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AN IDENTITY WITH DERIVATIONS ON RINGS AND BANACH ALGEBRAS

Abstract. The main purpose of this paper is to study the following: Let m, n, and k_i, $i = 1, 2, \ldots, n$, be positive integers and let R be a $2m(m+k_1+k_2+\ldots+k_n-1)!$-torsion free semiprime ring. Suppose that there exist derivations $D_i : R \to R$, $i = 1, 2, \ldots, n+1$, such that $D_1(x^m)x^{k_1+\ldots+k_n} + x^{k_1}D_2(x^m)x^{k_2+\ldots+k_n} + \ldots + x^{k_1+\ldots+k_n}D_{n+1}(x^m) = 0$ holds for all $x \in R$. Then we prove that $D_1 + D_2 + \ldots + D_{n+1} = 0$ and that the derivation $k_1D_2 + (k_1 + k_2)D_3 + \ldots + (k_1 + k_2 + \ldots + k_n)D_{n+1}$ maps R into its center. We also obtain a range inclusion result of continuous derivations on Banach algebras.

Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, if $nx = 0$, $x \in R$, implies $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies that $a = 0$. An additive mapping D is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. A mapping f of a ring R into itself is called centralizing on R if $[f(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[f(x), x] = 0$ holds for all $x \in R$, the mapping f is said to be commuting on R. For the explanation of maximal right ring of quotients of a semiprime ring, which will be denoted by Q, we refer the reader to [1]. Banach algebras in this paper will be over the complex field. We denote by $rad(A)$ the radical of a Banach algebra A and by $Q(A)$ the set of all quasinilpotent elements in A.

The third named author has recently proved the following result [17, Theorem 2.1].

THEOREM 1. *Let m, n be positive integers and let R be a $2mn(m+n-1)!$-torsion free semiprime ring. Suppose that there exist derivations $D, G : R \to R$, such that*

$$D(x^m)x^n + x^nG(x^m) = 0$$

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holds for all $x \in R$. In this case D and G map R into $Z(R)$. Besides $D+G = 0$.

It is our aim in this paper to prove the following result which fairly generalizes Theorem 1.

THEOREM 2. *Let m, n, and k_i, $i = 1, 2, \ldots, n$, be positive integers and let R be a $2m(m + k_1 + k_2 + \ldots + k_n - 1)!$-torsion free semiprime ring. Suppose that there exist derivations $D_i : R \to R, i = 1, 2, \ldots, n+1$, such that*

(1) $D_1(x^m)x^{k_1+\ldots+k_n} + x^{k_1}D_2(x^m)x^{k_2+\ldots+k_n} + \ldots + x^{k_1+\ldots+k_n}D_{n+1}(x^m) = 0$

holds for all $x \in R$. In this case the derivation $k_1D_2+(k_1+k_2)D_3+\ldots+(k_1+k_2+\ldots+k_n)D_{n+1}$ maps R into its center. Besides, $D_1+D_2+\ldots+D_{n+1} = 0$.

In the proof of Theorem 2 we shall use the fact that any semiprime ring R and its maximal right ring of quotients Q satisfy the same differential identities which is very useful since Q contains the identity element (see [9, Theorem 3]). For the explanation of differential identities we refer the reader to [6].

Proof of Theorem 2. The complete linearization of (1) gives us

$$\sum_{\pi \in S_{m+k_1+\ldots+k_n}} \Big(D_1(x_{\pi(1)} \cdots x_{\pi(m)})x_{\pi(m+1)} \cdots x_{\pi(m+k_1+\ldots+k_n)} + x_{\pi(1)} \cdots x_{\pi(k_1)}D_2(x_{\pi(1+k_1)} \cdots x_{\pi(m+k_1)})x_{\pi(m+k_1+1)} \cdots x_{\pi(m+k_1+\ldots+k_n)} + \ldots + x_{\pi(1)} \cdots x_{\pi(k_1+\ldots+k_n)}D_{n+1}(x_{\pi(1+k_1+\ldots+k_n)} \cdots x_{\pi(m+k_1+\ldots+k_n)})\Big) = 0 \tag{2}$$

for all $x_1, x_2, \ldots, x_{m+k_1+\ldots+k_n} \in R$. According to [9, Theorem 3] the above relation holds for all $x_1, x_2, \ldots, x_{m+k_1+\ldots+k_n} \in Q$ as well. Substituting $x_1 = x$, $x_2 = \ldots = x_{m+k_1+\ldots+k_n} = 1$ (here, 1 denotes the identity element), and applying the fact that $D_i(1) = 0$, $i = 1, 2, \ldots, n+1$, we obtain

$$\alpha(D_1(x) + D_2(x) + \ldots + D_{n+1}(x)) = 0$$

for all $x \in Q$, where $\alpha = m(m + k_1 + \ldots + k_n - 1)!$. Hence,

$$D_1 + D_2 + \ldots + D_{n+1} = 0. \tag{3}$$

Using again the relation (2) and substituting $x_1 = x_2 = x$, $x_3 = \ldots = x_{m+k_1+\ldots+k_n} = 1$ we obtain

$$0 = \beta((k_1 + \ldots + k_n)D_1(x)x + k_1xD_2(x) + (k_2 + \ldots + k_n)D_2(x)x + \ldots + (k_1 + \ldots + k_n)xD_{n+1}(x)) \tag{4}$$

for all $x \in Q$, where $\beta = 2m(m+k_1+\ldots+k_n-2)!$. Let D be a derivation on R defined by $D = k_1D_2+(k_1+k_2)D_3+\ldots+(k_1+k_2+\ldots+k_n)D_{n+1}$. Applying (3) and (4) we get $\beta[D(x), x] = 0$ for all $x \in Q$ and therefore $[D(x), x] = 0$ for all $x \in R$. In other words D is commuting on R. This yields that D

maps R into $Z(R)$ since any commuting derivation of a semiprime ring maps the ring into its center. ∎

We proceed with the following result.

THEOREM 3. *Let m, n, and k_i, $i = 1, 2, \ldots, n$, be positive integers and let R be a noncommutative prime ring such that $char(R) = 0$ or $char(R) > m + k_1 + k_2 + \ldots + k_n$. Suppose that there exist derivations $D_i : R \to R, i = 1, 2, \ldots, n+1$, satisfying (1). Then $D_i = 0$ for all i.*

For $m = 1$, $k_1 = k_2 = \ldots = k_n = 1$ the above result reduces to [3, Corollary 3.4.].

Proof of Theorem 3. Let us write $m + k_1 + k_2 + \ldots + k_n = N$ for brevity. Since D_i, $i = 1, 2, \ldots, n$, are derivations, the identity (1) can be written as

$$\begin{aligned} 0 = &D_1(x)x^{N-1} + xD_1(x)x^{N-2} + \ldots + x^{m-1}D_1(x)x^{N-m} \\ &+ x^{k_1}D_2(x)x^{N-k_1-1} + x^{k_1+1}D_2(x)x^{N-k_1-2} + \ldots \\ &+ x^{k_1+m-1}D_2(x)x^{N-k_1-m} + \ldots \\ &+ x^{N-m}D_{n+1}(x)x^{m-1} + x^{N-m+1}D_{n+1}(x)x^{m-2} + \ldots \\ &+ x^{N-1}D_{n+1}(x). \end{aligned}$$

Note that the above identity is actually the identity (1) in [3]. Namely, $f_1 = D_1$, $f_2 = D_2$ or $f_2 = D_1 + D_2$ or $f_2 = D_1$ (it depends on k_i). It is not difficult to verify that by [3, Corollary 3.4] the result follows. ∎

Theorem 3 will be used in the proof of our next result.

THEOREM 4. *Let m, n, and k_i, $i = 1, 2, \ldots, n$, be positive integers, let A be a Banach algebra, and let $D_i : A \to A, i = 1, 2, \ldots, n+1$ be continuous linear derivations. Suppose that*

$$D_1(x^m)x^{k_1+\ldots+k_n} + x^{k_1}D_2(x^m)x^{k_2+\ldots+k_n} + \ldots + x^{k_1+\ldots+k_n}D_{n+1}(x^m) \in rad(A)$$

holds for all $x \in A$. Then all derivations D_i map A into $rad(A)$.

Let us explain in somewhat more details the background of the theorem above. In 1955 Singer and Wermer [14] proved that a continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Johnson and Sinclair [7] have proved that any linear derivation on a semisimple Banach algebra is continuous. According to these two results, one can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra. Singer and Wermer conjectured in [14] that the continuity assumption in their result is superfluous. It took more than thirty years until this conjecture was finally proved by Thomas [15]. Obviously, from Thomas's result it follows directly that there are no nonzero linear derivations on a commutative semisimple Banach algebra. By our knowledge the first noncommutative extension of Singer-Wermer theorem has been proved by Yood [18] by showing that if for all pairs $x, y \in A$, where A is a noncommutative Banach algebra, the element $[D(x), y]$ lies in $rad(A)$, then D maps A into $rad(A)$. Brešar and the third named author [5] have generalized Yood's result by proving that in case $[D(x), x] \in rad(A)$ for all $x \in A$, then D maps A into $rad(A)$. The work of Mathieu and Murphy [11] and Runde [12] should be mentioned. Recently, Kim [8] has proved that in case $[D(x), x]D(x)[D(x), x] \in rad(A)$ for any $x \in A$, then a continuous derivation D maps A into $rad(A)$. Kim's result generalizes a result proved by the third named author [16]. Most results in the field of range inclusion theory deal with one derivation, while in the theorem above we have arbitrary number of derivations. The first result in this field with two derivations is, by our knowledge, the following result proved by Brešar and the third named author [5]. Let D and G be such continuous linear derivations on a noncommutative Banach algebra A, that $[D^2(x) + G(x), x] \in rad(A)$ holds for all $x \in A$. In this case both derivations D and G map A into $rad(A)$. The result we have just mentioned has been recently generalized by the third named author [17]. For references concerning range inclusion results of continuous derivations on noncommutative Banach algebras we refer the reader to [2], [4], and [10].

Proof of Theorem 4. By Lemma 3.2 in Sinclair's paper [13], every continuous linear derivation of a Banach algebra A leaves the primitive ideals invariant, which means that one can introduce for any primitive ideal $P \subset A$ derivations $D_{iP} : A/P \to A/P$, $i = 1, 2, \ldots, n+1$, where A/P is the factor algebra, by $D_{iP}(x^*) = D_i(x) + P$, $x^* = x + P$.

Let us first assume that A/P is noncommutative. In this case one can conclude from the assumptions of the theorem that

$$\begin{aligned} 0 = D_{1P}(x^{*m})x^{*k_1+\ldots+k_n} + x^{*k_1}D_{2P}(x^{*m})x^{*k_2+\ldots+k_n} \\ + \ldots + x^{*k_1+\ldots+k_n}D_{(n+1)P}(x^{*m}) \end{aligned}$$

holds for all $x^* \in A/P$, which gives $D_{iP} = 0$, $i = 1, 2, \ldots, n+1$, by Theorem 3 since A/P is prime.

In the case when A/P is commutative we have $D_{iP} = 0$ as well, since A/P is semisimple and since we know that there is no nonzero linear derivations on a commutative semisimple Banach algebra. Thus, for any $x \in A$ we have $D_i(x) \in P$, where P is any primitive ideal of A. Since $D_i(x)$, where x is any element from A, are in the intersection of all primitive ideals of A and since the intersection of all primitive ideals of A is the radical, one can conclude that $D_i(A) \subseteq rad(A)$, $i = 1, 2, \ldots, n+1$, which is a desired conclusion. Thereby the proof of the theorem is complete. ∎

The question arises whether Theorem 4 can be proved without the continuity assumption. This question leads to the problem whether Sinclair's result [13] which states that continuous linear derivation on a Banach algebra leaves any primitive ideal of the algebra invariant, can be proved without the continuity assumption. By our knowledge this problem is still open. However, in a special case, when a Banach algebra is semisimple, one can prove some results without the continuity assumptions.

THEOREM 5. *Let m, n, and k_i, $i = 1, 2, \ldots, n$, be positive integers, let A be a semisimple Banach algebra, and let $D_i : A \to A, i = 1, 2, \ldots, n+1$ be linear derivations. Suppose that*

$$D_1(x^m)x^{k_1+\ldots+k_n} + x^{k_1}D_2(x^m)x^{k_2+\ldots+k_n} + \ldots + x^{k_1+\ldots+k_n}D_{n+1}(x^m) = 0$$

holds for all $x \in A$. Then $D_1 = D_2 = \ldots = D_{n+1} = 0$.

Proof. It follows just by applying Theorem 4 because the derivations D_i, $i = 1, \ldots, n+1$, are automatically continuous [7]. ∎

Brešar and Vukman [5] have proved that in the case $[D(x), x]^2 \in rad(A)$ for all $x \in A$, where D is a continuous linear derivation of a Banach algebra A, we have $D(A) \subseteq rad(A)$. Brešar [2] has fairly generalized this result by proving that if $[D(x), x] \in Q(A)$ for all $x \in A$, then $D(A) \subseteq rad(A)$. This result together with Theorem 4 leads to the following conjecture.

CONJECTURE 6. *Let m, n, and k_i, $i = 1, 2, \ldots, n$, be positive integers, let A be a Banach algebra, and let $D_i : A \to A, i = 1, 2, \ldots, n+1$, be continuous linear derivations. Suppose that*

$$D_1(x^m)x^{k_1+\ldots+k_n} + x^{k_1}D_2(x^m)x^{k_2+\ldots+k_n} + \ldots + x^{k_1+\ldots+k_n}D_{n+1}(x^m) \in Q(A)$$

holds for all $x \in A$. In this case all derivations D_i map A into $rad(A)$.

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