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ON PRESERVERS OF DETERMINANT OVER A FIELD

Abstract. Operators preserving the determinant of real matrices were studied in [1], [2] and [5] under assumption that the operator is linear. In this paper the linearity of the operators is not assumed: we only assume that the operators are of the form $F = [f_{i,j}]$, where functions $f_{i,j} : \mathcal{F} \longrightarrow \mathcal{F}$ for $i, j = 1, 2, \ldots, n$ and $\mathcal{F}$ is a field. In the matrix space $M_n(\mathcal{F}) = \mathcal{F}^{n \times n}$, we characterize these operators from $M_{m,n}(\mathcal{F})$ into itself preserving the determinant. An operator preserving the determinant of matrices must be linear as in [4]. The form of operator F is presented.

1. Introduction

Let $\mathbb{N}$ denotes the set of positive integer numbers. For fixed $m, n \in \mathbb{N}$ by $M_{m,n}(\mathcal{F})$ we denote the set of $m \times n$ matrices over the field $\mathcal{F}$. Finally, let $M_n(\mathcal{F}) := M_{n,n}(\mathcal{F})$.

First of all let us introduce

DEFINITION 1. We say that an operator $F : M_n(\mathcal{F}) \longrightarrow M_n(\mathcal{F})$ defined by

(1) $\quad F(A) := [f_{i,j}(a_{i,j})], \quad \text{where } f_{i,j} : \mathcal{F} \longrightarrow \mathcal{F},\ i, j = 1, 2, \ldots, n,$

preserves the determinant of matrices from $M_n(\mathcal{F})$ if and only if for every matrix $A \in M_n(\mathcal{F})$, $A = [a_{i,j}]$, the equality

$$\det(F(A)) = \det(A)$$

holds, where the matrix $F(A) := [f_{i,j}(a_{i,j})]$ for $i, j = 1, 2, \ldots, n$.

2. Preservers of the determinant

In the part of the paper we characterize the operators of the form (1) preserving the determinant. At the begining we prove

LEMMA 1. *Let F from $M_n(\mathcal{F})$ into itself is an operator of the form* (1) *which preserves the determinant of matrices, then there exist constants $d_{i,j} \neq 0$ for*

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$i, j = 1, 2, \dots, n$ such that

$$f_{i,j}(x) = d_{i,j}x \quad \text{for} \quad x \in \mathcal{F}, \tag{2}$$

where

$$\prod_{i=1}^{n} d_{i,\sigma(i)} = 1 \tag{3}$$

for an arbitrary permutation $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ of the integer numbers $1, 2, \dots, n$.

Proof. For $n = 1$ if an operator F from $M_1(\mathcal{F})$ into itself of the form (1) preserves the determinant of matrices then $f_{1,1}(x) = x$ for $x \in \mathcal{F}$.

Let F from $M_n(\mathcal{F})$ into itself for $n \geq 2$ of the form (1) be an operator preserving the determinant of matrices.

First we prove that every function $f_{i,j}$ for $i, j = 1, 2, \dots, n$ is of the form

$$f_{i,j}(x) = d_{i,j}x + b_{i,j} \quad \text{for} \quad x \in \mathcal{F}, \tag{4}$$

where $d_{i,j}, b_{i,j} \in \mathcal{F}$.

Let us take an arbitrary fixed pair (k, l) such that $k, l \in \{1, 2, \dots, n\}$ and consider the function $f_{k,l}$.

Let $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ be a permutation of the integer numbers $1, 2, \dots, n$. Let $B_1 \in M_n(\mathcal{F})$ be a generalized permutation matrix with entries

$$b_{i,\sigma(i)} = \begin{cases} x & \text{for } (i, \sigma(i)) = (k, l) \\ 1 & \text{for } (r, \sigma(r)),\ r = 1, 2, \dots, n,\ r \neq i \\ 0 & \text{in other cases,} \end{cases}$$

for $x \in \mathcal{F}$.

From the property of a generalized permutation matrix we obtain $\det(B_1) = (-1)^{I_p}x$, where I_p denotes the number of inverses of the permutation σ.

Consider the matrix $F(B_1)$. Using Laplace's theorem expanding about k-th row we get

$$\det(F(B_1)) = f_{k,l}(x)D_{k,l} + R_{k,l},$$

where the $D_{k,l}$ is cofactor of the element $f_{k,l}(x)$, whereas $R_{k,l}$ is from Laplace's theorem the sum of the remaining products entries of the k-th row by the corresponding cofactor. Thus both $D_{k,l}$ and $R_{k,l}$ are constants and do not depend on the variable x.

Because F is an operator preserving the determinant and $\det(B_1) = (-1)^{I_p}x$, then we have $\det(F(B_1)) = \det(B_1)$ and

$$f_{k,l}(x)D_{k,l} + R_{k,l} = x \quad \text{for} \quad x \in \mathcal{F}. \tag{5}$$

Putting $x = 0$ in (5) we have

$$f_{k,l}(0)D_{k,l} + R_{k,l} = 0. \tag{6}$$

Subtracting (5) and (6) we obtain

$$\left(f_{k,l}(x) - f_{k,l}(0)\right) D_{k,l} = x \quad \text{for} \quad x \in \mathcal{F}$$

and $D_{k,l} \neq 0$. From the above

$$f_{k,l}(x) = D_{k,l}^{-1} x + f_{k,l}(0) \quad \text{for} \quad x \in \mathcal{F}$$

and functions $f_{k,l}$ are of the form (4), where $d_{k,l} \neq 0$ for $k, l = 1, 2, \ldots, n$.

Next we prove that all coefficients $b_{k,l}$ for $k, l = 1, 2, \ldots, n$ are equal to zero.

Let us consider a matrix $B_2 \in M_n(\mathcal{F})$ with entries $x_{i,j} \in \mathcal{F}$ for $i, j = 1, 2, \ldots, n$. Then using (4) the matrix $F(B_2)$ has entries $d_{i,j}x_{i,j} + b_{i,j}$ for $i, j = 1, 2, \ldots, n$. Because F is an operator preserving the determinant, we have $\det(B_2) = \det(F(B_2))$, i.e.,

$$\sum_{\sigma \in S_n} (-1)^{I_p} \prod_{i=1}^{n} x_{i,\sigma(i)} = \sum_{\sigma \in S_n} (-1)^{I_p} \prod_{i=1}^{n} \left(d_{i,\sigma(i)} x_{i,\sigma(i)} + b_{i,\sigma(i)}\right), \tag{7}$$

where S_n is the set of all permutations $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ of the integer numbers $1, 2, \ldots, n$; I_p denotes the number of inverses of the permutation σ. The formula (7) is an equality of two polynomials of n^2 variables $x_{i,j} \in \mathcal{F}$, $i, j = 1, 2, \ldots, n$. If two polynomials are equal then coefficients are equal, respectively. On the left side of (7) we have the term $1 \cdot \prod_{i=1}^{n} x_{i,\sigma(i)}$; on the right side there is $\prod_{i=1}^{n} d_{i,\sigma(i)} \cdot \prod_{i=1}^{n} x_{i,\sigma(i)}$. Hence we obtain (3) for every permutation σ.

Let $k \in \{1, 2, \ldots, n\}$ be fixed. On the right side of (7) we find the term of $n-1$ variables $\prod_{i=1\ i\neq k}^{n} d_{i,\sigma(i)} x_{i,\sigma(i)} b_{k,\sigma(k)}$. There is not this term on the left side of (7). Therefore $\prod_{i=1\ i\neq k}^{n} d_{i,\sigma(i)} x_{i,\sigma(i)} b_{k,\sigma(k)} = 0$ holds. From (3) we have $d_{i,\sigma(i)} \neq 0$ and we obtain $b_{k,\sigma(k)} = 0$ for arbitrary $k \in \{1, 2, \ldots, n\}$. Because σ was an arbitrary permutation, we obtain $b_{i,j} = 0$ for all $i, j = 1, 2, \ldots, n$. ∎

Let us denote by C the matrix

$$C = [c_{i,j}] \quad \text{with} \quad c_{i,j} = f_{i,j}(1) \quad \text{for} \quad i, j = 1, 2, \ldots, n. \tag{8}$$

REMARK 1. If an operator F from $M_n(\mathcal{F})$ into itself for $n \in \mathbb{N}$ of the form (1) preserves the determinant of matrices, then by Lemma 1, it follows that $c_{i,j} \neq 0$ and $f_{i,j}(0) = 0$ for $i, j = 1, 2, \ldots, n$.

LEMMA 2. *Let F from $M_n(\mathcal{F})$ into itself for $n \in \mathbb{N}$ be an operator of the form* (1) *which preserves the determinant of matrices. Then*

$$\operatorname{rank}(C) = 1, \tag{9}$$

where the matrix C is defined by (8).

Proof. For $n = 1$ the condition (9) is true.

For $n \geq 2$ let us consider the minor

$$B_3 = \begin{bmatrix} c_{i,k} & c_{i,l} \\ c_{r,k} & c_{r,l} \end{bmatrix}$$

of the matrix C with arbitrarily fixed indices i, r, k, l such that $1 \leq i < r \leq n$ and $1 \leq k < l \leq n$.

Let us consider the generalization permutation matrix $B_4 \in M_n(\mathcal{F})$ with entries 1 on positions

$$(1,\sigma(1)),(2,\sigma(2)),\ldots,(i-1,\sigma(i-1)),(i,k),(i+1,\sigma(i+1)),\ldots$$
$$\ldots,(r-1,\sigma(r-1)),(r,l),(r+1,\sigma(r+1)),\ldots,(n,\sigma(n))$$

and entries 0 on all other positions.

Let $B_5 \in M_n(\mathcal{F})$ be the matrix obtained from B_4 by interchanging the i-th and r-th columns. Then the determinants $\det(B_4) = (-1)^{I_q}$ and $\det(B_5) = (-1)^{I_q+1}$, where I_q denotes the number of inverses of the permutation $\sigma(1), \sigma(2), \ldots, \sigma(i-1), k, \sigma(i+1), \ldots, \sigma(r-1), l, \sigma(r+1), \ldots, \sigma(n)$ of the integer numbers $1, 2, \ldots, n$.

The $F(B_4)$ and $F(B_5)$ are generalized permutation matrix. The determinants

$$\det(F(B_4)) = (-1)^{I_q} c_{1,\sigma(1)} c_{2,\sigma(2)} \cdots c_{i-1,\sigma(i-1)} c_{i,k} c_{i+1,\sigma(i+1)} \cdots$$
$$c_{r-1,\sigma(r-1)} c_{r,l} c_{r+1,\sigma(r+1)} \cdots c_{n,\sigma(n)}$$

and

$$\det(F(B_5)) = (-1)^{I_q+1} c_{1,\sigma(1)} c_{2,\sigma(2)} \cdots c_{i-1,\sigma(i-1)} c_{i,l} c_{i+1,\sigma(i+1)} \cdots$$
$$c_{r-1,\sigma(r-1)} c_{r,k} c_{r+1,\sigma(r+1)} \cdots c_{n,\sigma(n)}.$$

Because F preserves determinants, we get $\det(F(B_4)) = (-1)^{I_q}$ and $\det(F(B_5)) = (-1)^{I_q+1}$. Then the equality holds $\det(F(B_4)) + \det(F(B_5)) = 0$. From the above it follows

$$c_{i,k} c_{r,l} - c_{i,l} c_{r,k} = 0.$$

This means that for an arbitrary minor B_3 of the matrix C of the order 2 we have $\det(B_3) = 0$. Then (9) is true. ∎

LEMMA 3 (see [3]). *A matrix $C \in M_{m,n}(\mathcal{F})$, $c_{i,j} \neq 0$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ has* rank $(C) = 1$, *if and only if there exist non-zero elements $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ in the field $\mathcal{F}$ such that for all entries $c_{i,j}$ of the matrix C the equalities*

(10) $$c_{i,j} = u_i v_j \text{ for } i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n$$

are satisfied.

We prove

THEOREM 1. *Let F from $M_n(\mathcal{F})$ into itself, where $n \in \mathbb{N}$, be an operator of the form* (1). *Then F preserves the determinant of matrices, if and only if there exist constants $u_i \neq 0$, $v_j \neq 0$ for $i, j = 1, 2, \ldots, n$ such that*

$$f_{i,j}(x) = u_i v_j x \quad for \quad x \in \mathcal{F}, \quad i, j = 1, 2, \ldots n, \tag{11}$$

where

$$\prod_{i=1}^{n} u_i v_i = 1. \tag{12}$$

Proof. From Lemma 1, we infer that functions $f_{i,j}$ have the form (2). For $x = 1$ we obtain that $d_{i,j} = c_{i,j}$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and (11) holds.

By Lemma 2, we obtain (9), and from Lemma 1, we get (3). Using Lemma 3 and (10) we obtain (12).

Conversely, assume that an operator F of the form (1) is defined by (11) and the assumptions (8), (9) and (12) are fulfiled.

Let $H = [h_{i,j}] \in M_n(\mathcal{F})$ be an arbitrary matrix. From the definition of the determinant and (11) we have

$$\det(F(H)) = \sum_{p \in S_n} (-1)^{I_q} \prod_{i=1}^{m} u_i v_{\sigma(i)} h_{i,\sigma(i)},$$

where S_n denotes the set of all permutations $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ of the set $\{1, 2, \ldots, n\}$.

Using properties of the determinants, (12) and the definition of σ it follows that

$$\det(F(H)) = \prod_{i=1}^{m} u_i v_{\sigma(i)} \sum_{\sigma \in S_n} h_{i,\sigma(i)} = 1 \cdot \det(H).$$

Thus the operator F from $M_n(\mathcal{F})$ into itself preserves the determinant of matrices. ■

The result of Theorem 1 may be written as

THEOREM 2. *Let F from $M_n(\mathcal{F})$ into itself be an operator of the form* (1). *Then F preserves the determinant of matrices if and only if there exist nonsingular diagonal matrices $M \in M_n(\mathcal{F})$, $M = \operatorname{diag}(u_1, u_2, \ldots, u_n)$, $N \in M_n(\mathcal{F})$, $N = \operatorname{diag}(v_1, v_2, \ldots, v_n)$ such that*

$$F(A) = MAN \quad \text{for all} \quad A \in M_n(\mathcal{F}) \quad \text{with} \quad \det(M) \cdot \det(N) = 1.$$

REMARK 2. The results of Theorems 1 and 2 were obtained with no regularity assumption on operator F (i.e. continuity, measurability or others).

In the paper [4] similar problem and results concerning the preservers of the determinant are presented. Let $\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a linear operator. In virtue [4, Theorem 3.1] we obtain that Φ is an operator preserving the determinant if and only if there exist invertible matrices $M, N \in M_n(\mathbb{R})$ such that $\Phi(A) = MAN$ or $\Phi(A) = MA^tN$ and $\det(MN) = 1$, where A^t denotes the transposition of the matrix A.

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