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ON A CERTAIN CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH A CONVOLUTION STRUCTURE

Abstract. Making use of a convolution structure, we introduce a new class of analytic functions defined in the open unit disc and investigate its various characteristics. Apart from deriving a set of coefficient bounds, we establish several inclusion relationships involving the (n, δ)-neighborhoods of analytic functions with negative coefficients belonging to this subclass.

1. Introduction and preliminaries

Let $A(n)$ denote the class of functions normalized by

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0, \ n \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and univalent in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C}, \ |z| < 1\}$.

For functions $f \in A(n)$ given by (1.1) and $g(z) \in A(n)$ given by

$$g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \quad (b_k \geq 0, \ n \in \mathbb{N}), \tag{1.2}$$

we recall the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z - \sum_{k=n+1}^{\infty} a_k b_k z^k \quad (z \in \mathcal{U}). \tag{1.3}$$

In terms of the Hadamard product (or convolution), we choose g as a fixed function in $A(n)$ such that $(f*g)(z)$ exists for any $f \in A(n)$, and for various choices of g we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (1.3), we consider the following examples.

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(1) If

$$(1.4)\qquad g(z)=z+\sum_{k=n+1}^{\infty}\frac{(\alpha_1)_{k-1}\dots(\alpha_p)_{k-1}}{(\beta_1)_{k-1}\dots(\beta_q)_{k-1}}\frac{z^k}{(k-1)!}=z+\sum_{k=n+1}^{\infty}\Gamma_k z^k,$$

where

$$\Gamma_k=\frac{(\alpha_1)_{k-1}\dots(\alpha_p)_{k-1}}{(\beta_1)_{k-1}\dots(\beta_q)_{k-1}}\frac{1}{(k-1)!},$$

then the convolution (1.3) gives the Dziok–Srivastava operator [6]:

$$\Lambda(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z)f(z)\equiv\mathcal{H}_q^p(\alpha_1,\beta_1)f(z),$$

where $\alpha_1,\cdots,\alpha_p$; $\beta_1,\cdots,\beta_q$ are positive real numbers, $p\leq q+1; p,q\in\mathbb{N}\cup\{0\}$, and $(a)_k$ denotes the familiar Pochhammer symbol (or shifted factorial).

Remark 1. When $p=1$, $q=1$; $\alpha_1=a$, $\alpha_2=1$; $\beta_1=c$, then the Dziok-Srivastava operator (1.4) corresponds to the operator due to Carlson-Shaffer operator[3] given by

$$\mathcal{L}(a,c)f(z):=(f*g)(z),$$

where

$$(1.5)\qquad \mathcal{L}(a,c)f(z)\equiv zF(a,1;c;z)*f(z)$$
$$:=z+\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}a_kz^k\quad(c\neq 0,-1,-2,\dots),$$

and $F(a,b;c;z)$ is the well known Gaussian hypergeometric function.

Remark 2. When $p=1$, $q=0$; $\alpha_1=m+1$, $\alpha_2=1$; $\beta_1=1$, then the the above Dziok-Srivastava operator yields the Ruscheweyh derivative operator [9] given by

$$(1.6)\qquad D^mf(z):=(f*g)(z)=z+\sum_{k=n+1}^{\infty}\binom{m+k-1}{k-1}a_kz^k.$$

(2) Furthermore, if

$$(1.7)\qquad g(z)=z+\sum_{k=n+1}^{\infty}k\left(\frac{k+\sigma}{1+\sigma}\right)^m z^k\quad(\sigma\geq 0;\ m\in\mathbb{Z}),$$

then the convolution (1.3) yields the operator $\mathcal{I}(\sigma,m)f: A(n)\to A(n)$ which was studied by Cho and Kim [4] (see also [5]).

(3) Lastly, if

$$(1.8)\qquad g(z)=z+\sum_{k=n+1}^{\infty}\left(\frac{k+\mu}{1+\mu}\right)^l z^k\quad(\mu\geq 0;\ l\in\mathbb{Z}),$$

then (1.3) gives the multiplier transformation $\mathcal{J}(\mu, l)f : A(n) \to A(n)$, which was introduced by Cho and Srivastava [5].

REMARK 3. For $\mu = 0$, the operator defined with (1.8) gives the *Sălăgean operator*

$$\mathcal{D}^l f(z) := z + \sum_{k=n+1}^{\infty} k^l z^k \quad (l \geq 0), \tag{1.9}$$

which was initially studied by Sălăgean [11].

Following Goodman [7], Ruscheweyh [10], Silverman [12] (see also [1, 2, 8]), we define the (n, δ)-*neighborhood* of a function $f \in A(n)$ by

$$N_{n,\delta}(f) := \left\{A(n) : h(z) = z - \sum_{k=n+1}^{\infty} c_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - c_k| \leq \delta\right\}. \tag{1.10}$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_{n,\delta}(e) := \left\{f \in A(n) : h(z) = z - \sum_{k=n+1}^{\infty} c_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|c_k| \leq \delta\right\}. \tag{1.11}$$

For the purpose of this paper, we introduce here a subclass of $A(n)$ denoted by $S_n(g; \lambda, b)$ which involves the convolution (1.3) and consist of functions of the form (1.1) satisfying the inequality:

$$\left|\frac{1}{b}\left(\frac{z((f*g)(z))'}{(1-\lambda)(f*g)(z) + \lambda z((f*g)(z))'} - 1\right)\right| < 1, \tag{1.12}$$

where $z \in \mathcal{U}$, $0 \leq \lambda \leq 1$, $b \in \mathbb{C} - \{0\}$.

The definition of the function class $S_n(g; \lambda, b)$ is essentially motivated by earlier investigations in [1] and [8] in each of which further details and references to other closely related subclasses can be found.

We deem it proper to mention below some of the function classes which emerge from the function class $S_n(g; \lambda, b)$ defined above. Indeed, we observe that if we specialize the function $g(z)$ by means of (1.4) to (1.9), and denote the corresponding reducible classes of functions of $S_n(g; \lambda, b)$, respectively, by $H_q^p(\lambda, b; \alpha_1, \beta_1)$, $P_c^a(\lambda, b)$, $Q_n^m(\lambda, b)$, $L_n^\sigma(\lambda, b)$, $M_n^\mu(\lambda, b, l)$, and $R_n^l(\lambda, b)$, then it follows that

1. $f(z) \in H_q^p(\lambda, b; \alpha_1, \beta_1) \Rightarrow$

$$\left|\frac{1}{b}\left(\frac{z(\mathcal{H}_q^p(\alpha_1, \beta_1)f(z))'}{(1-\lambda)\mathcal{H}_q^p(\alpha_1, \beta_1)f(z) + \lambda z(\mathcal{H}_q^p(\alpha_1, \beta_1)f(z))'} - 1\right)\right| < 1 \;\; (z \in \mathcal{U}), \tag{1.13}$$

2. $f(z) \in P_c^a(\lambda, b)$

$$\Rightarrow \left|\frac{1}{b}\left(\frac{z(\mathcal{L}(a,c)f(z))'}{(1-\lambda)\mathcal{L}(a,c)f(z)+\lambda z(\mathcal{L}(a,c)f(z))'}-1\right)\right|<1 \quad (z\in\mathcal{U}), \tag{1.14}$$

3. $f(z) \in Q_n^m(\lambda, b)$

$$\Rightarrow \left|\frac{1}{b}\left(\frac{z(D^m f(z))'}{(1-\lambda)D^m f(z)+\lambda z(D^m f(z))'}-1\right)\right|<1 \quad (z\in\mathcal{U}), \tag{1.15}$$

4. $f(z) \in L_n^\sigma(\lambda, b)$

$$\Rightarrow \left|\frac{1}{b}\left(\frac{z(\mathcal{I}(\sigma,m)f(z))'}{(1-\lambda)\mathcal{I}(\sigma,m)f(z)+\lambda z(\mathcal{I}(\sigma,m)f(z))'}-1\right)\right|<1 \quad (z\in\mathcal{U}), \tag{1.16}$$

5. $f(z) \in M_n^\mu(\lambda, b, l)$

$$\Rightarrow \left|\frac{1}{b}\left(\frac{z(\mathcal{J}(\mu,l)f(z))'}{(1-\lambda)\mathcal{J}(\mu,l)f(z)+\lambda z(\mathcal{J}(\mu,l)f(z))'}-1\right)\right|<1 \quad (z\in\mathcal{U}), \tag{1.17}$$

6. $f(z) \in R_n^l(\lambda, b)$

$$\Rightarrow \left|\frac{1}{b}\left(\frac{z\mathcal{D}^{l+1}f(z)}{(1-\lambda)\mathcal{D}^l f(z)+\lambda z\mathcal{D}^{l+1}f(z)}-1\right)\right|<1 \quad (z\in\mathcal{U}). \tag{1.18}$$

The purpose of the present paper is to investigate the various properties and characteristics of functions belonging to the above defined subclass $S_n(g;\lambda,b)$ of analytic functions in the open unit disk $\mathcal{U}$. Apart from deriving a set of coefficient bounds for this function class, we also establish several inclusion relationships involving the (n,δ)-neighborhoods of analytic functions with negative and missing coefficients belonging to this subclass. Special cases of some of these inclusion relations are also mentioned.

2. Coefficient inequalities

The following result gives the necessary and sufficient condition for the function $f(z) \in A(n)$ to be in the class $S_n(g;\lambda,b)$.

THEOREM 1. *Let the function $f \in A(n)$ be defined by* (1.1), *then $f(z)$ is in the class $S_n(g;\lambda,b)$ if and only if*

$$\sum_{k=n+1}^{\infty}\{[1+\lambda(k-1)](|b|-1)+k\}\,b_k a_k \le |b|. \tag{2.1}$$

Proof. Let a function $f(z)$ of the form (1.1) belong to the class $S_n(g;\lambda,b)$. Then in view of (1.12), we obtain the following inequality:

$$\left|\frac{\sum\limits_{k=n+1}^{\infty}(1-\lambda)(k-1)b_k a_k z^k}{z-\sum\limits_{k=n+1}^{\infty}[\lambda(k-1)+1]b_k a_k z^k}\right|<|b| \quad (z\in\mathcal{U}).$$

Thus putting $z = r$ $(0 \leq r < 1)$, we obtain

$$\text{(2.2)} \qquad \frac{\sum\limits_{k=n+1}^{\infty} (1-\lambda)(k-1)b_k a_k r^{k-1}}{1 - \sum\limits_{k=n+1}^{\infty} [\lambda(k-1)+1]b_k a_k r^{k-1}} < |b| \quad (z \in \mathcal{U}).$$

We observe that the expression in denominator on the left-hand side of (2.2) is positive for $r = 0$ and also for all $r(0 < r < 1)$. Thus after some simplification, we obtain

$$\text{(2.3)} \qquad \sum_{k=n+1}^{\infty} \{[1+\lambda(k-1)](|b|-1)+k\}\, b_k a_k r^{k-1} < |b|,$$

and letting $r \to 1^-$ through real values, (2.3) leads to the desired assertion (2.1) of Theorem 1.

Conversely, by applying (2.1), we find that

$$\begin{aligned}
&\left|z((f*g)(z))' - (1-\lambda)(f*g)(z) - \lambda z((f*g)(z))'\right| \\
&\qquad\qquad - |b|\left|(1-\lambda)(f*g)(z) + \lambda z((f*g)(z))'\right| \\
&= \left|\sum_{k=n+1}^{\infty} (1-\lambda)(k-1)b_k a_k z^k\right| - |b|\left|z - \sum_{k=n+1}^{\infty} [\lambda(k-1)+1]b_k a_k z^k\right| \\
&\leq \sum_{k=n+1}^{\infty} (1-\lambda)(k-1)b_k a_k |z|^k - |b||z| + |b| \sum_{k=n+1}^{\infty} [\lambda(k-1)+1]b_k a_k |z|^k \\
&< \sum_{k=n+1}^{\infty} \{[1+\lambda(k-1)](|b|-1)+k\}\, b_k a_k - |b| \leq 0.
\end{aligned}$$

Hence, by (1.12), we infer that $f(z) \in S_n(g; \lambda, b)$, which evidently completes the proof of Theorem 1. ∎

Corresponding to the various subclasses which arise from the function class $S_n(g; \lambda, b)$ by suitably choosing the function $g(z)$ as mentioned in (1.13) to (1.18), we arrive at the following corollaries giving the coefficient bound inequalities for these subclasses of functions.

COROLLARY 1. *Let the function $f \in A(n)$ be defined by* (1.1), *then $f(z)$ is in the class $H_q^p(\lambda, b; \alpha_1, \beta_1)$ if and only if*

$$\text{(2.4)} \qquad \sum_{k=n+1}^{\infty} \{[1+\lambda(k-1)](|b|-1)+k\}\, a_k \Gamma_k \leq |b|.$$

REMARK 4. For specific choices of parameters p, q, α_1, β_1 (as mentioned in the Remarks 1 and 2), Corollary 1 would yield the coefficient bound inequalities for the subclasses of functions $P_c^a(\lambda, b)$ and $Q_n^m(\lambda, b)$.

COROLLARY 2. *Let the function $f \in A(n)$ be defined by* (1.1), *then $f(z)$ is in the class $L_n^{\sigma}(\lambda, b)$ if and only if*

$$\sum_{k=n+1}^{\infty} \{[1+\lambda(k-1)](|b|-1)+k\}\, k\, (k+\sigma)^m a_k \le |b|\, (1+\sigma)^m. \tag{2.5}$$

COROLLARY 3. *Let the function $f \in A(n)$ be defined by* (1.1), *then $f(z)$ is in the class $M_n^{\mu}(\lambda, b, l)$ if and only if*

$$\sum_{k=n+1}^{\infty} \{[1+\lambda(k-1)](|b|-1)+k\}\, (k+\mu)^l a_k \le |b|\, (1+\mu)^l. \tag{2.6}$$

REMARK 5. When $\mu = 0$, Corollary 3 would give the coefficient bound inequality for the subclass of functions $R_n^l(\lambda, b)$.

3. Inclusion relations involving (n, δ)-neighborhoods

In this section, we establish several inclusion relations for the normalized analytic function class $S_n(g; \lambda, b)$ involving the (n, δ)-neighborhood defined by (1.11).

THEOREM 2. *If $b_k \ge b_n (k \ge n+1), |b| > 1$ and*

$$\delta := \frac{|b|(1+n)}{\{(|b|-1)(1+n\lambda)+n+1\}\, b_{n+1}}, \tag{3.1}$$

then

$$S_n(g; \lambda, b) \subset N_{n,\delta}(e). \tag{3.2}$$

Proof. Let $f(z) \in S_n(g; \lambda, b)$. Then, in view of the assertion (2.1) of Theorem 1, we obtain

$$\{(|b|-1)(1+n\lambda)+n+1\}\, b_{n+1} \sum_{k=n+1}^{\infty} a_k \le |b|$$

which readily implies that

$$\sum_{k=n+1}^{\infty} a_k \le \frac{|b|}{\{(1+n\lambda)(|b|-1)+n+1\}\, b_{n+1}}. \tag{3.3}$$

Making use of (2.1) again in conjunction with (3.3), we get

$$\begin{aligned} b_{n+1} \sum_{k=n+1}^{\infty} k a_k &\le |b| + (1+n\lambda)(1-|b|) b_{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\le |b| + (1+n\lambda)(1-|b|) b_{n+1} \frac{|b|}{((1+n\lambda)(|b|-1)+n+1)\, b_{n+1}} \\ &= \frac{|b|(1+n)}{(1+n\lambda)(|b|-1)+n+1}. \end{aligned}$$

Hence

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{|b|(1+n)}{\{(1+n\lambda)(|b|-1)+n+1\}b_{n+1}} =: \delta \quad (|b|>1) \tag{3.4}$$

which, by means of the definition (1.11), establishes the inclusion relation (3.2) asserted by Theorem 2. ∎

4. Neighborhoods for the class $S_n(g,\lambda,\eta,b)$

In this section, we determine the neighborhood properties for the function class $S_n(g,\lambda,\eta,b)$ which we define as follows.

A function $f \in A(n)$ is said to be in $S_n(g,\lambda,\eta,b)$ if there exists a function $h(z) \in S_n(g,\lambda,b)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1-\eta \qquad (z \in U; 0 \leq \eta < 1). \tag{4.1}$$

THEOREM 3. *If $h \in S_n(g,\lambda,b)$ and*

$$\eta = 1 - \frac{\delta\{(1+n\lambda)(|b|-1)+n+1\}b_{n+1}}{(n+1)\{(1+n\lambda)(|b|-1)+n+1\}b_{n+1}-|b|} \quad (|b|>1) \tag{4.2}$$

then

$$N_{n,\delta}(h) \subset S_n^{\eta}(g,\lambda,b). \tag{4.3}$$

Proof. Suppose that $f \in N_{n,\delta}(h)$. We then find from the definition (1.10) that

$$\sum_{k=n+1}^{\infty} k|a_k - c_k| \leq \delta$$

which readily implies the coefficient inequality:

$$\sum_{k=n+1}^{\infty} |a_k - c_k| \leq \frac{\delta}{n+1} \qquad (n \in \mathbb{N}).$$

Next, since $h \in S_n(g,\lambda,b)$, therefore we infer that

$$\sum_{k=n+1}^{\infty} c_k \leq \frac{|b|}{\{(1+n\lambda)(|b|-1)+n+1\}b_{n+1}}, \tag{4.4}$$

so that

$$\left|\frac{f(z)}{h(z)}-1\right| < \frac{\sum\limits_{k=n+1}^{\infty}|a_k-c_k|}{1-\sum\limits_{k=n+1}^{\infty}c_k}$$
$$\leq \frac{\delta}{(n+1)}\frac{\{(1+n\lambda)(|b|-1)+n+1\}b_{n+1}}{\{(1+n\lambda)(|b|-1)+n+1\}b_{n+1}-|b|}$$
$$=1-\eta,$$

provided that η is given by(4.2). Thus, by definition, $f\in S_n(g;\lambda,\eta,b)$ for η given by (4.2). This evidently completes our proof of Theorem 3. ∎

For various choices of $g(z)$, as detailed in Section 1; see also (1.13), (1.16) and (1.17), we arrive at the following corollaries giving the corresponding neighborhood properties for these subclasses of functions.

COROLLARY 4. *If* $h\in H_q^p(\lambda,b;\alpha_1,\beta_1)$ *and*

$$\eta_1 = 1-\frac{\delta\{(1+n\lambda)(|b|-1)+n+1\}\Gamma_{n+1}}{(n+1)\{(1+n\lambda)(|b|-1)+n+1\}\Gamma_{n+1}-|b|} \quad (|b|>1),$$

then

$$N_{n,\delta}(g)\subset H_q^p(\lambda,\eta_1,b;\alpha_1,\beta_1).$$

COROLLARY 5. *If* $h\in L_n^\sigma(\lambda,b)$ *and*

$$\eta_2 = 1-\frac{\delta\{(1+n\lambda)(|b|-1)+n+1\}(n+1)(n+\sigma+1)^m}{(n+1)\{(1+n\lambda)(|b|-1)+n+1\}(n+1)(n+1)^m-|b|(\sigma+1)^m}$$

$(|b|>1)$, *then*

$$N_{n,\delta}(g)\subset L_n^\sigma(\lambda,\eta_2,b).$$

COROLLARY 6. *If* $h\in M_n^\mu(\lambda,b,l)$ *and*

$$\eta_3 = 1-\frac{\delta\{(1+n\lambda)(|b|-1)+n+1\}(n+\mu+1)^l}{(n+1)\{(1+n\lambda)(|b|-1)+n+1\}(n+\mu+1)^l-|b|(1+\mu)^l},$$

$(|b|>1)$, *then*

$$N_{n,\delta}(g)\subset M_n^\mu(\lambda,\eta_3,b,l).$$

REMARK 6. Making use of Remarks 1 and 2, Corollaries 4 and 5 yield the neighborhood results for the subclasses $P_c^a(\lambda,b)$ and $Q_n^m(\lambda,b)$, respectively. Also, for $\mu=0$, Corollary 6 would evidently give the neighborhood result for the subclass of functions $R_n^l(\lambda,b)$.

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References

[1] O. Altintas, Ö. Özkan, H. M. Srivastava, *Neighborhoods of a class analytic functions with negative coefficients*, Appl. Math. Letters. 13 (2000), 63–67.
[2] O. Altintas, Ö. Özkan, H. M. Srivastava, *Neighborhoods of a certain family of multivalent functions with negative coefficients*, Comput. Math. Appl. 47 (2004), 1667–1672.
[3] B. C. Carlson, S. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM, J. Math. Anal. 15 (2002), 737–745.
[4] N. E. Cho, T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc. 40(3) (2003), 399–410.
[5] N. E. Cho, H. M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling 37 (2003), no. 1-2, 39–49.
[6] J. Dziok, H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transforms Spec. Funct. 14 (2003), 7–18.
[7] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. (8) (1957), 598–601.
[8] G. Murugusundaramoorthy, H. M. Srivastava, *Neighborhoods of certain classes of analytic functions of complex order*, J. Inequal. Pure Appl. Math. 5 (2) (2004), Art. 24. 8 pp.
[9] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. 49 (1975), 109–115.
[10] S. Rucheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. 81 (1981), 521–527.
[11] G. Ş. Sălăgean, *Subclasses of univalent functions*, in: Complex analysis—fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), 362–372, Lecture Notes in Math., 1013, Springer, Berlin.
[12] H. Silverman, *Neighborhoods of a classes of analytic function*, Far East J. Math. Sci. 3 (2) (1995), 165–169.

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