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COMMON FIXED POINT, BEST APPROXIMATION AND GENERALIZED f-WEAK CONTRACTION MULTIVALUED MAPPING IN p-NORMED SPACES

Abstract. We establish coincidence point and common fixed point results for multi-valued f-weak contraction mappings which assume closed values only. As an application, related common fixed point and invariant approximation are obtained in the setup of certain metrizable topological vector spaces. Our results provide extensions as well as substantial improvements of several well known results in the literature.

1. Introduction and preliminaries

Let X be a linear space. A p-norm on X is a real valued function $\|.\|_p$ on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0$ iff $x = 0$,
(ii) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$,
(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$,

for all $x, y \in X$ and all scalars λ. The pair $(X, \|.\|_p)$ is called a p-normed space. It is a metric linear space with $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$, and defines a translation invariant metric d_p on X. If $p = 1$, one obtains a normed linear space. It is well known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \leq 1$ (see e.g., [15]). The spaces l_p and $L_p[0, 1]$, $0 < p \leq 1$ are p-normed spaces. A p-normed space is not necessarily a locally convex space. Recall that the dual space X^* (the dual of X) separates points of X if, for each non zero x in X, there exists an f in X^* such that $f(x) \neq 0$. In this case the weak topology on X is well defined and is Hausdorff. We mention that, if X is not locally convex, then X^* need not separate points of X. For example if $X = L_p[0, 1]$, $0 < p < 1$ then $X^* = \{0\}$ ([19]). However there are some

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non locally convex spaces (such as the p-normed l_p, $0 < p < 1$) whose dual separates the points of X ([15]). In the sequel, we shall assume that X^* separates the points of the p-normed space X whenever the weak topology is under consideration. Let X be a p-normed space. We denote by $CB(X)$ and $CL(X)$, the families of all nonempty closed bounded and nonempty closed subsets of X, respectively. Let $A, B \in CL(X)$. Set, $N_\varepsilon(A) = \{x \in X : d_p(x, A) < \varepsilon\}$ where $d_p(x, A) = \inf\{\|x - y\|_p : y \in A\}$ for each $x \in X$. We define a generalized Hausdorff metric H_p on $CL(X)$ induced by the p-norm of X by

$$H_p(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \emptyset \\ \infty & \text{if } E_{A,B} = \emptyset \end{cases}$$

where $E_{A,B} = \{\varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A)\}$. Following [5], a mapping $T : X \longrightarrow CL(X)$ is said to be a multivalued weak contraction iff there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H_p(Tx, Ty) \leq \theta \|x - y\|_p + L d_p(y, Tx)$$

for every $x, y \in X$. Following Kamran [12], let f be a self map on X. A mapping $T : X \longrightarrow CL(X)$ is said to be a multivalued $f-$ weak contraction or multivalued (f, θ, L)-weak contraction iff there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H_p(Tx, Ty) \leq \theta \|fx - fy\|_p + L d_p(fy, Tx) \qquad (1.1)$$

for every $x, y \in X$.

DEFINITION 1.1. Let $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$. A point x in X is said to be

(1) a *fixed point of* f if $f(x) = x$;

(2) a *fixed point of* T if $x \in T(x)$;

(3) a *coincidence point of the pair* (f, T) if $fx \in Tx$;

(4) a *common fixed point of the pair* (f, T) if $x = fx \in Tx$;

(5) f is said to be T-weakly commuting at x if $ffx \in Tfx$. $F(f), C(f, T)$ and $F(f, T)$ denote set of all fixed points of f, set of all coincidence points of the pair (f, T) and the set of all common fixed points of the pair (f, T), respectively.

Recently, Abbas et al. [1], obtained the following result, which will be needed in the sequel.

THEOREM 1.2. *Let X be a metric space, $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$ be an (f, θ, L)- weak contraction with $\overline{T(X)} \subseteq f(X)$. Suppose that $\overline{T(X)}$ is complete. Then $C(f, T) \neq \emptyset$. Moreover, $F(f, T) \neq \emptyset$ if one of the following conditions holds:*

(a) *For some* $x \in C(f,T)$, f *is* T*-weakly commuting at* x *and* $f^2x = fx$.

(b) f *and* T *are weakly compatible on* $C(f,T)$, f *is continuous, and* $\lim_{n\to\infty} f^n x$ *exists for some* $x \in C(f,T)$.

(c) *For some* $z \in C(f,T)$, f *is continuous at* z, *and* $\lim_{n\to\infty} f^n y = z$ *for some* $y \in X$.

(d) $f(C(f,T))$ *is a singleton subset of* $C(f,T)$.

DEFINITION 1.3. A subset F of a p-normed space X is called (6) *q-starshaped or starshaped with respect to q* if $\lambda x+(1-\lambda)q \in F$ for all $x \in F$ and $\lambda \in [0,1]$.

DEFINITION 1.4. Let f be a self map on a p-normed space X and $F \subseteq X$. f is called

(7) *affine on* F if F is convex and

$$f(\lambda x + (1-\lambda)y) = \lambda fx + (1-\lambda)fy$$

for all $x, y \in F$ and $\lambda \in [0,1]$;

(8) *q-affine on* F if F is starshaped with respect to q and

$$f(\lambda x + (1-\lambda)q) = \lambda fx + (1-\lambda)q$$

for all $x \in F$ and $\lambda \in [0,1]$.

DEFINITION 1.5. Let $f : F \longrightarrow F$ and $T : F \longrightarrow CL(F)$. The pair (f,T) is called

(9) *commuting* if $Tfx = fTx$ for all $x \in F$;

(10) *R-weakly commuting* if $fTx \in CL(F)$ and

$$H_p(fTx, Tfx) \le Rd_p(fx, Tx)$$

for all $x \in F$ and for some $R > 0$;

(11) *weakly compatible* [10] if they commute at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f,T)$. If F is a q-starshaped with $q \in F(f)$, then the pair (f,T) is called

(12) *C_q-commuting* if $fTx = Tfx$ for all

$$x \in C_q(f,T) = \cup\{C(f,T_\lambda) : \lambda \in [0,1]\},$$

where $T_\lambda x = \lambda Tx + (1-\lambda)q$;

(13) *R-subweakly commuting on F (with respect to q)* if $fTx \in CL(F)$ and

$$H_p(fTx, Tfx) \le R \inf\{d_p(fx, T_\lambda x) : 0 \le \lambda \le 1\}$$

for each $x \in F$ and for some $R > 0$;

(14) *R-subcommuting on F (with respect to q [18])* if $fTx \in CL(F)$ and $H_p(fTx, Tfx) \le \frac{R}{\lambda} d_p(fx, T_\lambda x)$ for all $x \in F, \lambda \in (0,1]$ and some $R > 0$.

DEFINITION 1.6. Let $f : X \longrightarrow X$, $T : X \longrightarrow CL(X)$ and $F \subseteq X$. $f - T$ is called

(15) *demiclosed at* 0 if whenever a sequence $\{x_n\}$ in F converges weakly to x_0 in F and $y_n \in (f-T)x_n$ such that $\{y_n\}$ converges to 0 strongly, then $0 \in (f-T)x_0$.

It is well known that R-subweakly commuting maps are R-weakly commuting and R-weakly commuting maps are weakly compatible. It is also known that, if the pair (f,T) is weakly compatible at $x \in C(f,T)$, then f is T-weakly commuting at x, and hence $f^n(x) \in C(f,T)$. However the converse is not true in general (For a detailed discussion on the above mentioned notions and their implications, we refer the reader to [4], [8], [7], [9], [10], [21] and [22]).

Let X be a p-normed space and M be any subset of X. If there exists a $y_0 \in M$ such that $\|x-y_0\|_p = d_p(x,M)$, then y_0 is called a *best approximation to x out of M*. We denote by $P_M(x)$, the set of all best approximations to x from M.

Let M be a q-starshaped subset of a p-normed space X, $f: M \longrightarrow M$ and $T: M \longrightarrow CL(M)$. Following [4], the pair $\{f,T\}$ satisfies the coincidence point condition on a closed subset A of M if, whenever $\{x_n\}$ is a sequence in A such that $d_p(fx_n, Tx_n) \longrightarrow 0$, then $fu \in Tu$ for some $u \in A$. A map T satisfies the fixed point condition on $A \in CL(M)$ if, whenever $\{x_n\}$ is a sequence in A such that $d_p(x_n, Tx_n) \longrightarrow 0$, then $u \in Tu$ for some $u \in A$. We also define, $\delta_p(fy,Tx) = \inf\{d_p(fy, T_\lambda x) : 0 \le \lambda \le 1\}$.

The purpose of this paper is to obtain general coincidence point results for f-weak contractive multivalued mapping in the frame work of locally bounded topological vector spaces and locally convex topological vector spaces. As an application, an invariant approximation result is established, which in turn extends and strengthens various known results.

2. Common fixed point and approximation results

Based upon Theorem 1.2, we present the following theorem.

THEOREM 2.1. *Let M be a subset of a p-normed space X, $f: M \longrightarrow M$ and $T: M \longrightarrow CL(M)$. Suppose that M is q-starshaped, $f(M) = M$ [resp. f is q-affine on M], $T(M)$ is bounded, $\overline{T(M)}$ is complete, $\overline{T(M)} \subseteq f(M)$, the pair $\{f,T\}$ satisfies the coincidence point condition on M and*

$$H_p(Tx,Ty) \le \|fx-fy\|_p + \delta_p(fy,Tx) \tag{2.1}$$

for all $x,y \in M$. Then $C(f,T) \neq \emptyset$. Moreover, $F(f,T) \neq \emptyset$ if one of the conditions (a) – (d) of Theorem 1.2 holds.

Proof. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\lambda_n \to 1$. For $n \ge 1$, let

$$T_n(x) = T_{\lambda_n}(x) = \lambda_n Tx + (1-\lambda_n)q$$

for all x in M. As M is q-starshaped, $\overline{T(M)}$ is complete, $\overline{T(M)} \subseteq f(M)$, and $f(M) = M$ [resp. f is q-affine on M], we have $\overline{T_n(M)} \subseteq f(M)$ and $\overline{T_n(M)}$ is complete for each $n \geq 1$. Now consider,

$$\begin{aligned} H_p(T_nx, T_ny) &= (\lambda_n)^p H_p(Tx, Ty) \\ &\leq (\lambda_n)^p \|fx - fy\|_p + (\lambda_n)^p \delta_p(fy, Tx) \\ &\leq (\lambda_n)^p \|fx - fy\|_p + (\lambda_n)^p d_p(fy, T_nx) \end{aligned}$$

for all $x, y \in M$, which implies that each T_n is an $(f, (\lambda_n)^p, (\lambda_n)^p)$-weak contraction on M for every $p \in (0, 1]$. Hence, from Theorem 1.2 we conclude that $fx_n \in Tx_n = \lambda_n Tx_n + (1 - \lambda_n)q$ for some $x_n \in M$. As $fx_n = \lambda_n y_n + (1 - \lambda_n)q$ for some $y_n \in Tx_n \subseteq T(M)$, $T(M)$ is bounded, $\lambda_n \to 1$ and

$$\|fx_n - y_n\|_p = (1 - \lambda_n)^p \|q - y_n\|_p \leq (1 - \lambda_n)^p (\|q\|_p + \|y_n\|_p),$$

it follows that, $\|fx_n - y_n\|_p \to 0$, and hence $d_p(fx_n, Tx_n) \leq \|fx_n - y_n\|_p \to 0$. Since the pair $\{f, T\}$ satisfies the coincidence point condition on M, there exists a $u \in M$ such that $fu \in Tu$. Thus $C(f, T) \neq \emptyset$. Using arguments similar to those given in the proof of Theorem 1.2, it can be shown that $F(f, T) \neq \emptyset$ if one of the conditions $(a) - (d)$ of Theorem 1.2 holds.

Clearly an f-nonexpansive multivalued map T satisfies inequality (2.1), so Theorem 2.1 improves , extends and generalizes Corollary 2.5 of Hussain and Jungck [6], Corollaries 3.2, 3.4 of Jungck [8], Theorem 6 due to Jungck and Sessa [11], Theorems 2.2-2.5 of Latif and Tweddle [17], Theorem 3 due to Rhoades [18] and Theorems 2.1, 2.2, 2.4, 2.6, 2.7, 2.8 of Shahzad and Hussain [22].

COROLLARY 2.2. *Let M be a subset of a p-normed space X, $f : M \longrightarrow M$ and $T : M \longrightarrow CL(M)$. Suppose that M is q-starshaped, $f(M) = M$ [resp. f is q-affine on M], $T(M)$ is bounded, $\overline{T(M)}$ is complete, $\overline{T(M)} \subseteq f(M)$, and f and T satisfy* (2.1) *for all $x, y \in M$. Then $C(f, T) \neq \emptyset$ if one of the following conditions holds:*

(e) *$(f - T)(M)$ is closed.*

(f) *M is weakly compact and $f - T$ is demiclosed at* 0.

(g) *T and f satisfy for all $x, y \in M$,*

$$\text{(2.2)} \quad \begin{aligned} &H_p^r(Tx, Ty) \\ &\quad \leq \theta_1(d_p(fx, Tx)) d_p^r(fx, Tx) + \theta_2(d_p(fy, Ty)) d_p^r(fy, Ty), \end{aligned}$$

where $\theta_i : R \to [0, 1)$ $(i = 1, 2)$ and $r \geq 1$ is some fixed positive real number. Moreover $F(f, T) \neq \emptyset$ if one of the conditions (a)–(d) *of Theorem* 1.2 *holds.*

Proof. (e) As in the proof of Theorem 2.1, $fx_n - y_n \to 0$ as $n \to \infty$ where $y_n \in Tx_n$. As $(f - T)(M)$ is closed, so $0 \in (f - T)(M)$. Hence the pair

$\{f, T\}$ satisfies the coincidence point condition on M and the result follows from Theorem 2.1.

(f) As in the proof of Theorem 2.1, $fx_n - y_n \to 0$ as $n \to \infty$ where $y_n \in Tx_n$. By the weak compactness of M, there is a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ such that $\{x_m\}$ converges weakly to $y \in M$ as $m \to \infty$. Since $f - T$ is demiclosed at 0, we obtain $0 \in (f - T)y$. Hence the pair $\{f, T\}$ satisfies the coincidence point condition on M and the result follows from Theorem 2.1.

(g) Following an argument similar to that given in Theorem 2.11 [13], the pair $\{f, T\}$ satisfies the coincidence point condition on M and the result follows from Theorem 2.1.

As an application of Theorem 2.1, we obtain the following result, which improves, generalizes and extends Theorems 3.2, 3.3 of Al-Thagafi [2] for $D = P_M(p)$, Theorem 3.1, 3.3 due to Al-Thagafi and Shahzad [3], Theorem 7 of Jungck and Sessa [11], Theorem 3.14 of Kamran [12], Theorem 3 of Latif and Bano [16], Theorem 3 of Sahab, Khan and Sessa [20], Theorems 2.12, 2.13 of Shahzad and Hussain [22], and many others.

THEOREM 2.3. *Let M be a subset of a p-normed space X, $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$, $T(M \cap \partial M) \subseteq M$ and $x_0 \in X$. Suppose that $P_M(x_0)$ is closed and q-starshaped, $f(P_M(x_0)) = P_M(x_0)$, $\overline{T(P_M(x_0))}$ is complete, the pair $\{f, T\}$ satisfies the coincidence point condition on $P_M(x_0)$,*

$$\sup_{y \in Tx} \|y - x_0\|_p \leq \|fx - x_0\|_p$$

for all $x \in P_M(x_0)$, and the pair $\{f, T\}$ satisfies (2.1) for all $x, y \in P_M(x_0)$. Then, $C(f, T) \cap P_M(x_0) \neq \emptyset$. Moreover $F(f, T) \cap P_M(x_0) \neq \emptyset$, provided one of the following conditions holds:

(a) *For some $x \in C(f, T) \cap P_M(x_0)$, f is T-weakly commuting at x and $f^2x = fx$.*

(b) *f and T are weakly compatible on $C(f, T) \cap P_M(x_0)$, f is continuous, and $\lim_{n \to \infty} f^n x$ exists for some $x \in C(f, T) \cap P_M(x_0)$.*

(c) *For some $x \in C(f, T) \cap P_M(x_0)$, f is continuous at x, and $\lim_{n \to \infty} f^n y = x$ for some $y \in P_M(x_0)$.*

(d) *$f(C(f, T)) \cap P_M(x_0)$ is a singleton subset of $C(f, T) \cap P_M(x_0)$.*

Proof. For any $x \in P_M(x_0)$,

$$\|(1 - \lambda)x + \lambda x_0 - x_0\|_p = (1 - \lambda)^p \|x - x_0\|_p < \|x - x_0\|_p = d_p(p, M)$$

for all $\lambda \in (0, 1)$. Thus, $\{(1 - \lambda)x + \lambda x_0 : \lambda \in (0, 1)\} \cap M = \emptyset$ and so $x \in M \cap \partial M$. Since, $T(M \cap \partial M) \subseteq M$, it follows that $Tx \subseteq M$. Now let

$z \in Tx$. As $fx \in P_M(x_0)$,

$$\begin{aligned} \|z - x_0\|_p &\le \sup_{y \in Tx} \|y - x_0\|_p \\ &\le \|fx - x_0\|_p = d_p(x_0, M). \end{aligned}$$

Thus $z \in P_M(x_0)$ and hence $Tx \subseteq P_M(x_0)$. Moreover,

$$\overline{T(P_M(x_0))} \subseteq P_M(x_0) = f(P_M(x_0)).$$

The result now follows from Theorem 2.1.

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