. Consider the family $(A(x))_{x\in G}$ of matrices in $\mathrm{M}_n(K)$. By

$$A(x)A(y) = A(x+y) = A(y+x) = A(y)A(x)$$

any two members of this family commute.

Inspecting the proof of Theorem 7 in ([J, p. 134]) one observes that the hypothesis made there, namely that K be algebraically closed, may be replaced by the (weaker) assumption that all $\chi_{A(x)}$ split over K without violating the conclusion of this theorem.

Thus we may find some $S \in \mathrm{Gl}_n(K)$ such that B defined by $B(x) := SA(x)S^{-1}$ is of the form $B(x) = \mathrm{diag}(B_1(x), B_2(x), \ldots, B_r(x))$ where all $B_i\colon G \to \mathrm{M}_{n_i}(K)$ are of the form

$$B_i(x) = \begin{pmatrix} \pi_i(x) & & & 0 \\ & \pi_i(x) & & \\ & & \ddots & \\ * & & & \pi_i(x) \end{pmatrix}, \quad x \in G.$$

Since all $A(x)$ are regular so are all $B(x)$ and thus also all $B_i(x)$. $A(x+y) = A(x)A(y)$ obviously implies that B (and all B_i) satisfy the same functional equation. Since the B_i are lower triangular this in particular means that all π_i are exponential functions. By the regularity of B_i all $\pi_i \neq 0$.

Thus $B_i = \pi_i \cdot (E_i + B_i^*)$ where E_i is the identity matrix of $\mathrm{M}_{n_i}(K)$. Then $B_i^*(x)$ has the form

$$B_i^*(x) = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix}.$$

Thus $B_i^*(x) = (1/\pi_i(x)) \cdot B_i(x) - E_i$ is a lower triangular matrix which is also nilpotent, $B_i^*(x)^{n_i} = 0$. Let

$$C_i(x) := \ln\left(E_i + B_i^*(x)\right) := \sum_{j=1}^{n_i-1} \frac{(-1)^{j-1}}{j} B_i^*(x)^j.$$

The formal power series $\ell(X) := \ln(1+X) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} X^j$ satisfies $\ell((1+X)(1+Y)-1) = \ell(X)+\ell(Y)$. $B_i^*(x)$ and $B_i^*(y)$ commute since $B_i(x)$ and $B_i(y)$ do. This implies

$$\begin{aligned} C_i(x)+C_i(y) &= \ell(B_i^*(x)) + \ell(B_i^*(y)) = \ell\left((E_i + B_i^*(x))(E_i + B_i^*(y)) - E_i\right) \\ &= \ell\left(\frac{B_i(x)}{\pi_i(x)}\frac{B_i(y)}{\pi_i(y)} - E_i\right) = \ell\left(\frac{B_i(x+y)}{\pi_i(x+y)} - E_i\right) \\ &= \ell\left(B_i^*(x+y)\right) = C_i(x+y), \qquad i = 1,2,\ldots,r,\ x,y \in G. \end{aligned}$$

Since, for formal series, $\exp(\ell(X)) = 1 + X$ and since all $C_i(x)$ are lower triangular and nilpotent we infer that $\exp\left(C_i(x)\right) := \sum_{j=0}^{n_i-1} \frac{1}{j!} C_i(x)^j = E_i + B_i^*(x) = (1/\pi_i(x))B_i(x)$. Thus $B_i(x) = \pi_i(x)\exp(C_i(x))$ where $C_i\colon G \to \mathrm{M}_{n_i}(K)$ has the desired properties. ($C_i(x)$ and $C_i(y)$ commute since $B_i^*(x)$ and $B_i^*(y)$ commute.) ∎

COROLLARY 2. *Let all assumptions of Lemma* 5 *be satisfied. Then all entries of A are of the form $a_{ij} = \sum_{l=1}^{s} \pi_l^* \cdot q_{ij}^{(l)}$ where $\pi_1^*, \ldots, \pi_s^*\colon G \to K$ are s distinct exponential functions, all $\neq 0$, and where all $q_{ij}^{(l)}$ are elements of $\mathcal{P}^*(G,K)$.*

P r o o f. The entries of the matrices C_k are additive functions. Thus the entries of $x \mapsto \exp(C_k(x))$ are contained in $\mathcal{P}^*(G,K)$. This implies that the entries of B_k, and then also of B, are of the form $\pi_{ij} \cdot r_{ij}$ with non vanishing exponential functions π_{ij} and $r_{ij} \in \mathcal{P}^*(G,K)$. From $A(x) = S^{-1}B(x)S$ we infer that the entries of A are linear combinations of the $\pi_{ij} \cdot r_{ij}$. The assertion follows by defining $\pi_1^*, \ldots, \pi_s^*$ to be s different exponential functions ($\neq 0$) such that $\{\pi_1^*, \ldots, \pi_s^*\} = \{\pi_{ij} \mid 1 \le i,j \le n\}$. ∎

Now we are ready to prove Theorem 8.

P r o o f. (Theorem 8) Let, first, $p \in \mathcal{P}^*(G,K)$. Thus $p = \sum_{\nu \in \mathbb{N}_0^n, |\nu| \le N} g_1^{\nu_1} \cdot \ldots \cdot g_n^{\nu_n} a_\nu$ for certain additive functions $g_i\colon G \to K$ and certain $a_\nu \in K$, $|\nu| \le N$, where $n, N \in \mathbb{N}$. It is easily verified that $T_y g_1^{\nu_1} \cdot \ldots \cdot g_n^{\nu_n} \in \mathcal{L}_K\left(\left\{g^\mu := g_1^{\mu_1} \cdot \ldots \cdot g_n^{\mu_n} \mid |\mu| \le |\nu|\right\}\right)$. Thus $T_y p$ is contained in the space generated by the functions $g^\mu a_\nu$, $|\mu|, |\nu| \le N$. This space is of finite dimension. So $\mathcal{T}(p)$ as a subspace of this space is also of finite dimension.

Now, let $p \in \mathcal{P}(G,K)$ and assume that $\mathcal{T}(p)$ has finite dimension. Of course we may assume $p \neq 0$. Let $p_1 = p, p_2, \ldots, p_n$ be a basis of $\mathcal{T}(p)$. Since $T_x q \in \mathcal{T}(p)$ for all $q \in \mathcal{T}(p)$ and all $x \in G$ the $T_x p_i$ are linear combinations of the p_j,

$$T_x p_i = \sum_{j=1}^{n} a_{ij}(x) p_j,\ x \in G,\ 1 \le i \le n. \tag{13}$$

This together with $T_{x+y} = T_x \circ T_y$ implies that $A\colon G \to \mathrm{M}_n(K)$ defined by $A(x) = (a_{ij}(x))_{1\le i,j\le n}$ satisfies $A(x+y) = A(y)A(x)$ for all $x, y \in G$. $T_0 = \mathrm{id}$ implies $A(0) = E = E_n$ showing (since $A(0) = A(x)A(-x)$) that $A\colon G \to \mathrm{Gl}_n(K)$.

Let $\overline{K}$ be any extension of K such that all $\chi_{A(x)}$ split over $\overline{K}$. By Lemma 5 we may find some $S \in \mathrm{Gl}_n(\overline{K})$ such that $B\colon G \to \mathrm{Gl}_n(\overline{K})$ defined by $B(x) := SA(x)S^{-1}$ has the properties described in Lemma 5 (with $\overline{K}$ instead of K). By this lemma and its corollary, Corollary 2, we see that $a_{ij} = \sum_{l=1}^{s} \pi_l^* q_{ij}^{(l)}$ with distinct exponential functions $0 \ne \pi_l^*\colon G \to \overline{K}$ and polynomials $q_{ij}^{(l)} \in \mathcal{P}^*(G, \overline{K})$.

(13) implies $p(x) = p_1(x) = T_x p_1(0) = \sum_{j=1}^{n} a_{1j}(x) p_j(0)$. Inserting here the specific form of the a_{1j} shows that $p = \sum_{l=1}^{s} \pi_l^* q_l$ with certain $q_l \in \mathcal{P}^*(G, \overline{K})$.

In this representation one of the π_l^* has to be identical 1. Otherwise we would write

$$p \cdot 1 + \sum_{l=1}^{s} \pi_l^* \cdot (-q_l) = 0$$

with $s+1$ distinct exponential functions $1, \pi_1^*, \ldots, \pi_s^*$, all $\ne 0$, and polynomials $p, -q_1, \ldots, -q_s \in \mathcal{P}(G, \overline{K})$. This, by Lemma 4, implies $0 = p = -q_1 = \ldots = -q_s$. Thus in particular $p = 0$, a contradiction.

So, say, $\pi_1^* = 1$. Then $p = \sum_{l=1}^{s} \pi_l^* q_l$ may be written as

$$1 \cdot (q_1 - p) + \sum_{l=2}^{s} \pi_l^* q_l = 0.$$

Applying Lemma 4 once more now gives $p = q_1$. Thus $p \in \mathcal{P}^*(G, \overline{K})$.

We still have to show that $p \in \mathcal{P}^*(G, K) \subseteq \mathcal{P}^*(G, \overline{K})$. Let $u_i := \sum_{k=1}^{n} s_{ik} p_k$ where $s_{ik} \in \overline{K}$ are the entries of the matrix $S := (s_{ik})_{1\le i,k\le n}$. Then all u_i are contained in $\overline{\mathcal{T}}(p) := \mathcal{L}_{\overline{K}}\left(\{T_x p \mid x \in G\}\right)$. Equality $S \cdot A(x) = B(x) \cdot S$ and (13) imply $T_x u_i = \sum_{k=1}^{n} b_{ik}(x) u_k$. Since $B(x) = \mathrm{diag}(B_1(x), \ldots, B_r(x))$ and since the $B_l(x)$ are lower triangular matrices with $\pi_l(x)$ as entries along the main diagonal we recognize that $T_x u_{j_l} = \pi_l(x) u_{j_l}$ for $l = 1, 2, \ldots, s$, where $j_1 := 1, j_2 := n_1 + 1, \ldots, j_r := n_1 + \ldots + n_{r-1} + 1$.

All u_i are different from 0. In fact, all vectors $\sigma_i := (s_{i1}, s_{i2}, \ldots, s_{in})$ are different from the zero vector in $\overline{K}^n$. They even are linearly independent over $\overline{K}$ since S is regular. Moreover $p_1, p_2, \ldots, p_n$ are linearly independent over K. Let D be a basis of $\overline{K}$ over K. Then $s_{ik} = \sum_{d\in D} s_{ik}^{(d)} d$, where all $s_{ik}^{(d)} \in K$ and where for any fixed i and k the number of those $d \in D$ such that $s_{ik}^{(d)} \neq 0$ is finite. Suppose $u_i = 0$, then

$$0 = \sum_{k=1}^{n}\Big(\sum_{d\in D} s_{ik}^{(d)} d\Big)p_k(x) = \sum_{d\in D}\Big(\sum_{k=1}^{n} s_{ik}^{(d)} p_k(x)\Big)d,\ x \in G.$$

But $p_k(x), s_{ik}^{(d)} \in K$. Thus $\sum_{k=1}^{n} s_{ik}^{(d)} p_k(x) = 0$ for all $x \in G$ and all $d \in D$, i.e., $\sum_{k=1}^{n} s_{ik}^{(d)} p_k = 0$. This is a linear relation for the p_k over K. Thus $s_{ik}^{(d)} = 0$ for all k and all $d \in D$. But then $\sigma_i = 0$, a contradiction.

In particular all $u_{j_l} \neq 0$. Evaluating $T_x u_{j_l} = \pi_l \cdot u_{j_l}$ at some $y \in G$ with $u_{j_l}(y) \neq 0$ then shows $\pi_i = (1/u_{j_l}(y))T_y u_{j_l}$, i.e., $\pi_i \in \overline{\mathcal{T}}(p)$ for all $i = 1, \ldots, r$. But $\overline{\mathcal{T}}(p) \subseteq \mathcal{P}(G, \overline{K})$. This implies $\pi_1 = \pi_2 = \ldots = \pi_r = 1$, since by Lemma 4 only the exponential functions 0 and 1 are polynomials.

But then all $\chi_{A(x)}$ already split over K. Thus we may choose $\overline{K} = K$ which finally gives the desired result. ∎

We can even prove more.

THEOREM 9. *Let K be an arbitrary field of characteristic* 0 *and V any vector space over K. Then for any abelian group G a generalized polynomial $p \in \mathcal{P}(G, V)$ is a polynomial in additive functions, $p \in \mathcal{P}^*(G, V)$, if and only if the subspace $\mathcal{T}(p) := \mathcal{L}_K\left(\{T_x p \,|\, x \in G\}\right)$ of $\mathcal{P}(G, V)$, generated by all translates $T_x p$ of p, is of finite dimension.*

Proof. If $p \in \mathcal{P}^*(G, V)$ the space $\mathcal{T}(p)$ must be of finite dimension. This can be seen in (almost) exactly the same way as the corresponding part in Theorem 8.

Now assume $0 \neq p \in \mathcal{P}(G, V)$ and that $\mathcal{T}(p)$ is of finite dimension. Again we may proceed first as in the proof of the preceding theorem. $T_x p = T_x p_1 = \sum_{j=1}^{n} a_{1j}(x) p_j$, where $p_1 = p, p_2, \ldots, p_n$ constitute a basis of $\mathcal{T}(p)$, implies $p(x) = \sum_{j=1}^{n} a_{1j}(x) p_j(0)$, which shows that $\mathcal{L}_K(p(G))$ is of finite dimension. Obviously $\varphi \circ p \in \mathcal{P}(G, K)$ for any given $\varphi \in V^*$. Since $T_x(\varphi \circ p) = \varphi \circ (T_x p)$ for all $x \in G$ we obtain $\mathcal{T}(\varphi \circ p) = \mathcal{L}_K(\{\varphi \circ p_1, \ldots, \varphi \circ p_n\})$ showing that $\mathcal{T}(\varphi \circ p)$ is of finite dimension for all $\varphi \in V^*$. Thus $\varphi \circ p \in \mathcal{P}^*(G, K)$ for all $\varphi \in V^*$ by Theorem 8. Then Theorem 6 implies $p \in \mathcal{P}^*(G, V)$. ∎

4. Polynomials in additive functions and multi-Jensen functions

Assuming that V and W are $\mathbb{Q}$-vecor spaces, this section deals with another characterization of the elements of $\mathcal{P}^*(V, W) \subseteq \mathcal{P}(V, W)$. For $n \in \mathbb{N}$ let $\delta_n : V \to V^n$ be the diagonalization mapping, which assigns to $x \in V$ the vector $(x, \ldots, x) \in V^n$ with x in each component. In [PS] it has been shown that for every symmetric n-Jensen function $f : V^n \to W$ the diagonalization $p := f \circ \delta_n$ is an element of $\mathcal{P}_n(V, W)$ and that for every $p \in \mathcal{P}_n(V, W)$ there exists a uniquely determined symmetric n-Jensen function $f : V^n \to W$, such that $p = f \circ \delta_n$. This one-to-one correspondence will be utilized now for that characterization.

LEMMA 6. *Let $P \in \mathcal{Q}(K^m, W)$ be the polynomial*

$$P(y_1, \ldots, y_m) = \sum_{k=0}^{n} \sum_{j_1+\cdots+j_m=k} y_1^{j_1} \cdots y_m^{j_m} a_{j_1 \ldots j_m},$$

$a_{j_1 \ldots j_m} \in W$, let $g_i : V \to K$, $i = 1, \ldots, m$, be additive functions and let the function $f : V^n \to W$ be defined by

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{n} \sum_{1 \le j_1 < \cdots < j_i \le n} M_i(x_{j_1}, \ldots, x_{j_i}), \tag{14}$$

where

$$M_i(x_1, \ldots, x_i) = \binom{n}{i}^{-1} \sum_{k_1+\cdots+k_m=i} \prod_{j=1}^{m} \prod_{\ell=1}^{k_j} g_j\left(x_{\sum_{q=1}^{j-1} k_q+\ell}\right) a_{k_1 \ldots k_m}, \tag{15}$$

$i = 0, \ldots, n$. Then f is n-Jensen and for every generalized polynomial $p \in \mathcal{P}_n(V, W)$ the following holds: $p = P \circ (g_1, \ldots, g_m)$ if and only if $p = f \circ \delta_n$.

Proof. At first we show, that the functions M_i defined by (15) are i-additive. Given $r \in \{1, \ldots, i\}$, for any partition $k_1 + \cdots + k_m = i$ of i there are uniquely determined numbers $\mu \in \{1, \ldots, m\}$ with $k_\mu \ge 1$ and $\ell_r \in \{1, \ldots, k_\mu\}$, such that $r = \sum_{q=1}^{\mu-1} k_q + \ell_r$. For $y = (y_1, \ldots, y_i) \in V^i$ and $\hat{y}_r \in V$, let $\tilde{y} \in V^i$ be the vector with the components $\tilde{y}_r = y_r + \hat{y}_r$ and $\tilde{y}_j = y_j$ for all $j \neq r$. By additivity of g_μ one obtains

$$\begin{aligned} &\prod_{\ell=1}^{k_\mu} g_\mu(\tilde{y}_{\sum_{q=1}^{\mu-1} k_q+\ell}) \\ &= \prod_{\ell=1}^{k_\mu} g_\mu(y_{\sum_{q=1}^{\mu-1} k_q+\ell}) + \prod_{\ell=1}^{\ell_r-1} g_\mu(y_{\sum_{q=1}^{\mu-1} k_q+\ell}) \cdot g_\mu(\hat{y}_r) \cdot \prod_{\ell=\ell_r+1}^{k_\mu} g_\mu(y_{\sum_{q=1}^{\mu-1} k_q+\ell}). \end{aligned}$$

Multiplying this sum with the remaining products $\prod_{\ell=1}^{k_j} g_j(y_{\sum_{q=1}^{j-1} k_q+\ell})$, $j \neq \mu$, then taking the weighted sum over all partitions $k_1 + \cdots + k_m$ of i and multiplying with $\binom{n}{i}^{-1}$, it follows

$$\begin{aligned} &M_i(y_1, \ldots, y_{r-1}, y_r + \hat{y}_r, y_{r+1}, \ldots, y_i) \\ &\qquad = M_i(y_1, \ldots, y_i) + M_i(y_1, \ldots, y_{r-1}, \hat{y}_r, y_{r+1}, \ldots, y_i), \end{aligned}$$

showing i-additivity of M_i, $i = 1, \ldots, n$. Since f is the sum of i-additive mappings, $i = 1, \ldots, n$, plus a constant, it is n-Jensen by Theorem 2 of [PS]. The diagonalization of M_i, $i = 1, \ldots, n$, renders

$$M_i\big(\delta_i(x)\big) = \binom{n}{i}^{-1} \sum_{k_1+\cdots+k_m=i} \prod_{j=1}^{m} \prod_{\ell=1}^{k_j} g_j(x) a_{k_1\ldots k_m}$$
$$= \binom{n}{i}^{-1} \sum_{k_1+\cdots+k_m=i} g_1(x)^{k_1} \cdots g_m(x)^{k_m} a_{k_1\ldots k_m}, \quad x \in V,$$

that is, a homogeneous polynomial of degree i in the variables $g_j(x)$, $j = 1, \ldots, m$. Furthermore,

$$f\big(\delta_n(x)\big) = M_0 + \sum_{i=1}^{n} \sum_{1\le j_1<\cdots<j_i\le n} M_i\big(\delta_i(x)\big) = M_0 + \sum_{i=1}^{n} \binom{n}{i} M_i\big(\delta_i(x)\big)$$
$$= \sum_{i=0}^{n} \sum_{k_1+\cdots+k_m=i} g_1(x)^{k_1} \cdots g_m(x)^{k_m} a_{k_1\ldots k_m} = P\big(g_1(x), \ldots, g_m(x)\big).$$

Since by Corollary 1 of [PS] the diagonalization of an n-Jensen function is a generalized polynomial of degree $\le n$ (or, as shown in Section 6 of [RS], an ordinary polynomial in additive functions is a generalized polynomial), it follows for an arbitrary generalized polynomial $p \in \mathcal{P}_n(V, W)$, that $p = f \circ \delta_n$ if and only if $p = P \circ (g_1, \ldots, g_m)$. ∎

REMARK 3. A representation of the uniquely determined symmetric n-Jensen function f, whose diagonalization is rendering a generalized polynomial p of degree $\le n$ satisfying $p = P \circ (g_1, \ldots, g_m)$ with P and g_i as in the lemma, is easily obtained by substituting M_i in (14) by

$$\frac{1}{i!} \sum_{\pi \in \mathcal{S}_i} M_i(y_{\pi(1)}, \ldots, y_{\pi(i)}), \quad i = 1, \ldots, n,$$

where the M_i in these expressions are given by (15) and $\mathcal{S}_i$ denotes the symmetric group of order i.

THEOREM 10. *Let be $p \in \mathcal{P}_n(V, W)$. Then there exist an $m \in \mathbb{N}$, additive functions $g_1, \ldots, g_m : V \to K$ and a polynomial $P \in \mathcal{Q}(K^m, W)$ such that $p = P \circ (g_1, \ldots, g_m)$ if and only if there exist an $L \in \mathbb{N}$, Jensen functions $h_1^{(\ell)}, \ldots, h_n^{(\ell)} : V \to K$ and $a_\ell \in W$ for $\ell = 1, \ldots, L$, such that $p = f \circ \delta_n$, with $f : V^n \to W$,*

$$f(x_1, \ldots, x_n) = \sum_{\ell=1}^{L} \prod_{j=1}^{n} h_j^{(\ell)}(x_j) a_\ell. \tag{16}$$

Proof. Suppose that the generalized polynomial has a representation $p = P \circ (g_1, \ldots, g_m)$ with additive functions g_i and let $f : V^n \to W$ be the n-Jensen function defined by (14) and (15), satisfying $p = f \circ \delta_n$ by Lemma 6. We are going to transform the products of the additive functions appearing in (15), where, in dependence upon k_j, a specific g_j may appear repeatedly or even not at all, into a product of n Jensen functions, where each of them appears exactly once. For that let $S = \emptyset$ or $S = \{j_1, \ldots, j_i\} \subseteq \{1, \ldots, n\} =: \mathbf{n}$, let $k_1 + \cdots + k_m = i$ be a partition of $i = |S|$ and take a $j \in \mathbf{n}$. In case $j \in S$ there is an $\ell \in \mathbf{i}$, such that $j = j_\ell$ and we define

$$h_j^{S,\ k_1 \ldots k_m}(x) := g_{\mu_\ell}(x), \quad x \in V,$$

where $\mu_\ell = \max\{\nu \mid \sum_{q=1}^{\nu-1} k_q + 1 \leq \ell\}$. In case $j \notin S$ we define

$$h_j^{S,\ k_1 \ldots k_m}(x) := 1 \in K, \quad x \in V.$$

All functions $h_j^{S,\ k_1 \ldots k_m}$ are Jensen and

$$\prod_{j=1}^{n} h_j^{S,\ k_1 \ldots k_m}(x_j) = \prod_{j \in S} h_j^{S,\ k_1 \ldots k_m}(x_j) = \prod_{\ell=1}^{i} h_{j_\ell}^{\{j_1, \ldots, j_i\},\ k_1 \ldots k_m}(x_{j_\ell})$$
$$= \prod_{\ell=1}^{k_1+\cdots+k_m} g_{\mu_\ell}(x_{j_\ell}) = \prod_{s=1}^{m} \prod_{t=1}^{k_s} g_s(x_{j_{\sum_{q=1}^{s-1} k_q + t}}),$$

since $\mu_\ell = s$ is equivalent to $\sum_{q=1}^{s-1} k_q + 1 \leq \ell \leq \sum_{q=1}^{s} k_q$. Therefore we can rewrite (15) as

$$M_i(x_{j_1}, \ldots, x_{j_i}) = \binom{n}{i}^{-1} \sum_{k_1+\cdots+k_m=i} \prod_{j=1}^{n} h_j^{S,\ k_1 \ldots k_m}(x_j) a_{k_1 \ldots k_m},$$

where $S = \{j_1, \ldots, j_i\}$ as before. Observing $h_j^{\emptyset,\ 0 \ldots 0} \equiv 1$ for $j = 1, \ldots, n$, we get

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq \mathbf{n}} \binom{n}{|S|}^{-1} \sum_{k_1+\cdots+k_m=|S|} \prod_{j=1}^{n} h_j^{S,\ k_1 \ldots k_m}(x_j) a_{k_1 \ldots k_m}.$$

With a suitable renaming of the indices of summation and of the W-valued factors $\binom{n}{|S|}^{-1} a_{k_1 \ldots k_m}$ we obtain an $L \in \mathbb{N}$, $a_\ell \in W$, $\ell = 1, \ldots, L$, and $n \cdot L$ Jensen functions $h_1^{(\ell)}, \ldots, h_n^{(\ell)}$, $\ell = 1, \ldots, L$, such that f may be represented by (16).

Conversely, suppose that $p = f \circ \delta_n$ for the generalized polynomial p of degree $\leq n$, where f is given by (16) with an $L \in \mathbb{N}$, $a_\ell \in W$ and Jensen functions $h_1^{(\ell)}, \ldots, h_n^{(\ell)}$, $\ell = 1, \ldots, L$. For each of these Jensen functions there is an additive function $b_j^{(\ell)} : V \to K$ and a constant $c_j^{(\ell)} \in K$, such that

$$h_j^{(\ell)}(x) = b_j^{(\ell)}(x) + c_j^{(\ell)}, \quad j = 1, \dots, n, \quad \ell = 1, \dots, L, \quad x \in V.$$

f as a sum of products of n Jensen functions is an n-Jensen function and may be written as

$$\begin{aligned} f(x_1, \dots, x_n) &= \sum_{\ell=1}^{L} \prod_{j=1}^{n} (b_j^{(\ell)}(x_j) + c_j^{(\ell)}) a_\ell \\ &= \sum_{\ell=1}^{L} \sum_{S \subseteq \mathbf{n}} \prod_{j \in \mathbf{n} \setminus S} c_j^{(\ell)} \prod_{j \in S} b_j^{(\ell)}(x_j) a_\ell, \qquad x_1, \dots, x_n \in V. \end{aligned}$$

Diagonalization of this function renders

$$p(x) = f(x, \dots, x) = \sum_{\ell=1}^{L} \sum_{S \subseteq \mathbf{n}} \prod_{j \in \mathbf{n} \setminus S} c_j^{(\ell)} \prod_{j \in S} b_j^{(\ell)}(x) a_\ell, \quad x \in V,$$

hence there is a natural number $m := n \cdot L$ and a polynomial $P \in \mathcal{Q}(K^m, W)$, such that, after a suitable renaming like $b_j^{(\ell)} =: g_{n(\ell-1)+j}$, $j = 1, \dots, n$, $\ell = 1, \dots, L$, the generalized polynomial may be represented as $p = P \circ (g_1, \dots, g_m)$. ∎

For an example below we need the following lemma, which sharpens the result quoted from [AD] at the begin of the proof of Theorem 1.

LEMMA 7. *Let V be a vector space of arbitrary dimension and let $\mathcal{B}$ be a Hamel basis of V. If $g_1, \dots, g_m : V \to \mathbb{Q}$ are m linearly independent additive functions, then there are $b_1, \dots, b_m \in \mathcal{B}$ such that the vectors $(g_1(b_i), \dots, g_m(b_i)) \in \mathbb{Q}^m$, $i = 1, \dots, m$, are linearly independent.*

Proof. The proof is by induction on m. Observing $g_1 \neq 0$, there is a $b_1 \in \mathcal{B}$ such that $g_1(b_1) \neq 0$, hence the assertion is true in case $m = 1$. Suppose that there are $b_1, \dots, b_{m-1} \in \mathcal{B}$, such that

$$\det \begin{pmatrix} g_1(b_1) & \cdots & g_{m-1}(b_1) \\ \vdots & & \vdots \\ g_1(b_{m-1}) & \cdots & g_{m-1}(b_{m-1}) \end{pmatrix} \neq 0.$$

The functions $g_1, \dots, g_m$ are linearly independent, so there is an $x \in V$ such that

$$\det\begin{pmatrix} g_1(b_1) & \cdots & g_{m-1}(b_1) & g_m(b_1) \\ \vdots & & \vdots & \vdots \\ g_1(b_{m-1}) & \cdots & g_{m-1}(b_{m-1}) & g_m(b_{m-1}) \\ g_1(x) & \cdots & g_{m-1}(x) & g_m(x) \end{pmatrix} = \sum_{i=1}^{m} g_i(x) d_i(b_1, \ldots, b_{m-1}) \neq 0,$$

where the determinant was expanded in the last row with $d_i(b_1, \ldots, b_{m-1})$, $i = 1, \ldots, m$, being the corresponding $(m-1) \times (m-1)$ subdeterminants. Writing $x \in V$ as

$$x = \sum_{j=1}^{m-1} \alpha_j b_j + \sum_{b \in \mathcal{B} \setminus \{b_1, \ldots, b_{m-1}\}} \alpha_b b,$$

additivity of the g_i renders

$$\sum_{i=1}^{m} g_i(x) d_i(b_1, \ldots, b_{m-1}) = \sum_{j=1}^{m-1} \alpha_j \sum_{i=1}^{m} g_i(b_j) d_i(b_1, \ldots, b_{m-1})$$
$$+ \sum_{b \in \mathcal{B} \setminus \{b_1, \ldots, b_{m-1}\}} \alpha_b \sum_{i=1}^{m} g_i(b) d_i(b_1, \ldots, b_{m-1}).$$

Since $\sum_{i=1}^{m} g_i(b_j) d_i(b_1, \ldots, b_{m-1}) = 0$ for all $j = 1, \ldots, m-1$, there is a $b \in \mathcal{B} \setminus \{b_1, \ldots, b_{m-1}\}$ such that $\sum_{i=1}^{m} g_i(b) d_i(b_1, \ldots, b_{m-1}) \neq 0$. ∎

The following example may serve as an illustration for several results in this paper. It is a generalization of an example, originally presented in [RS, Section 7], of a generalized polynomial, which is not a polynomial in additive functions.

EXAMPLE 2. Let V be a $\mathbb{Q}$-vector space of infinite dimension with a Hamel Basis $\mathcal{B}$ and let W be a $\mathbb{Q}$-vector space with $\dim W \geq 1$. For a fixed nonzero $w \in W$ and a natural number $n \geq 2$ let the mapping $f : V^n \to W$ be defined by

$$f(x_1, \ldots, x_n) = f(\sum_{b \in \mathcal{B}} \beta_{1,b} b, \ldots, \sum_{b \in \mathcal{B}} \beta_{n,b} b) := (\sum_{b \in \mathcal{B}} \prod_{i=1}^{n} \beta_{i,b}) w. \tag{17}$$

Since f is n-linear and symmetric, its diagonalization $p : V \to W$, $p(x) := f(x, \ldots, x)$, is a generalized polynomial of degree $\leq n$.

(A) Assume that there are an $m \in \mathbb{N}$, a polynomial $P \in \mathcal{Q}(\mathbb{Q}^m, W)$ and additive functions $g_1, \ldots, g_m : V \to \mathbb{Q}$ such that

$$p = P \circ (g_1, \ldots, g_m). \tag{18}$$

We will derive a contradiction to (A) in two ways, firstly by a result of Section 3 and secondly using results of Section 2 and of the present section.

1. By Theorem 9 it follows $\dim \mathcal{T}(p) = N < \infty$, hence there are $b_i \in \mathcal{B}$ and $\lambda_i \in \mathbb{Q}$, $i = 1, \ldots, N+1$, $(\lambda_1, \ldots, \lambda_{N+1}) \neq (0, \ldots, 0)$, such that $\sum_{i=1}^{N+1} \lambda_i T_{b_i} p = 0$. Evaluating this sum at $b \in \mathcal{B} \setminus \{b_1, \ldots, b_{N+1}\}$ we obtain, by definition of f,

$$0 = \sum_{i=1}^{N+1} \lambda_i p(b_i + b) = \sum_{i=1}^{N+1} \lambda_i f(b_i + b, \ldots, b_i + b) = 2 \sum_{i=1}^{N+1} \lambda_i w.$$

Hence, replacing b by b_j and evaluating as before, we get

$$\begin{aligned} 0 &= \sum_{i=1}^{N+1} \lambda_i p(b_i + b_j) = \lambda_j f(2b_j, \ldots, 2b_j) + \sum_{i=1, i\neq j}^{N+1} \lambda_i f(b_i + b_j, \ldots, b_i + b_j) \\ &= 2^n \lambda_j w + 2 \sum_{i=1, i \neq j}^{N+1} \lambda_i w = (2^n - 2)\lambda_j w \end{aligned}$$

for $j = 1, \ldots, N+1$. Consequently, $\lambda_1 = \cdots = \lambda_{N+1} = 0$, a contradiction.

2. By Remark 2 we may assume that m is the minimal number such that p has a representation of the form (18). By Theorem 2 of [PS] the n-Jensen function f admits a representation (14) with $M_0 = f(0, \ldots, 0)$ and

$$M_i(x_{j_1}, \ldots, x_{j_i}) = \sum_{T \subseteq \{j_1, \ldots, j_i\}} (-1)^{|\{j_1, \ldots, j_i\} \setminus T|} f(x_T), \quad i = 1, \ldots, n,$$

where $1 \leq j_1 < j_2 < \cdots < j_i \leq n$, and $x_T \in V^n$ being the vector with components $(x_T)_j = 0$, if $j \notin T$, and $(x_T)_j = x_j$, if $j \in T$, $j = 1, \ldots, n$. By definition of f we have in the present case $M_i = 0$ for $i = 0, \ldots, n-1$ and therefore $M_n(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. Because of $p(rx) = r^n p(x)$ for all $r \in \mathbb{Q}$, we also have $P(g_1(rx), \ldots, g_m(rx)) = r^n P(g_1(x), \ldots, g_m(x))$, therefore the degree of P has to be n. So Lemma 6 renders under the assumption (18)

$$f(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{k_1 + .. + k_m = n} \sum_{\pi \in \mathcal{S}_n} \prod_{j=1}^{m} \prod_{\ell=1}^{k_j} g_j(x_{\pi(\sum_{q=1}^{j-1} k_q + \ell)}) a_{k_1 \ldots k_m} \tag{19}$$

with $a_{k_1 \ldots k_m} \in W$, and we will derive a contradiction to (18), showing that the uniquely determined symmetric n-Jensen function f associated with p cannot have a representation (19).

Diagonalization of (19) yields

$$p(x) = f(x, \ldots, x) = \sum_{k_1 + \cdots + k_m = n} g_1(x)^{k_1} \cdots g_m(x)^{k_m} a_{k_1 \ldots k_m}, \quad x \in V, \tag{20}$$

and minimality of m implies by Theorem 3 linear independence of $\{g_1, \ldots, g_m\}$, hence, by Lemma 7, there are $b_1, \ldots, b_m \in \mathcal{B}$ such that $g_1\big|_U, \ldots, g_m\big|_U$ are linearly independent, where $U := \mathcal{L}_{\mathbb{Q}}(\{b_1, \ldots, b_m\})$. Moreover the $m \times m$-matrix $A := \big(g_i(b_j)\big)$ is regular and with $A^{-1} = C = (c_{ij})$ the functions $\tilde{g}_i : V \to K$, $\tilde{g}_i(x) := \sum_{j=1}^m c_{ij} g_j(x)$, $i = 1, \ldots, m$, are additive and satisfy $\tilde{g}_i(b_j) = \delta_{ij}$ for $i, j = 1, \ldots, m$. Therefore (20) may be written as

$$p(x) = \sum_{k_1+\cdots+k_m=n} \tilde{g}_1(x)^{k_1} \cdots \tilde{g}_m(x)^{k_m} \tilde{a}_{k_1 \ldots k_m}, \quad x \in V, \tag{21}$$

with $\tilde{a}_{k_1 \ldots k_m} \in W$. On the other hand, by definition of f, we also have

$$p(x) = \sum_{i=1}^m \tilde{g}_i(x)^n w, \quad x \in U. \tag{22}$$

Inserting $x = \sum_{i=1}^m \lambda_i b_i \in U$ in (21) and (22), we obtain because of $\tilde{g}_i(x) = \lambda_i$, $i = 1, \ldots, m$, that

$$\sum_{k_1+\cdots+k_m=n} \lambda_1^{k_1} \cdots \lambda_m^{k_m} \tilde{a}_{k_1 \ldots k_m} = p(x) = \sum_{i=1}^m \lambda_i^n w,$$

implying that $\tilde{a}_{0 \ldots 0 n 0 \ldots 0} = w$, n being at the i-th position in the multiindex, $i = 1, \ldots, m$, and $\tilde{a}_{k_1 \ldots k_m} = 0$ otherwise. Hence the representation of $p(x)$ given by (22) is valid for all $x \in V$.

Taking now an arbitrary $b \in \mathcal{B} \setminus \{b_1, \ldots, b_m\}$ and $j \in \{1, \ldots, m\}$, it follows for all $r \in \mathbb{Q}$

$$(r^n + 1)w = p(rb + b_j) = (\sum_{i=1}^m \tilde{g}_i(rb + b_j)^n)w = \sum_{i=1}^m (r\tilde{g}_i(b) + \delta_{ij})^n w$$

$$= (r^n \sum_{i=1, i \neq j}^m \tilde{g}_i(b)^n + (r\tilde{g}_j(b) + 1)^n)w = (r^n + 1)w + \sum_{\ell=1}^{n-1} \binom{n}{\ell} r^\ell \tilde{g}_j(b)^\ell w,$$

hence $\sum_{\ell=1}^{n-1} \binom{n}{\ell} r^\ell \tilde{g}_j(b)^\ell$ is a polynomial in r which vanishes identically. It follows $\tilde{g}_j(b) = 0$ for all $b \in \mathcal{B} \setminus \{b_1, \ldots, b_m\}$ and for $j = 1, \ldots, m$, in contradiction to $\sum_{i=1}^m \tilde{g}_i^n(b) w = p(b) = w$ for all $b \in \mathcal{B}$.

THEOREM 11. *Let V, W be $\mathbb{Q}$-vector spaces, $W \neq \{0\}$. Then* $\dim V < \infty$ *if and only if $\mathcal{P}(V, W) = \mathcal{P}^*(V, W)$.*

Proof. Suppose $\dim V = m < \infty$ and denote by $\mathcal{B} = \{b_1, \ldots, b_m\}$ a basis of V. Let be $p \in \mathcal{P}(V, W)$, then there is an $n \in \mathbb{N}_0$ such that $p \in \mathcal{P}_n(V, W)$ and there are symmetric, i-additive mappings $\hat{p}_i : V^i \to W$, $i = 0, \ldots, n$, such that $p = \sum_{i=0}^n \hat{p}_i \circ \delta_i$. For $j = 1, \ldots, m$ we define the projections $h_j : V \to \mathbb{Q}$, $h_j(x_k) = h_j(\sum_{l=1}^m \lambda_{kl} b_l) = \lambda_{kj}$. The h_j are Jensen-functions and

$$\hat{p}_i(x_1, \ldots, x_i) = \sum_{j_1, \ldots, j_i = 1}^{m} \lambda_{1j_1} \cdots \lambda_{ij_i} \hat{p}_i(b_{j_1}, \ldots, b_{j_i})$$

$$= \sum_{j_1, \ldots, j_i = 1}^{m} h_{j_1}(x_1) \cdots h_{j_i}(x_i) \hat{p}_i(b_{j_1}, \ldots, b_{j_i}), \quad i = 0, \ldots, n,$$

hence, by Theorem 10, $\hat{p}_i \circ \delta_i \in \mathcal{P}_i^*(V, W) \subseteq \mathcal{P}^*(V, W)$ for $i = 0, \ldots, n$, and therefore $p \in \mathcal{P}^*(V, W)$. The inclusion $\mathcal{P}^*(V, W) \subseteq \mathcal{P}(V, W)$ is obvious.

If $\dim V = \infty$, then $p := f \circ \delta_n$ with f defined by (17) is in $\mathcal{P}(V, W)$, but not in $\mathcal{P}^*(V, W)$, as was shown in Example 1. ∎

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