DEMONSTRATIO MATHEMATICA
Vol. XLI No 1 2008

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A CLOSEDNESS THEOREM FOR NORMED SPACES

Abstract. For spaces X, Y, for which some algebraic operations are defined and in some cases topologies for X, Y are defined too, we define for the space X a dual space X^d with respect to the space Y. If σ is a topology for Y, (compatible with the algebraic operations of Y), then the pointwise topology τ_p for Y^X is defined. We show that X^d is (algebraically)τ_p-closed in Y^X. For normed spaces is shown that suitable subspaces of X^d are τ_p-closed in a product space $K \subseteq Y^X$. As a corollary we obtain a generalization of Alaoglu's theorem.

1. Introduction

By $\mathbb{R}$ and $\mathbb{C}$ we denote the reals and the complex numbers respectively and by $\mathbb{K}$ we mean $\mathbb{R}$ or $\mathbb{C}$.

Now let $(X, \|\cdot\|)$ be a normed $\mathbb{K}$-vector space and $X' := \{f : X \to \mathbb{K} | f$ linear and continuous$\}$ the (first) dual space (dual) of X. Hence X' consists of functions which at the same time are an algebraic homomorphism (a linear map) and a topological homomorphism (a continuous map). Of course X and $\mathbb{K}$ are normed spaces, meaning that X and $\mathbb{K}$ belong to the same class of spaces. We will also set $X^d := X'$.

Now we consider another quite different example. Let X, Y be rings and for simplicity let us assume that X, Y are commutative rings with units. Then we can consider $X^d := \{h \in Y^X | \ h$ is a ring homomorphism$\}$ as the first dual space of X (w.r.t.Y.)

REMARKS. 1. $(X, \|\cdot\|)$ and $(\mathbb{K}, |\cdot|)$ and $X = (X(+,*))$, $Y = (Y(+,*))$ respectively not only belong to the same class of spaces, but we also have a natural correspondence between the (algebraic) operations. In the first case the vector space structure of X correspondes to the vector space structure of $\mathbb{K}$ and the norm topologies $\tau_{\|\cdot\|}$ and $\tau_{|\cdot|}$ (Euclidian topology) correspond to each other. Concerning the rings X, Y we let correspond: the addition

Key words and phrases: generalized dual space, rings, pointwise topology, normed spaces, Alaoglu's theorem.
2000 *Mathematics Subject Classification*: 46B10, 46B50, 46E15.

of X to the addition of Y and the multiplication of X to the multiplication of Y.

2. By the usual operator norm, $(X', \|\cdot\|) = (X^d, \|\cdot\|)$ again becomes a normed space (even a Banach space) meaning that also X and X' belong to the same class of spaces. For our two rings X and Y in the usual pointwise manner we can in Y^X define an addition and a multiplication but then in general X^d is not closed under these operations, i.e. $f+g$, $f*g$ may not be homomorphisms if f, g are. This different behaviour becomes important if we want to define second dual spaces: $X^{dd} = (X', \|\cdot\|)' = (X^d, \|\cdot\|)^d$ and X^{dd} (w.r.t.Y) for the rings X, Y respectively. Such constructions we will present in a forthcoming paper. If we have defined X^{dd} then we can consider the canonical embedding map $J : X \to X^{dd}$, $\forall x \in X$, $J(x) = w(x, \cdot)$, where $w(x,h) = h(x)\ \forall h \in X^d$. Here we used the evaluation map $w : X \times Y^X \to Y$, $w(x,f) = f(x)$, $(x,f) \in X \times Y^X$.

And of course the choice of a suitable space Y for the Y-dual of the space X depends on the fact, whether or not the canonical map J is well-defined and has nice properties. In our paper we want to prove a τ_p-closedness theorem for X^d, where τ_p means the pointwise topology for Y^X and X, Y are normed spaces. As a corollary we get a generalization of Alaoglu's theorem for normed spaces.

2. Definition of an abstract dual space

Starting from our two examples we will define the dual space of a given space X with respect to a suitable space Y.

DEFINITION 2.1. Let X, Y be spaces with finitely many algebraic operations such that X, Y belong to the same class of such spaces. We assume that we can assign to each algebraic operation in X an algebraic operation in Y (in a natural manner). $X^d := \{h : X \to Y \mid h$ is a homomorphism with respect to each pair of corresponding algebraic operations in X and in Y respectively$\}$; if both Y and X have topologies we assume in addition that each $h \in X^d$ is continuous. X^d is called the Y-dual of X or the dual space of X with respect to Y.

EXAMPLES. 1. Let X, Y be $\mathbb{K}$-normed spaces, then $X^d = L(X,Y) = \{h : X \to Y \mid h$ is linear and h is continuous$\}$ is the ("natural") Y-dual of X.

2. We know that a Boolean ring is a ring each element of which is idempotent. We also know that there exists the smallest nontrivial (commutative) ring with unit $F_2 = \{0,1\}$ which is at the same time a field. For this ring we have for instance $0 \cdot 0 = 0$, $1 \cdot 1 = 1$ and hence F_2 is a Boolean ring too.

Now let X be a Boolean ring with unit. To get a Y-dual for X we set $Y = F_2$ and define:

$$X^d = \{h : X \to F_2 \mid h \text{ is a ring homomorphism}\}$$

to be the first dual space of X.

REMARKS. 1. If X is a normed space then X' consists of a special family of functionals, where X^d from our definition 2.1. consists of a special family of operators.

2. If X, Y are vector spaces (over $\mathbb{K}$) then a linear map from X to Y we consider as a homomorphism from X to Y. If we assume that Y has a topology σ and the algebraic operations in Y are continuous then this includes that the scalar multiplication from $(\mathbb{K}, \tau_{|\cdot|}) \times (Y, \sigma)$ to (Y, σ) is continuous too.

3. The closedness theorem

At first we must show that X^d is algebraically closed in Y^X with respect to the pointwise topology τ_p.

PROPOSITION 3.1. *We assume that Y has a Hausdorff topology σ such that all algebraic operations in Y are continuous with respect to σ.*

Then X^d is τ_p-closed in Y^X.

Proof. Let be $f \in \overline{X^d}^{\tau_p}$, then we find a net $(f_i)_{i\in t}$ from X^d such that $f_i \xrightarrow{\tau_p} f \in Y^X$; now we consider an arbitrary pair of (multiplicatively written) algebraic operations, for example binary ones:

$$\forall (x,y) \in X \times X, \ f_i(x) \to f(x), \ f_i(y) \to f(y), \ f_i(xy) \to f(xy)$$

implying $f_i(x)f_i(y) \to f(x)f(y)$ by the continuity of algebraic operations in Y; but $f_i \in X^d$ implies that $f_i(x)f_i(y) = f_i(xy)$, hence $f_i(xy) \to f(xy)$ and $f_i(xy) \to f(x)f(y)$ implying $f(xy) = f(x)f(y)$ since Y is Hausdorff. Since by this way we can treat unitary operations and operations of arbitrary arity and we also can include linear maps from X to Y, so finally we have $f \in X^d$. ∎

REMARK. We can denote by $(X^d)_{\text{alg}} = \{h : X \to Y \mid h$ is a homomorphism w.r.t. each pair of corresponding algebraic operations in X and Y respectively$\}$. If X has no topology or Y has no topology then $X^d = (X^d)_{\text{alg}}$; otherwise by definition 2.1 $X^d = (X^d)_{\text{alg}} \cap C(X,Y)$, where $C(X,Y)$ are the continuous functions. In 3.1 we used $(X^d)_{\text{alg}}$, since to force the function f to be continuous too, we must (in addition) assume that X^d is equicontinuous or evenly continuous.

We now can prove the closedness theorem.

THEOREM 3.2. *Let X, Y be $\mathbb{K}$-normed spaces, and let in addition exist finitely many pairs of corresponding algebraic operations in X and in Y*

respectively; if $\|\cdot\|$ is the norm for Y, let all algebraic operations in Y be $\tau_{\|\cdot\|}$- continuous; let X^d be defined as in Definition 2.1 and let $L(X,Y)$ be the set of all linear and continuous maps from X to Y, as we already mentioned; hence $X^d \subseteq L(X,Y)$ and for $L(X,Y)$ we consider the operator norm:

$$\|h\| = \sup_{\|x\|\leq 1} \|h(x)\|$$

and we restrict $\|\cdot\|$ to X^d. Let be

$$\forall c \in \mathbb{R},\ c > 0,\ H_c := \{h \in X^d |\ \|h\| \leq c\};$$
$$\forall x \in X,\ \forall c > 0,\ K_{x,c} := \{y \in Y |\ \|y\| \leq c\|x\|\};$$

$\forall c$ for $\prod_{x\in X} K_{x,c}$ we consider the Tychonoff-topology.

Then $H_c \subseteq \prod_{x\in X} K_{x,c}$ and H_c is closed in $\prod_{x\in X} K_{x,c}$.

Proof. Let $h \in H_c$, then $\|h\| \leq c$;

$$\forall x \in X,\ \|h(x)\| \leq \|h\|\ \|x\| \leq c\|x\| \Rightarrow h(x) \in K_{x,c} \Rightarrow h \in \prod_{x\in X} K_{x,c}.$$

Let $g \in \overline{H}_c^{\prod K_{x,c}}$ then there exists a net $(h_i)_{i\in I}$ from H_c such that $h_i \to g$ in $\prod_{x\in X} K_{x,c}$ implying $h_i \xrightarrow{\tau_p} g$ since $\prod_{x\in X} K_{x,c} \subseteq Y^X$ and $h_i \xrightarrow{\tau_p} g$ in Y^X. $(Y, \tau_{\|\cdot\|})$ is Hausdorff and all algebraic operations in Y are $\tau_{\|\cdot\|}$-continuous; since $h_i \in X^d$ $\forall i \in I$ by 3.1. we get that g is linear and that g is a homomorphism for each pair of algebraic operations in X and Y respectively. Now $g \in \prod K_{x,c}$

$$\Rightarrow (g(x))_{x\in X} \in \prod K_{x,c} \Rightarrow g(x) \in K_{x,c}\ \forall x \in X \Rightarrow \|g(x)\| \leq c\|x\|\ \forall x \in X$$
$$\Rightarrow g \text{ is continuous and } g \in L(X,Y),\ \|g\| \leq c \text{ implying } g \in H_c. \ \blacksquare$$

COROLLARY 3.3. *If X, Y have the properties mentioned in Theorem 3.2 and $(Y, \|\cdot\|)$ is a finite-dimensional normed space, then (H_c, τ_p) is a compact Hausdorff space for each $c > 0$.*

Proof. If $(Y, \|\cdot\|)$ is finite-dimensional then each $K_{x,c}$ is compact because $K_{x,c}$ is bounded and closed in Y; then $\prod_{x\in X} K_{x,c}$ is compact by the Tychonoff theorem $\forall c > 0$. Hence H_c being closed in $\prod_{x\in X} K_{x,c}$ is compact too. Since $(Y, \tau_{\|\cdot\|})$ is Hausdorff, each $K_{x,c}$ is Hausdorff and hence $\prod_{x\in X} K_{x,c}$ is Hausdorff too and so H_c is Hausdorff. ■

REMARK 3.4. If X is a normed space alone, if $Y = \mathbb{K}$ and $c = 1$ then we get the Alaoglu theorem (for normed spaces; see for instance [1]). Thus the corollary is a generalization of Alaoglu's theorem.

References

[1] J. B. Conway, *A Course in Functional Analysis*, Second Edition, New York, Springer 1990.

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Received February 9, 2007; revised version August 14, 2007.