Chang Bum Kim, Hee Sik Kim

ON BG-ALGEBRAS

Abstract. In this paper we introduce the notion of BG-algebras which is a generalization of B-algebras. We construct a BG-algebra from a non-empty set, which is non-group-derived. Moreover, using the notion of normal subalgebra, we obtain several isomorphism theorems of BG-algebra and related properties.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3,4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1,2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([8]) introduced the notion of d-algebras which is another generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. They also introduced B-algebras ([9, 10]), i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y \in X$. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([5]) introduced a new notion, called a BH-algebra which is a generalization of $BCH/BCI/BCK$-algebras, i.e., (I); (II) and (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$, for any $x, y \in X$. In this paper we introduce the notion of BG-algebras which is a generalization of B-algebras. We construct a BG-algebra from a non-empty set, which is non-group-derived. Moreover, using the notion of normal subalgebra, we obtain several isomorphism theorems of BG-algebra and related properties.

2. BG-algebras

A BG-algebra is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(1) $\qquad\qquad\qquad x * x = 0,$

(2) $\qquad\qquad\qquad x * 0 = x,$

(3) $\qquad\qquad\qquad (x * y) * (0 * y) = x,$

for all $x, y \in X$.

EXAMPLE 2.1. Let $X := \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then $(X; *, 0)$ is a BG-algebra.

THEOREM 2.2. *If $(X; *, 0)$ is a B-algebra, then $(X; *, 0)$ is a BG-algebra.*

P r o o f. Since $(X; *, 0)$ is B-algebra, the axioms (1) and (2) for BG-algebra are satisfied and $(x * y) * (0 * y) = x * ((0 * y) * (0 * y)) = x * 0 = x$ for any $x, y \in X$. Hence $(X; *, 0)$ is a BG-algebra. ∎

REMARK. The converse of Theorem 2.2 does not hold. The BG-algebra $(X; *, 0)$ given in Example 2.1 is not a B-algebra, since $(0 * 2) * 1 = 2 * 1 = 2$ and $0 * (1 * (0 * 2)) = 0 * (1 * 2) = 0 * 1 = 1$ imply $(0 * 2) * 1 \neq 0 * (1 * (0 * 2))$.

Thus the class of B-algebras is a proper subclass of BG-algebras.

PROPOSITION 2.3. *Let $(X; \circ, 0)$ be a group. If we define $x * y = x \circ y^{-1}$, then $(X; *, 0)$ is a BG-algebra.*

P r o o f. We see that $x * x = x \circ x^{-1} = 0$ and $x * 0 = x \circ 0^{-1} = x \circ 0 = x$. For any $x, y \in X$, we have $(x * y) * (0 * y) = (x \circ y^{-1}) \circ (0 \circ y^{-1})^{-1} = x \circ (y^{-1} \circ y) = x \circ 0 = x$. Hence $(X; *, 0)$ is a BG-algebra. ∎

From Proposition 2.3 we can see that every group $(X; \circ, 0)$ can determine a BG-algebra $(X; *, 0)$, called a *group-derived BG-algebra*. It is then a question of interest to determine whether or not all BG-algebras are group-derived. Example 2.1 is an example of a non-group-derived BG-algebra, since the only group of order 3 is $(Z_3; +, 0)$. Indeed, if we assume that Example 2.1 is group-derived, then $2 = 2 * 1 = 2 + 1^{-1} = 2 + 2 = 1$, a contradiction.

LEMMA 2.4. *Let $(X; *, 0)$ be a BG-algebra. Then*

   (i) *the right cancellation law holds in X, i.e., $x * y = z * y$ implies $x = z$,*
   (ii) *$0 * (0 * x) = x$ for all $x \in X$,*

(iii) *if $x * y = 0$, then $x = y$ for any $x, y \in X$,*

(iv) *if $0 * x = 0 * y$, then $x = y$ for any $x, y \in X$,*

(v) *$(x * (0 * x)) * x = x$ for all $x \in X$.*

P r o o f. (i). Assume that $x * y = z * y$. Then $x = (x * y) * (0 * y) = (z * y) * (0 * y) = z$. (ii). In axiom (3) for BG-algebra, replacing y by x, we have that $(x * x) * (0 * x) = x$ implies $0 * (0 * x) = x$. (iii). If $x * y = 0$, then we have $x * y = 0 = y * y$. By applying (i) we obtain, $x = y$. (iv). If $0 * x = 0 * y$, then $x = (x * x) * (0 * x) = (y * y) * (0 * y) = y$ by the axiom (3) for BG-algebra. (v). $(x * (0 * x)) * x = (x * (0 * x)) * (0 * (0 * x)) = x$ by the axiom (3) and Lemma 2.4-(ii). ∎

THEOREM 2.6. *Let $(X; *, 0)$ be a BG-algebra with the identity $(x * y) * z = x * (0 * ((0 * y) * z))$ for all $x, y, z \in X$. Then $(X; *, 0)$ is group-derived.*

P r o o f. Define a binary operation " $\circ$ " on X by

$$x \circ y := x * (0 * y).$$

Then $x \circ 0 = x * (0 * 0) = x * 0 = x$ and $0 \circ x = 0 * (0 * x) = x$ by Lemma 2.4-(ii). Thus 0 acts like an identity element on X. Also $x \circ (0 * x) = x * (0 * (0 * x)) = x * x = 0$ and $(0 * x) \circ x = (0 * x) * (0 * x) = 0$. So $0 * x$ behaves like a multiplicative inverse for x with respect to the operation " $\circ$ ". We claim that $(X; \circ)$ is a semigroup. Indeed,

$$
\begin{aligned}
x \circ (y \circ z) &= x * (0 * (y * (0 * z))) \\
&= x * (0 * ((0 * (0 * y)) * (0 * z))) && \text{[by Lemma 2.4-(ii)]} \\
&= (x * (0 * y)) * (0 * z) && \text{[by hypothesis]} \\
&= (x \circ y) \circ z.
\end{aligned}
$$

Note that $x \circ y^{-1} = x * (0 * y^{-1}) = x * (0 * (0 * y)) = x * y$. Hence $(X; *, 0)$ is also a group-derived BG-algebra. This completes the proof. ∎

The condition $(x*y)*z = x*(0*((0*y)*z))$ in Theorem 2.6 does not hold in general. In Example 2.1 we have $(2*1)*2 = 0$, while $2*(0*((0*1)*2)) = 2$.

EXAMPLE 2.7. Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X; *, 0)$ is a BG-algebra satisfying the identity $(x * y) * z = x * (0 * ((0 * y) * z))$. So $(X; *, 0)$ is a group-derived BG-algebra.

PROPOSITION 2.8. *Every BG-algebra is a BH-algebra.*

P r o o f. It follows immediately from Lemma 2.4-(iii). ∎

The converse of Proposition 2.8 need not be true in general.

EXAMPLE 2.9. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

*	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X; *, 0)$ is a *BH*-algebra ([5]), but it is not a *BG*-algebra, since $(2 * 3) * (0 * 3) = 1 \neq 2$.

By Theorem 2.2 and Proposition 2.8, we know the following relation:

$$
\begin{array}{ccccc}
\text{The class} & & \text{The class} & & \text{The class} \\
\text{of} & \subset & \text{of} & \subset & \text{of} \\
B\text{-algebras} & & BG\text{-algebras} & & BH\text{-algebras}
\end{array}
$$

We construct a *BG*-algebra for any non-empty set as follows:

THEOREM 2.10. *Let X be a set with $0 \in X$. If we define a binary operation " $*$ " on X by*

$$
x * y := \begin{cases} y & \text{if } x = 0, \\ 0 & \text{if } x = y, \\ x & \text{otherwise,} \end{cases}
$$

for any $x, y \in X$, then $(X; *, 0)$ is a BG-algebra.

P r o o f. For any $x, y \in X$, if $x = 0$, then $(x*y)*(0*y) = (0*y)*(0*y) = 0$. Assume $x \neq 0$. If $y = x$, then $(x*y)*(0*y) = (x*x)*(0*x) = 0*(0*x) = x$. If $y \neq x$, then $(x*y)*(0*y) = x*(0*y) = x*y = x$, proving that $(X; *, 0)$ is a *BG*-algebra. ∎

Using Theorem 2.10 we can construct an infinitely many *BG*-algebras. The following example is constructed by Theorem 2.10.

EXAMPLE 2.11. Let $X := \{0,1,2,3\}$ be a set with the following table:

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	1	0

Then $(X; *, 0)$ is a BG-algebra.

THEOREM 2.12. *The BG-algebra $(X; *, 0)$, $|X| \geq 3$, obtained by Theorem 2.10 is non-group-derived.*

Proof. Let $a, b, c \in X$ be distinct elements and $c \neq 0$. Then $c * a = c * b = c$. If we assume that $(X; *, 0)$ is a group-derived BG-algebra obtained from the group $(X; \circ)$ by Theorem 2.10, then $c = c * a = c \circ a^{-1}$ and $c = c * b = c \circ b^{-1}$, i.e., $a = b$, a contradiction. This proves the theorem. ∎

3. Homomorphisms and quotient BG-algebras

Let $(X; *_X, 0_X)$ and $(Y; *_Y, 0_Y)$ be BG-algebras. A mapping $\varphi : X \to Y$ is called a *BG-homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for any $x, y \in X$. A BG-homomorphism $\varphi : X \to Y$ is called a *BG-isomorphism* if φ is a bijection, and denote it by $X \cong Y$. Let $\varphi : X \to Y$ be a BG-homomorphism. Then the subset $\{x \in X | \varphi(x) = 0_Y\}$ of X is called the *kernel* of the BG-homomorphism φ, and denote it by $Ker\varphi$. In this section, we discuss several isomorphism theorems discussed in [9] in view of BG-algebras, and also obtain some consequences of structure theorems.

DEFINITION 3.1. Let $(X; *, 0)$ be a BG-algebra. A non-empty subset S of X is called a *subalgebra* of X if $x * y \in S$ for any $x, y \in S$.

In Example 2.11, $S_1 = \{0, 1\}$ and $S_2 = \{0, 1, 2\}$ are subalgebras of X. We know that any subalgebra of a BG-algebra is also a BG-algebra.

THEOREM 3.2. *Let $(X; *, 0)$ be a BG-algebra and $\emptyset \neq S \subseteq X$. Then the following are equivalent:*

(a) *S is a subalgebra of X;*
(b) *$x * (0 * y), 0 * y \in S$ for any $x, y \in S$.*

Proof. (a)⇒(b). Since $S \neq \emptyset$, there exists an element $x \in S$ and so $0 = x * x \in S$. Since S is closed under "$*$", $0 * y \in S$ and thus $x * (0 * y) \in S$.

(b)⇒(a). Since $x * y = x * (0 * (0 * y))$, $x * y \in S$ for any $x, y \in S$. ∎

A non-empty subset N of X is said to be *normal* ([9]) of X if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$.

EXAMPLE 3.3. Let $X := \{0,1,2,3\}$ be a set with the following table:

*	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then $(X; *, 0)$ is a BG-algebra and $N = \{0,3\}$ is normal of X.

THEOREM 3.4. *Every normal subset N of a BG-algebra X is a subalgebra of X.*

P r o o f. If $x, y \in N$, then $x * 0, y * 0 \in N$. Since N is normal, $x * y = (x * y) * (0 * 0) \in N$. Thus N is a subalgebra of X. ∎

REMARK. The converse of Theorem 3.4 does not hold. Indeed, in Example 2.1, $N = \{0,2\}$ is a subalgebra of X, but it is not normal, since $0*2, 1*1 \in N$, while $(0 * 1) * (2 * 1) = 1 \notin N$.

LEMMA 3.5. *Let N be a normal subalgebra of a BG-algebra X and let $x, y \in N$. If $x * y \in N$, then $y * x \in N$.*

P r o o f. Let $x * y \in N$. Since $y * y = 0 \in N$ and N is normal, $y * x = (y * x) * (y * y) \in N$. ∎

Let $(X; *, 0)$ be a BG-algebra and let N be a normal subalgebra of X. Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if $x*y \in N$, where $x, y \in X$. Then it is easy to show that $\sim_N$ is an equivalence relation on X. Denote the equivalence class containing x by $[x]_N$, i.e., $[x]_N = \{y \in X | x \sim_N y\}$ and let $X/N = \{[x]_N | x \in X\}$.

THEOREM 3.6. *Let N be a normal subalgebra of a BG-algebra X. Then X/N is a BG-algebra.*

P r o o f. If we define $[x]_N * [y]_N := [x * y]_N$, then the operation " $*$ " is well-defined, since if $x \sim_N p$ and $y \sim_N q$, then $x * p \in N$ and $y * q \in N$ implies $(x * y) * (p * q) \in N$ by normality of N. So $x * y \sim_N p * q$ and so $[x * y]_N = [p * q]_N$. Note that $[0]_N = \{x \in X | x \sim_N 0\} = \{x \in X | x * 0 \in N\} = \{x \in X | x \in N\} = N$. Checking remaining axioms is trivial and we omit the proof. ∎

The BG-algebra X/N discussed in Theorem 3.6 is called the *quotient BG-algebra* of X by N. The proofs of Theorems 3.7 and 3.10 follow from the Homomorphism Theorem for algebras ([6, p. 28–29]), and we omit the proofs.

THEOREM 3.7. *Let N be a normal subalgebra of a BG-algebra X. Then the mapping γ $X \to X/N$ given by $\gamma(x) := [x]_N$ is a surjective BG-homomorphism, and $Ker\gamma = N$.*

The mapping γ discussed in Theorem 3.7 is called the *natural* (or *canonical*) *BG*-homomorphism of X onto X/N.

THEOREM 3.8. *Let $\varphi : X \to Y$ be a BG-homomorphism. Then φ is injective if and only if $Ker\varphi = \{0_X\}$.*

P r o o f. Let $x, y \in X$ with $\varphi(x) = \varphi(y)$. Then $\varphi(x) * \varphi(y) = 0_Y$. So $x * y \in Ker\varphi$. Since $Ker\varphi = \{0_X\}$, $x * y = 0_X$. Thus $x = y$ by Lemma 2.4-(iii). Hence φ is injective. The converse is trivial, and we omit the proof. ∎

THEOREM 3.9. *Let $\varphi : X \to Y$ be a BG-homomorphism. Then the kernel $Ker\varphi$ is a normal subalgebra of X.*

P r o o f. Since $0_X \in Ker\varphi$, $Ker\varphi \neq \emptyset$. Let $x * y, a * b \in Ker\varphi$. Then $\varphi(x) * \varphi(y) = 0 = \varphi(a) * \varphi(b)$. Thus by Lemma 2.3-(iii), $\varphi(x) = \varphi(y)$ and $\varphi(a) = \varphi(b)$. It follows that $\varphi((x * a) * (y * b)) = \varphi(x * a) * \varphi(y * b) = (\varphi(x) * \varphi(a)) * (\varphi(y) * \varphi(b)) = (\varphi(x) * \varphi(a)) * (\varphi(x) * \varphi(a)) = 0$. So $(x * a) * (y * b) \in Ker\varphi$. Hence $Ker\varphi$ is a normal subalgebra of X. ∎

By Theorem 3.7 and Theorem 3.9, if $\varphi : X \to Y$ is a BG-homomorphism, then $X/Ker\varphi$ is a BG-algebra.

THEOREM 3.10. *Let $\varphi : X \to Y$ be a BG-homomorphism. Then $X/Ker\varphi \cong Im\varphi$. In particular, if φ is surjective, then $X/Ker\varphi \cong Y$.*

THEOREM 3.11. *Let $\varphi : X \to Y$ be a BG-epimorphism, and let K be a normal subalgebra of Y. Then $X/\varphi^{-1}(K) \cong Y/K$.*

P r o o f. Let $\gamma : Y \to Y/K$ be a natural BG-homomorphism, and let $\mu = \gamma \circ \varphi$. Then μ is a BG-homomorphism and $Ker\mu = Ker(\gamma \circ \varphi) = \{x \in X | (\gamma \circ \varphi)(x) = [\varphi(x)]_K = K\} = \{x \in X | \varphi(x) \in K\} = \{x \in X | x \in \varphi^{-1}(K)\} = \varphi^{-1}(K)$. By Theorem 3.10, we have $X/\varphi^{-1}(K) \cong Y/K$. ∎

The proofs of Theorems 3.12, 3.13 and 3.15 follow from the Second Isomorphism Theorem ([6, p. 149–150]), and we omit the proofs.

THEOREM 3.12. *Let N and K be normal subalgebras of a BG-algebra X, and let $K \subseteq N$. Then $X/N \cong \frac{X/K}{N/K}$.*

THEOREM 3.13. *Let X, Y and Z be BG-algebras, and $h : X \to Y$ be a BG-epimorphism and $g : X \to Z$ be a BG-homomorphism. If $Ker(h) \subseteq Ker(g)$, then there exists a unique BG-homomorphism $f : Y \to Z$ satisfying $f \circ h = g$.*

THEOREM 3.14. *Let X, Y and Z be BG-algebras, and $g : X \to Z$ be a BG-homomorphism and $h : Y \to Z$ be a BG-monomorphism. If $Im(g) \subseteq Im(h)$, then there is a unique BG-homomorphism $f : X \to Y$ satisfying $h \circ f = g$.*

Proof. For each $x \in X, g(x) \in Im\ (g) \subseteq Im\ (h)$. Since h is a BG-monomorphism, there exists a unique $y \in Y$ such that $h(y) = g(x)$. Define a map $f : X \to Y$ by $f(x) = y$. Then $h \circ f = g$. Let $x_1, x_2 \in X$, then $g(x_1 * x_2) = h(f(x_1 * x_2)) = h(f(x_1) * f(x_2))$. Since h is injective, $f(x_1 * x_2) = f(x_1) * f(x_2)$. So f is a BG-homomorphism. The uniqueness of f follows from the fact that h is a BG-monomorphism. This completes the proof. ∎

THEOREM 3.15. *Let X and Y be BG-algebras and let $f : X \to Y$ be a BG-homomorphism. If N is a normal subalgebra of X such that $N \subseteq Ker(f)$, then $\bar{f} : X/N \to Y$ defined by $\bar{f}([x]_N) := f(x)$ for all $x \in X$ is a unique BG-homomorphism such that $\bar{f} \circ \gamma = f$. where $\gamma : X \to X/N$ is natural BG-homomorphism.*

COROLLARY 3.16. *Let X and Y be BG-algebras and let $f : X \to Y$ be a BG-homomorphism. If N is a normal subalgebra of X such that $N \subseteq Ker(f)$, then the following are equivalent:*

  (i) *there exists a unique BG-homomorphism $\bar{f} : X/N \to Y$ such that $\bar{f} \circ \gamma = f$, where $\gamma : X \to X/N$ is the natural BG-homomorphism;*
 (ii) $N \subseteq Ker(f)$.

Furthermore, $\bar{f}$ is a BG-monomorphism if and only if $N = Ker(f)$.

Proof. (ii)⇒(i). By Theorem 3.13.
     (i)⇒(ii). If $x \in N$, then
$$f(x) = (\bar{f} \circ \gamma)(x) = \bar{f}([x]_N) = \bar{f}([0]_N) = f(0) = 0.$$
Hence $x \in Ker(f)$.

     Furthermore, $\bar{f}$ is a monomorphism if and only if $Ker\ \bar{f} = \{N\}$ if and only if $f(x) = 0$ implies $[x]_N = [0]_N = N$ if and only if $Ker(f) \subseteq N$. This completes the proof. ∎

THEOREM 3.17. *Let $f : X \to Y$ be a BG-homomorphism, and let M, N be normal subalgebras of X and Y, respectively, such that $f(M) \subseteq N$. Then there exists a unique BG-homomorphism $h : X/M \to Y/N$ such that $h \circ p = q \circ f$, where $p : X \to X/M$ and $q : Y \to Y/N$ are natural BG-homomorphisms.*

Proof. Define a map $h : X/M \to Y/N$ by $h([x]_M) = [f(x)]_N$. Then h is well-defined. Indeed, if $[x]_M = [y]_M$ $(x, y \in X)$, then $x * y \in M$. Thus $f(x) * f(y) = f(x * y) \in f(M) \subseteq N$. So $[f(x)]_N = [f(y)]_N$. If

$[x]_M, [y]_M \in X/M$, then $h([x]_M * [y]_M) = h([x * y]_M) = [f(x * y)]_N = [f(x)]_N * [f(y)]_N = h([x]_M) * h([y]_M)$. So h is a BG-homomorphism. On the other hand, if x is any element of X, then since $(h \circ p)(x) = h([x]_M) = [f(x)]_N = (q \circ f)(x)$, we obtain that $h \circ p = q \circ f$. To show the uniqueness of h, let $k : X/M \rightarrow Y/N$ be a BG-homomorphism such that $k \circ p = q \circ f$. Then $k([x]_M) = k(p(x)) = q(f(x)) = h(p(x)) = h([x]_M)$ for any $[x]_M \in X/M$. Thus $k = h$. This completes the proof. ■

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Chang Bum Kim
DEPARTMENT OF MATHEMATICS
KOOKMIN UNIVERSITY
SEOUL 136-702, KOREA
e-mail: cbkim@kookmin.ac.kr

Hee Sik Kim
DEPARTMENT OF MATHEMATICS
HANYANG UNIVERSITY
SEOUL 133-763, KOREA
e-mail: heekim@hanyang.ac.kr