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STRONG MAXIMUM PRINCIPLES FOR INFINITE SYSTEMS OF PARABOLIC DIFFERENTIAL-FUNCTIONAL INEQUALITIES WITH NONSTANDARD INITIAL INEQUALITIES WITH INTEGRALS

Abstract. In this paper we consider infinite systems of parabolic differential-functional inequalities with nonstandard initial inequalities with integrals. For that systems we give strong maximum principles in relatively arbitrary $(n+1)$-dimensional time-space sets more general than the cylindrical domain.

1. Introduction

We shall consider an infinite system of parabolic type differential-functional inequalities of the following form

$$u_t^i(x,t) \le F_i(x,t,u^i(x,t),u_x^i(x,t),u_{xx}^i(x,t),u) \quad (i \in \mathbb{N}), \tag{1.1}$$

where $x=(x_1,\ldots,x_n)$, $(x,t) \in D$ and $D \subset \mathbb{R}^n \times (t_0,t_0+T]$.

The symbol u denotes the mapping

$$u : \mathbb{N} \times \widetilde{D} \ni (i,x,t) \to u^i(x,t) \in \mathbb{R},$$

where $\widetilde{D}$ is an arbitrary set such that

$$\overline{D} \subset \widetilde{D} \subset \mathbb{R}^n \times (-\infty, t_0+T].$$

The right-hand sides F_i $(i \in \mathbb{N})$ of system (1.1) are functionals of u, $u_x^i(x,t) = grad_x u^i(x,t)$ and $u_{xx}^i(x,t)$ denote the matrices of second order derivatives with respect to x of $u^i(x,t)$ $(i \in \mathbb{N})$.

In this paper we give theorems on strong maximum principles for problems with inequalities (1.1) and with the nonstandard inequalities

$$(u^j(x,t_0)-K^j)+\sum_{i\in I_*} h_i(x)(u^j(x,T_i)-K^j) \le 0 \text{ for } x \in S_{t_0} \ (j \in \mathbb{N}),$$

where K^i $(i \in \mathbb{N})$ are constant functions such that $(K^1,K^2,\ldots) \in l^\infty$.

The results obtained in this paper are generalization of some thesis from publications: L. Byszewski [3] and J. Chabrowski [8].

Comparison theorems for infinite systems of parabolic functional-differential equations were considered by D. Jaruszewska-Walczak in [5]. The results obtained in [5] are in the case when the solutions are defined on bounded sets. In this paper, the situation is different. Consequently, the assumptions on the right- hand sides of the equations and inequalities, in this paper, are different than in [5].

Infinite and finite systems of hyperbolic functional differential inequalities were considered by Z. Kamont in [6] and [7]. The monograf [6] is a self-contained exposition of hyperbolic functional differential inequalities and their applications, on which topic the present author initiated research. It aims to give a systematic and unified presentation of recent developments in the following problems: functional differential inequalities generated by initial and mixed problems; existence theory of local and global solutions; functional integral equations generated by hyperbolic equations; numerical methods of lines for hyperbolic problems; and difference methods for initial and initial-boundary value problems. Besides classical solutions, some classes of weak solutions are also treated, such as Carathéodory solutions for quasilinear equations, entropy solutions and viscosity solutions for nonlinear problems, and solutions in the Friedrichs sense for almost linear equations. The theory of difference and differential difference equations generated by original problems and its applications to the construction of numerical methods for functional differential problems is also discussed.

In paper [7], Z. Kamont presents general comparison theorems for hyperbolic functional-differential infinite systems. He gives an estimate of functions of several variables satisfying an infinite system of functional differential inequalities by means of solutions of suitable systems of ordinary functional-differential equations. As a consequence he obtains a general theorem of the Perron type on the uniqueness of classical solutions of initial value problems. Next he proves a comparison result for infinite systems with initial-boundary value conditions. A general uniqueness result with nonlinear estimates of the Perron type is also obtained.

2. Preliminaries

We shall use the following notations:

$$\mathbb{R} = (-\infty, +\infty), \quad \mathbb{R}_- = (-\infty, 0], \ \mathbb{N} = \{1, 2, \dots\},$$
$$x = (x_1, \dots, x_n) \in \mathbb{R}^n (n \in \mathbb{N}).$$

By l^∞ we denote the Banach space of real sequences $\xi = (\xi^1, \xi^2, \dots)$ such that $\sup\{|\xi^j| : j = 1, 2, \dots\} < \infty$ and $\|\xi\|_{l^\infty} = \sup\{|\xi^j| : j = 1, 2, \dots\}$. For

$\xi = (\xi^1, \xi^2, \ldots)$, $\eta = (\eta^1, \eta^2, \ldots) \in l^\infty$ we write $\xi \le \eta$ in the sense $\xi^i \le \eta^i$ $(i \in \mathbb{N})$.

By $M_{n \times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n \times n}$.

We write $r \ge 0$ if

$$\sum_{j,k=1}^{n} r_{jk}\lambda_j\lambda_k \ge 0$$

for all $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$.

Let t_0 be an arbitrary real finite number and let $T \in (0, \infty)$.

A set $D \subset \{(x,t) : x \in \mathbb{R}^n, t_0 < t \le t_0 + T\}$ is called *a set of type (P)* if:

(a) the projection of the interior of set D on the t-axis is the interval $(t_0, t_0 + T)$,

(b) for every $(\widetilde{x}, \widetilde{t}) \in D$ there exists a positive number $\delta = \delta(\widetilde{x}, \widetilde{t})$ such that

$$\Big\{(x,t) : \sum_{i=1}^{n}(x_i - \widetilde{x}_i)^2 + (t - \widetilde{t})^2 < \delta,\ t < \widetilde{t}\Big\} \subset D,$$

(c) all the boundary points $(\widetilde{x}, \widetilde{t})$ of D for which there is a positive number $\delta = \delta(\widetilde{x}, \widetilde{t})$ such that

$$\Big\{(x,t) : \sum_{i=1}^{n}(x_i - \widetilde{x}_i)^2 + (t - \widetilde{t})^2 < \delta,\quad t \le \widetilde{t}\Big\} \subset D$$

belong to D.

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t = \begin{cases} int\{x \in \mathbb{R}^n : (x, t_0) \in \overline{D}\} & \text{for } t = t_0, \\ \{x \in \mathbb{R}^n : (x,t) \in D\} & \text{for } t \ne t_0, \end{cases}$$

$$\sigma_t = \begin{cases} int[\overline{D} \cap (\mathbb{R}^n \times \{t_0\})] & \text{for } t = t_0, \\ D \cap (\mathbb{R}^n \times \{t\}) & \text{for } t \ne t_0. \end{cases}$$

Let $\widetilde{D}$ be an arbitrary set such that

$$\overline{D} \subset \widetilde{D} \subset \mathbb{R}^n \times (-\infty, t_0 + T].$$

We introduce the following sets:

$$\partial_p D := \widetilde{D} \backslash D \quad \text{and} \quad \Gamma := \partial_p D \setminus \sigma_{t_0}.$$

For an arbitrary fixed point $(\widetilde{x}, \widetilde{t}) \in D$, we denote by $S^-(\widetilde{x}, \widetilde{t})$ the set of points $(x,t) \in D$, that can be joined to $(\widetilde{x}, \widetilde{t})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (x,t) to $(\widetilde{x}, \widetilde{t})$.

Let $Z_\infty(\widetilde{D})$ denote the linear space of mappings

$$w : \mathbb{N} \times \widetilde{D} \ni (i, x, t) \to w^i(x,t) \in \mathbb{R},$$

where functions

$$w^i : \widetilde{D} \ni (x,t) \to w^i(x,t) \in \mathbb{R}$$

are continuous in $\overline{D}$ and

$$\sup\{|w^i(x,t)| : (x,t) \in \widetilde{D},\ i \in \mathbb{N}\} < \infty.$$

In the set of mappings w belonging to $Z_\infty(\widetilde{D})$ we define the functional $[\cdot]_{t,\infty}$ by the formula

$$[w]_{t,\infty} = \sup\{0, w^i(x,\widetilde{t}) : (x,\widetilde{t}) \in \widetilde{D},\ \widetilde{t} \le t,\ i \in \mathbb{N}\},$$

where $t \le t_0 + T$.

By $Z^{2,1}_\infty(\widetilde{D})$ we denote the linear subspace of $Z_\infty(\widetilde{D})$. A mapping w belongs to $Z^{2,1}_\infty(\widetilde{D})$ if w^i_t, $w^i_x = (w^i_{x_1}, \dots, w^i_{x_n})$, $w^i_{xx} = [w^i_{x_j x_k}]_{n\times n}$ $(i \in \mathbb{N})$ are continuous in D.

For each $i \in \mathbb{N}$ by F_i we denote the mapping

$$F_i : D \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n}(\mathbb{R}) \times Z_\infty(\widetilde{D}) \ni (x,t,z,q,r,w) \to F_i(x,t,z,q,r,w) \in \mathbb{R} \quad (i \in \mathbb{N}),$$

where $q = (q_1, \dots, q_n)$ and $r = [r_{jk}]$.

By P_i $(i \in \mathbb{N})$ we denote an operator given by the formula

$$(P_i w)(x,t) := w^i_t(x,t) - F_i(x,t,w^i(x,t),w^i_x(x,t),w^i_{xx}(x,t),w) \quad (i \in \mathbb{N}),$$

for $w \in Z^{2,1}_\infty(\widetilde{D})$ and $(x,t) \in D$.

A function $u \in Z^{2,1}_\infty(\widetilde{D})$ is called *a solution of the system of the functional-differential inequalities*

$$(P_i u)(x,t) \underset{(\ge)}{\le} 0 \qquad (i \in \mathbb{N})$$

in D, if they satisfy the system for all $(x,t) \in D$.

For a given subset $E \subset D$, and a given mapping $w \in Z^{2,1}_\infty(\widetilde{D})$ and a fixed index $i \in \mathbb{N}$ the function F_i is called *uniformly parabolic with respect to w in E*, if there is a constant $\varkappa > 0$ (depending on E) such that for any two matrices $r = [r_{jk}] \in M_{n\times n}(\mathbb{R})$, $\widetilde{r} = [\widetilde{r}_{jk}] \in M_{n\times n}(\mathbb{R})$ and for $(x,t) \in E$ we have

$$(2.1) \quad r \le \widetilde{r} \Longrightarrow F_i(x,t,w^i(x,t),w^i_x(x,t),\widetilde{r},w) - F_i(x,t,w^i(x,t),w^i_x(x,t),r,w) \ge \varkappa \sum_{j=1}^{n} (\widetilde{r}_{jj} - r_{jj}).$$

If (2.1) is satisfied for $\varkappa = 0$ and $r = w^i_{xx}(x,t)$, where $(x,t) \in E$, and for $\widetilde{r} = w^i_{xx}(x,t) + \widehat{r}$, where $(x,t) \in E$ and $\widehat{r} \ge 0$, then F_i is called *parabolic with respect to w* in E.

Let $J = \mathbb{N}$ or J is a finite set of mutually different natural numbers.

Let us define the following set:

$$S = \bigcup_{i \in J} (\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}),$$

where, in case if $J = \mathbb{N}$, the following conditions are satisfied:

(i) $t_0 < T_{2i-1} < T_{2i} \le t_0 + T$ for $i \in J$ and $T_{2i-1} \neq T_{2j-1}$, $T_{2i} \neq T_{2j}$ for $i, j \in J, i \neq j$;

(ii) $\mathbb{T}_0 := \inf\{T_{2i-1} : i \in J\} > t_0$;

(iii) $S_t \supset S_{t_0}$ for every $t \in \bigcup_{i \in J} [T_{2i-1}, T_{2i}]$;

(iv) $S_t \supset S_{t_0}$ for every $t \in [\mathbb{T}_0, t_0 + T]$,

and in case if J is a finite set of mutually different natural numbers, the conditions (i), (iii) are satisfied.

An unbounded set D of type (P) is called *a set of type* $(P_{S\Gamma})$, if:

(a) $S \neq \emptyset$,

(b) $\Gamma \cap \overline{\sigma}_{t_0} \neq \emptyset$.

Let S_* denote a non-empty subset of S. We define the following set:

$$J_* = \{i \in J : (\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}) \subset S_*\}.$$

A bounded set D of type (P) satisfying condition (a) of the definition of a set of type $(P_{S\Gamma})$ is called *a set of type* (P_{SB}).

Observe,that if D is a set of type (P_{SB}), then D satisfies condition (b) of definition of a set of type $(P_{S\Gamma})$. Observe also, that if D_0 is an arbitrary bounded subset of $\mathbb{R}^n$, then $D = D_0 \times (t_0, t_0 + T]$ is a set of type (P_{SB}). In the case, if D_0 is an arbitrary unbounded proper subset of $\mathbb{R}^n$, then $D = D_0 \times (t_0, t_0 + T]$ is a set of type $(P_{S\Gamma})$.

For every set $A \subset \widetilde{D}$ and for each function $w \in Z_\infty(\widetilde{D})$ we apply the notation:

$$\max_{(x,t) \in A} w(x,t) := \Big(\max_{(x,t) \in A} w^1(x,t), \max_{(x,t) \in A} w^2(x,t), \ldots\Big).$$

3. Strong maximum principles with nonstandard inequalities with integrals in sets of types $(P_{S\Gamma})$ and (P_{SB})

THEOREM 3.1. *Assume that:*

(1) $D \subset \mathbb{R}^n \times (t_0, t_0 + T]$ *is a set of type* $(P_{S\Gamma})$ *or* (P_{SB});

(2) F_i $(i \in \mathbb{N})$ *are the mappings as in Section* 2 *and there exists a constant* $L > 0$ *such that*

$$F_i(x,t,z,q,r,w) - F_i(x,t,\widetilde{z},\widetilde{q},\widetilde{r},\widetilde{w})$$
$$\leq L(|z-\widetilde{z}| + |x|\sum_{j=1}^{n}|q_j-\widetilde{q}_j| + |x|^2\sum_{j,k=1}^{n}|r_{jk}-\widetilde{r}_{jk}| + [w-\widetilde{w}]_{t,\infty}) \quad (i\in\mathbb{N}),$$

for all $(x,t)\in D$, $z,\widetilde{z}\in\mathbb{R}$, $q,\widetilde{q}\in\mathbb{R}^n$, $r,\widetilde{r}\in M_{n\times n}(\mathbb{R})$, $w,\widetilde{w}\in Z_\infty(\widetilde{D})$;

(3) $u\in Z^{2,1}_\infty(\widetilde{D})$ *and the maximum of function* u *on* Γ *is attained. Moreover,*

$$K^i := \max_{(x,t)\in\Gamma} u^i(x,t) \quad (i\in\mathbb{N}), \tag{3.1}$$

and $K\in l^\infty$ *is defined by formulae*

$$K:\mathbb{N}\times\widetilde{D}\ni(i,x,t)\to K^i;$$

(4) *The following inequalities hold*

$$(u^j(x,t_0)-K^j) + \sum_{i\in J_*} h_i(x)\left(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x,\tau)d\tau - K^j\right) \leq 0 \quad \text{for } x\in S_{t_0}\ (j\in\mathbb{N}), \tag{3.2}$$

where $h_i : S_{t_0}\to\mathbb{R}_-$ $(i\in J_*)$ *are given functions such that*

$$-1\leq\sum_{i\in J_*}h_i(x)\leq 0 \qquad \text{for } x\in S_{t_0}$$

and, additionally, if $\mathrm{card}J_* = \aleph_0$, *then the series*

$$\sum_{i\in I_*} h_i(x)\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x,\tau)d\tau \quad (j\in\mathbb{N})$$

are convergent for $x\in S_{t_0}$;

(5) *There exists a point* $(x^*,t^*)\in\widetilde{D}$ *such that*

$$u(x^*,t^*) = \max_{(x,t)\in\widetilde{D}} u(x,t);$$

Moreover,

$$M^i := u^i(x^*,t^*) \quad (i\in\mathbb{N}) \tag{3.3}$$

and $M\in l^\infty$ *is defined by*

$$M:\mathbb{N}\times\widetilde{D}\ni(i,x,t)\to M^i;$$

(6) $F_i(x,t,M^i,0,0,M)\leq 0$ *for* $(x,t)\in D$ $(i\in\mathbb{N})$;

(7) *The function* u *is a solution of system*

$$(P_iu)(x,t)\leq 0 \qquad \text{for } (x,t)\in D \quad (i\in\mathbb{N}).$$

(8) *The mappings* F_i $(i\in\mathbb{N})$ *are parabolic with respect to* u *in* D *and uniformly parabolic with respect to* M *in any compact subset of* D.

Then

$$\max_{(x,t)\in\widetilde{D}} u(x,t) = \max_{(x,t)\in\Gamma} u(x,t). \tag{3.4}$$

Moreover, if there is a point $(\widetilde{x},\widetilde{t}) \in D$ *such that*

$$u(\widetilde{x},\widetilde{t}) = \max_{(x,t)\in\widetilde{D}} u(x,t),$$

then

$$u(x,t) = \max_{(x,t)\in\Gamma} u(x,t) \qquad \textit{for} \;\; (x,t) \in S^-(\widetilde{x},\widetilde{t}).$$

Proof. We prove Theorem 3.1 only for a set of type $(P_{S\Gamma})$, because the proof of this theorem for a set of type (P_{SB}) is similar.

It is obvious that a set of type $(P_{S\Gamma})$ is a set of type (P_Γ) from [4], hence,in the case where $\sum_{i\in J_*} h_i(x) = 0$ for $x \in S_{t_0}$, Theorem 3.1 is a consequence of Theorem 4.1 from [4]. Therefore, we shall give the proof of Theorem 3.1 only in the case where

$$-1 \le \sum_{i\in J_*} h_i(x) < 0 \qquad \text{for} \;\; x \in S_{t_0}. \tag{3.5}$$

We argue by contradiction. Suppose that (3.5) is satisfied and let

$$M \neq K. \tag{3.6}$$

From (3.1) and (3.3), it follows

$$K^i \le M^i \qquad (i \in \mathbb{N}). \tag{3.7}$$

Consequently (3.6), (3.7) imply that

$$\text{There is } \; l \in \mathbb{N} \text{ such that } K^l < M^l. \tag{3.8}$$

From assumption (5), we deduce that

$$\text{There is a point } (x^*,t^*) \in \widetilde{D} \text{ such that} \tag{3.9}$$

$$u(x^*,t^*) = M := \max_{(x,t)\in\widetilde{D}} u(x,t).$$

By (3.9), by assumption (3) and by (3.8) we obtain

$$(x^*,t^*) \in \widetilde{D} \setminus \Gamma = D \cup \sigma_{t_0}. \tag{3.10}$$

An argument analogous to the proof of Theorem 4.1 from [4] yields

$$(x^*,t^*) \notin D. \tag{3.11}$$

Consequently

$$(x^*,t^*) \in \sigma_{t_0}. \tag{3.12}$$

On account of the definition of sets J and J_*, we distinguish the following cases:

(A) J_* is a finite set, i.e., without loss generality there is a number $p \in \mathbb{N}$ such that $J_* = \{1, \dots, p\}$.

(B) $card J_* = \aleph_0$.

We consider first the case (A). By (3.2) and by the inequalities

$$u(x^*, t) < u(x^*, t_0) \quad \text{for} \quad t \in \bigcup_{i=1}^{p} [T_{2i-1}, T_{2i}],$$

which are consequences of (3.9) and (3.11) and of conditions (a)(i), (a)(iii) of the definition of a set of type $(P_{S\Gamma})$, we have

$$\begin{aligned} 0 &\geq (u^j(x^*, t_0) - K^j) + \sum_{i=1}^{p} h_i(x^*) \Big(\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(x^*, \tau) - K^j \Big) \\ &\geq (u^j(x^*, t_0) - K^j) + \sum_{i=1}^{p} h_i(x^*) \Big(\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(x^*, t_0) d\tau - K^j \Big) \\ &= (u^j(x^*, t_0) - K^j) \Big(1 + \sum_{i=1}^{p} h_i(x^*) \Big) \qquad (j \in \mathbb{N}). \end{aligned}$$

Thus

$$u(x^*, t_0) \leq K, \quad \text{if} \quad 1 + \sum_{i=1}^{p} h_i(x^*) > 0. \tag{3.13}$$

Obviously, from (3.8) and (3.11), we obtain a contradiction of (3.12) with (3.9). Assume now that

$$\sum_{i=1}^{p} h_i(x^*) = -1. \tag{3.14}$$

By the mean-value integral theorem, it is obviously that for every $j \in \mathbb{N}$ and $i \in \{1, \dots, p\}$ there is

$$\widetilde{T}_i^j \in [T_{2i-1}, T_{2i}]$$

such that

$$u^j(x^*, \widetilde{T}_i^j) = \frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(x^*, \tau) d\tau. \tag{3.15}$$

Simultaneously, we see that for every $j \in \mathbb{N}$ there is a number $l_j \in \{1, \dots, p\}$ such tha

$$u^j(x^*, \widetilde{T}_{l_j}^j) = \max_{i=1,\dots,p} u^j(x^*, \widetilde{T}_i^j). \tag{3.16}$$

Hence, by (3.14),(3.16),(3.15) and by (3.2), we have

$$\begin{aligned}
&u^j(x^*,t_0) - u^j(x^*,\widetilde{T}^j_{l_j})\\
&= (u^j(x^*,t_0) - K^j) - (u^j(x^*,\widetilde{T}^j_{l_j}) - K^j)\\
&= (u^j(x^*,t_0) - K^j) + \sum_{i=1}^{p} h_i(x^*)(u^j(x^*,\widetilde{T}^j_{l_j}) - K^j)\\
&\leq (u^j(x^*,t_0) - K^j) + \sum_{i=1}^{p} h_i(x^*)(u^j(x^*,\widetilde{T}^j_{i}) - K^j)\\
&= (u^j(x^*,t_0) - K^j) + \sum_{i=1}^{p} h_i(x^*)\Big(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,\tau)d\tau - K^j\Big)\\
&\leq 0 \qquad (j\in\mathbb{N}).
\end{aligned}$$

It implies that

$$u^j(x^*,t_0) \leq u^j(x^*,\widetilde{T}^j_{l_j}) \;\; (j\in\mathbb{N}), \quad \text{if} \quad \sum_{i=1}^{p} h_i(x^*) = -1. \tag{3.17}$$

Since, condition (a)(i) of the definition of a set type $(P_{S\Gamma})$ implies inequalities $\widetilde{T}^j_{l_j} > t_0 \quad (j\in\mathbb{N})$, then from (3.12) we observe that (3.17) contradicts (3.9). This completes the proof of formula (3.4) in case (A).

Next consider the case (B). Similarly to the proof of (3.4) in case (A), by assumption (4) and by the inequalities

$$u(x^*,t) < u(x^*,t_0) \quad \text{ for } \quad t\in \bigcup_{i\in J_*}[T_{2i-1},T_{2i}],$$

being a consequence of (3.9), (3.12) and of conditions (a)(i), (a)(iii) of the definition of a set of type $(P_{S\Gamma})$, we have

$$\begin{aligned}
0 &\geq (u^j(x^*,t_0) - K^j) + \sum_{i\in J_*} h_i(x^*)\Big(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,\tau)d\tau - K^j\Big)\\
&\geq (u^j(x^*,t_0) - K^j) + \sum_{i\in J_*} h_i(x^*)\Big(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,t_0)d\tau - K^j\Big)\\
&= (u^j(x^*,t_0) - K^j)\Big(1+\sum_{i\in J_*} h_i(x^*)\Big) \quad (j\in\mathbb{N}).
\end{aligned}$$

Hence

$$u(x^*,t_0) \leq K, \qquad \text{if} \;\; 1+\sum_{i\in J_*} h_i(x^*) > 0. \tag{3.18}$$

It follows from (3.8)and (3.12) that a condition (3.18) is at a contradiction with condition (3.9).We assume here

$$\sum_{i=J_*} h^i(x^*) = -1. \tag{3.19}$$

Applying the mean-value integral theorem we have that for every $j \in \mathbb{N}$ and $i \in J_*$ there is $\widetilde{T}_i^j \in [T_{2i-1}, T_{2i}]$ such that

$$u^j(x^*, \widetilde{T}_i^j) = \frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(x^*, \tau)d\tau. \tag{3.20}$$

Define

$$\widetilde{T}_*^j = \inf_{i \in J_*} \widetilde{T}_i^j \quad (j \in \mathbb{N}). \tag{3.21}$$

Because $u^i \in C(\overline{D})$ $(i \in \mathbb{N})$ and $x^* \in S_t$ for every $t \in [\mathbb{T}_0, t_0 + T]$, if $cardJ = \aleph_0$ (see: (3.12) and (a)(iv), (a)(ii) of the definition of a set of type $(P_{S\Gamma})$), it follows from (3.21) that for every $j \in \mathbb{N}$ there is $\widehat{t}_j \in [\widetilde{T}_*^j, t_0 + T]$ such that

$$u^j(x^*, \widehat{t}_j) = \max_{t \in [\widetilde{T}_*^j, t_0 + T]} u^j(x^*, t). \tag{3.22}$$

Consequently, by(3.19), (3.22), (3.20) and by assumption (4), we obtain

$$\begin{aligned}
&u^j(x^*, t_0) - u^j(x^*, \widehat{t}_j) \\
&= (u^j(x^*, t_0) - K^j) - (u^j(x^*, \widehat{t}_j) - K^j) \\
&= (u^j(x^*, t_0) - K^j) + \sum_{i \in J_*} h_i(x^*)(u^j(x^*, \widehat{t}_j) - K^j) \\
&\leq (u^j(x^*, t_0) - K^j) + \sum_{i \in J_*} h_i(x^*)(u^j(x^*, \widetilde{T}_i^j) - K^j) \\
&= (u^j(x^*, t_0) - K^j) + \sum_{i \in J_*} h_i(x^*)\left(\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(x^*, \tau)d\tau - K^j\right) \\
&\leq 0 \quad (j \in \mathbb{N}).
\end{aligned}$$

From the last inequality we have

$$u^j(x^*, t_0) \leq u^j(x^*, \widehat{t}_j) \quad (j \in \mathbb{N}), \quad \text{if } \sum_{i \in J_*} h_i(x^*) = -1. \tag{3.23}$$

Since, condition (a)(ii) of the definition of a set of type $(P_{S\Gamma})$ implies inequalities $\widehat{t}_j > t_0$ $(j \in \mathbb{N})$, hence, we see from (3.12) that (3.23) contradicts (3.9). This proves of equality (3.4).

The second part of Theorem 3.1 is a consequence of (3.4) and Lemma 3.1 from [4]. Therefore, the proof of Theorem 3.1 is complete.

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