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LINE GRAPHS, THEIR DESARGUESIAN CLOSURES, AND CORRESPONDING GROUPS OF AUTOMORPHISMS

Abstract. The notion of Desarguesian closure of an arbitrary graph was introduced in [7], and basic properties of Desarguesian closure of complete graphs were also presented in [7]. Then, in [4], the Desarguesian closure of binomial graphs (cf. [5]) was studied. In this paper we shall be mainly concerned with the line graphs associated with complete graphs, their Desarguesian closure, horizon, and automorphisms.

Introduction

The notion of *Desarguesian closure* of a (complete) graph was introduced in [7]. Roughly speaking, the construction of such a closure consists in adding to every edge of a graph an "improper point", and collecting new points into new improper lines, determined by triangles ("planes") of the underlying graph. The resulting structure, which is a partial Steiner triple system, can be considered as a "generalized Desargues configuration" representing a perspective of two simplices. As an interesting feature of a generalized Desargues configuration we can note that it satisfies the (projective) Desargues axiom: every Desargues configuration contained in it closes, and every triangle in such a structure can be completed to a Desargues configuration. Some basic properties of generalized Desargues configurations were established in [7], in particular, their automorphisms were described.

Clearly, the construction of Desarguesian closure can be applied to an arbitrary graph. In [4] it was applied to binomial graphs, and then the resulting geometry was studied. The aim of this paper is to characterize the geometry which arises when we apply our construction to line graphs. For a given graph $\mathfrak{G}$ its *line graph* $\mathfrak{G}^*$ (cf. [8]) describes the structure of the neighborhood of edges of $\mathfrak{G}$. Evidently, the construction of the line graph can be iterated, and on every level we can apply the construction of the

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Desarguesian closure. What are then the resulting geometries, and what relationships between them may hold? For an arbitrary graph $\mathfrak{G}$ the family of improper objects of its Desarguesian closure yields a substructure, called the *horizon* of $\mathfrak{G}$. One can note a similarity between the construction of the horizon of $\mathfrak{G}$ and the construction of the graph $\mathfrak{G}^*$ – how can this similarity be formally stated? In this paper we try to solve these questions.

Section 1 consists, primarily, of definitions of the notions investigated in this paper. Then, in section 2 we remind some general properties of (generalized) Desargues configurations, called Desargues spaces. In what follows, we show that the horizon of every graph can be embedded into the projective space over $GF(2)$ (2.4(iv)). In section 3 we examine in some details horizons of graphs. After some preliminary observations (which can be also of some interest on their own, e.g. in 3.3 we characterize the cliques on the horizon of an arbitrary graph) we determine the geometry of the horizon of the line graph associated with a complete graph (3.7) and then in section 4 we determine the group of its automorphisms (4.6, 4.7). More general, proposition 4.2 characterizes the groups of automorphisms of the horizons of the line graphs associated with a wider class of graphs. In the appendix (section A) we give recursive and direct formulas for parameters of iterated line graphs and Desarguesian closures of such graphs.

1. Definitions and preliminary results

Let X be an arbitrary set. We write $\wp_k(X)$ for the family of all k-element subsets of X. Then a *graph* is a structure $\mathfrak{G} = \langle S, \mathcal{E}\rangle$ with $\emptyset \neq \mathcal{E} \subseteq \wp_2(S)$, i.e. a partial linear space with 2-element lines (in this context lines are usually called *edges*).

With an arbitrary graph $\mathfrak{G} = \langle S, \mathcal{E}\rangle$ we associate its Desarguesian closure defined as follows. With every edge $a \in \mathcal{E}$ we associate an element a^∞ in such a way that $a^\infty \notin S$ and $a_1{}^\infty \neq a_2{}^\infty$ for distinct $a_1, a_2 \in \mathcal{E}$. Set $S^\infty = \{a^\infty \colon a \in \mathcal{E}\}$ and $\overline{S} = S \cup S^\infty$. For $a \in \mathcal{E}$ we put $\overline{a} = a \cup \{a^\infty\}$. A subset $Z \in \wp_3(S)$ is *a triangle* of $\mathfrak{G}$ if $\wp_2(Z) \subseteq \mathcal{E}$; let $\mathcal{T} = \mathcal{T}_{\mathfrak{G}}$ be the set of all triangles of $\mathfrak{G}$. Then for any $Z \in \mathcal{T}$ we define a new "line" $Z^\infty = \{a^\infty \colon a \in \wp_2(Z)\}$. Finally, we define $\mathcal{L} = \{\overline{a} \colon a \in \mathcal{E}\} \cup \{Z^\infty \colon Z \in \mathcal{T}\}$. The *Desarguesian closure of* $\mathfrak{G}$ is the structure

$$\mathbf{D}(\mathfrak{G}) := \langle \overline{S}, \mathcal{L}\rangle$$

(cf. [7]). It is seen that $\mathbf{D}(\mathfrak{G})$ is a partial linear space with line size equal to 3, i.e. it is a partial Steiner triple system (cf. [1, 2, 6]). Elements of the set S^∞ are frequently called *improper points* of $\mathbf{D}(\mathfrak{G})$, and elements of the set S are *proper points*. Similarly, lines of the form $\overline{a}$ are *proper*, and those of the form Z^∞ are *improper*. Note that the set of improper points yields a subspace of $\mathbf{D}(\mathfrak{G})$, which will be referred to as *the (Desarguesian) horizon of* $\mathfrak{G}$, and will be denoted by $\mathbf{H}(\mathfrak{G})$.

Let, as above, $\mathfrak{G} = \langle S, \mathcal{E}\rangle$ be an arbitrary graph. With $\mathfrak{G}$ we associate its *line graph* $\mathfrak{G}^*$ as follows. For any two edges $a, b \in \mathcal{E}$ we say that a and b are neighbors if $a \cap b \neq \emptyset$ and $a \neq b$; in such a case we write $a \sim b$. Then we put $S^* = \mathcal{E}$ and $\mathcal{E}^* = \big\{\{a, b\} \in \wp_2(\mathcal{E})\colon a \sim b\big\}$. Finally (cf. [8, 2]),

$$\mathfrak{G}^* := \langle S^*, \mathcal{E}^*\rangle.$$

For an arbitrary finite partial linear space $\mathfrak{M} = \langle X, \mathcal{L}\rangle$ with $\mathcal{L} \subset \wp(X)$ we use the following notation:

- $\boldsymbol{v} = \boldsymbol{v}_{\mathfrak{M}} = |X|$ is the number of points of $\mathfrak{M}$;
- $\boldsymbol{b} = \boldsymbol{b}_{\mathfrak{M}} = |\mathcal{L}|$ is the number of lines of $\mathfrak{M}$;
- $\boldsymbol{\tau} = \boldsymbol{\tau}_{\mathfrak{M}}$ is the number of triangles in $\mathfrak{M}$;
- $\boldsymbol{r}_{\mathfrak{M}}(x) = \boldsymbol{r}(x)$ for $x \in X$ is the number of lines of $\mathfrak{M}$ which pass through x; if $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ for all $x, y \in X$ we write simply $\boldsymbol{r} = \boldsymbol{r}_{\mathfrak{M}} = \boldsymbol{r}_{\mathfrak{M}}(x)$;
- $\boldsymbol{k}_{\mathfrak{M}}(l) = \boldsymbol{k}(l)$ for $l \in \mathcal{L}$ is the number of points of $\mathfrak{M}$ which are on l; if $\boldsymbol{k}(l) = \boldsymbol{k}(m)$ for all $l, m \in \mathcal{L}$ we write simply $\boldsymbol{k} = \boldsymbol{k}_{\mathfrak{M}} = \boldsymbol{k}_{\mathfrak{M}}(l)$.

If X is an arbitrary set, two disjoint families $\mathcal{L}_1$ and $\mathcal{L}_2$ of subsets of X *yield a partial net* if

> for every $x \in X$ there is exactly one $l_1 \in \mathcal{L}_1$ and there is exactly one $l_2 \in \mathcal{L}_2$ with $x \in l_1, l_2$;

note that then the structure $\langle X, \mathcal{L}_1 \cup \mathcal{L}_2\rangle$ is a partial linear space, and elements of $\mathcal{L}_i$ are pairwise disjoint, both for $i = 1$ and for $i = 2$.

LEMMA 1.1. *Let $\mathfrak{G} = \langle S, \mathcal{E}\rangle$ be an arbitrary graph. Then the parameters of $\mathfrak{G}^*$ are calculated as follows:*

(i) $\boldsymbol{v}_{\mathfrak{G}^*} = \boldsymbol{b}_{\mathfrak{G}}$.
(ii) $\boldsymbol{b}_{\mathfrak{G}^*} = \sum_{x \in S} \binom{\boldsymbol{r}_{\mathfrak{G}}(x)}{2}$.
(iii) *If $a = \{x, y\} \in S^*$ then $\boldsymbol{r}_{\mathfrak{G}^*}(a) = \boldsymbol{r}_{\mathfrak{G}}(x) + \boldsymbol{r}_{\mathfrak{G}}(y) - 2$.*
(iv) $\boldsymbol{\tau}_{\mathfrak{G}^*} = \boldsymbol{\tau}_{\mathfrak{G}} + \sum_{x \in S} \binom{\boldsymbol{r}_{\mathfrak{G}}(x)}{3}$.

Proof. (i) is evident.
(ii): Every edge $\{a, b\}$ of $\mathfrak{G}^*$ can be uniquely associated with the pair $(x, Z) \in S \times \wp_2(S)$ such that $x \in a \cap b$ and $\{x, t\} \in \mathcal{E}$ for $t \in Z$. Thus Z is a subset of the set of all points collinear with x in $\mathfrak{G}$ so, Z can be chosen in $\binom{\boldsymbol{r}_{\mathfrak{G}}(x)}{2}$ ways.
(iii): Let $p \in S^*$ be arbitrary. From definition, $a \sim p$ iff, either p goes through x, or p goes through y, and $p \neq a$. This justifies our formula.
(iv): Note that a triple of points of $\mathfrak{G}^*$ is a triangle in $\mathfrak{G}^*$ if, either, it has the form $\big\{\{x, y\}, \{y, z\}, \{z, x\}\big\}$, or it has the form $\big\{\{t, x\}, \{t, y\}, \{t, z\}\big\}$ for

some $x, y, z, t \in S$. Triangles of the first type correspond to triangles of $\mathfrak{G}$ – there are $\boldsymbol{\tau}_{\mathfrak{G}}$ triangles of this type. Triangles of the second type are formed (point after point) by edges of $\mathfrak{G}$ with one end fixed. □

To avoid some trivial cases, we assume the following condition

Cx : Every connected component of $\mathfrak{G}$ has at least 3 elements.

As a consequence of Cx, no edge of $\mathfrak{G}$ is its connected component. It is seen that Cx states, equivalently, that $\mathfrak{G}^*$ has no isolated (i.e. degree 0) point.

Immediately from definition we get parameters of the Desarguesian closure of a graph.

FACT 1.2. *Let $\mathfrak{G}$ be an arbitrary graph, set $\mathfrak{D} = \mathbf{D}(\mathfrak{G})$.*

(i) *The number $\boldsymbol{v}^{\infty}_{\mathfrak{G}}$ of improper points of $\mathfrak{D}$ is $\boldsymbol{b}_{\mathfrak{G}}$; consequently, $\boldsymbol{v}_{\mathfrak{D}} = \boldsymbol{v}_{\mathfrak{G}} + \boldsymbol{v}^{\infty}_{\mathfrak{G}} = \boldsymbol{v}_{\mathfrak{G}} + \boldsymbol{b}_{\mathfrak{G}}$.*

(ii) *The number $\boldsymbol{b}^{\infty}_{\mathfrak{G}}$ of improper lines of $\mathfrak{D}$ is the number $\boldsymbol{\tau}_{\mathfrak{G}}$ of triangles of $\mathfrak{G}$, and then $\boldsymbol{b}_{\mathfrak{D}} = \boldsymbol{b}_{\mathfrak{G}} + \boldsymbol{b}^{\infty}_{\mathfrak{G}} = \boldsymbol{b}_{\mathfrak{G}} + \boldsymbol{\tau}_{\mathfrak{G}}$.*

(iii) *If q is a proper point, then $\boldsymbol{r}_{\mathfrak{D}}(q) = \boldsymbol{r}_{\mathfrak{G}}(q)$; if $q = a^{\infty}$, where $a = \{x, y\}$ is an edge of $\mathfrak{G}$, then $\boldsymbol{r}_{\mathfrak{D}}(q) = \boldsymbol{r}^{\infty}_{\mathfrak{G}}(a) + 1$, where $\boldsymbol{r}^{\infty}_{\mathfrak{G}}(a) = \boldsymbol{r}_{\mathbf{H}(\mathfrak{G})}(q)$ is the number of points z of $\mathfrak{G}$ such that $\{x, y, z\}$ is a triangle in $\mathfrak{G}$.*

(iv) *The size of every line l of $\mathfrak{D}$ is $\boldsymbol{k}_{\mathfrak{D}}(l) = 3$.*

Combining 1.1 with 1.2 we obtain the following.

COROLLARY 1.3. *Let $\mathfrak{G} = \langle S, \mathcal{E} \rangle$ be an arbitrary graph, let $\mathfrak{D} = \mathbf{D}(\mathfrak{G}^*)$.*

(i) $\boldsymbol{v}_{\mathfrak{D}} = \boldsymbol{v}_{\mathfrak{G}^*} + \boldsymbol{b}_{\mathfrak{G}^*} = \boldsymbol{b}_{\mathfrak{G}} + \sum_{x \in S} \binom{\boldsymbol{r}_{\mathfrak{G}}(x)}{2}$.

(ii) $\boldsymbol{b}_{\mathfrak{D}} = \boldsymbol{b}_{\mathfrak{G}^*} + \boldsymbol{b}^{\infty}_{\mathfrak{G}^*} = \sum_{x \in S} \binom{\boldsymbol{r}_{\mathfrak{G}}(x)}{2} + \boldsymbol{\tau}_{\mathfrak{G}} + \sum_{x \in S} \binom{\boldsymbol{r}_{\mathfrak{G}}(x)}{3}$.

(iii) *If $A = \{a, b\} \in \mathcal{E}^*$, then $\boldsymbol{r}_{\mathfrak{D}}(A^{\infty}) = \boldsymbol{r}^{\infty}_{\mathfrak{G}^*}(A) + 1$ and $\boldsymbol{r}^{\infty}_{\mathfrak{G}^*}(A) = \boldsymbol{r}_{\mathfrak{G}}(a \cap b) - 2 + \sigma(a, b)$, where $\sigma(a, b) = 1$ if a, b can be completed to a triangle in $\mathfrak{G}$, and $\sigma(a, b) = 0$ otherwise.*

Proof. (i) and (ii) are evident. To prove (iii) it suffices to determine, with the same reasoning as in the proof of 1.1(iv), all the triangles of $\mathfrak{G}^*$ with one fixed side A. □

The construction of the line graph can be, evidently, iterated. For an arbitrary graph $\mathfrak{G}$ we define inductively: $\mathfrak{G}^{(0)} = \mathfrak{G}$, $\mathfrak{G}^{(m+1)} = \mathfrak{G}^{(m)*}$ for $m = 0, 1, \ldots$. In section A we shall show some properties of this iteration.

2. Complete graphs

Recall that the *complete graph on n vertices* is the structure $K_n = \langle X, \wp_2(X) \rangle$, where X is any set with $|X| = n$. In the sequel, for short we write $K_n^{(m)} = (K_n)^{(m)}$. For convenience, we take $X = \{0, \ldots, n-1\}$. The following is evident

FACT 2.1. *For every integer n with $n \geq 2$ it holds*

$$\boldsymbol{v}_{K_n} = n, \quad \boldsymbol{b}_{K_n} = \binom{n}{2}, \quad \boldsymbol{\tau}_{K_n} = \binom{n}{3}, \quad \boldsymbol{r}_{K_n} = n-1,$$

$$\sigma(a,b) = 1 \text{ for every pair } (a,b) \text{ of neighbor edges of } K_n.$$

As an immediate consequence of 2.1 and 1.3 we obtain

COROLLARY 2.2. *Let $\mathfrak{D} = \mathbf{D}({K_n}^*)$ and $\mathfrak{H} = \mathbf{H}({K_n}^*)$.*

(i) *$\mathfrak{D}$ has $(n-1)\binom{n}{2}$ points, $(n+1)\binom{n}{3}$ lines, its proper points have degree $2n-4$, and improper points have degree $n-1$.*

(ii) *$\mathfrak{H}$ has $n\binom{n-1}{2}$ points, $(n-2)\binom{n}{3}$ lines, and its points are of the degree $n-2$.*

Proof. We have in turn: $\boldsymbol{v}_{\mathfrak{D}} = \boldsymbol{b}_{K_n} + \boldsymbol{v}_{K_n} \cdot \binom{\boldsymbol{r}_{K_n}}{2} = \binom{n}{2} + n\binom{n-1}{2} = (n-1)\binom{n}{2}$. Then $\boldsymbol{b}_{\mathfrak{D}} = \boldsymbol{v}_{K_n} \cdot \binom{\boldsymbol{r}_{K_n}}{2} + \boldsymbol{\tau}_{K_n} + \boldsymbol{v}_{K_n} \cdot \binom{\boldsymbol{r}_{K_n}}{3} = n\binom{n-1}{2} + \binom{n}{3} + n\binom{n-1}{3}$.
A proper point $a = \{x,y\}$ of $\mathfrak{D}$ has degree $2 \cdot \boldsymbol{r}_{K_n}(x) - 2 = 2(n-1) - 2$.
An improper point has degree $\boldsymbol{r}_{K_n} - 2 + \sigma + 1 = (n-1) + 0$. □

Complete graphs and their Desarguesian closures are, in a sense, most important. We write $\mathbf{D}_n = \mathbf{D}(K_n)$ and we call $\mathbf{D}_n$ *the Desargues space (of dimension $n-1$)*. It is seen that $\mathbf{D}_n$ can be visualized as the perspective with center 0 of the simplex $\{1,\ldots,n\}$ onto $\big\{\{0,i\}^\infty : i = 1\ldots,n\big\}$. This observation yields

FACT 2.3. *Let $n \geq 4$ and let $\mathfrak{P}$ be an at least $(n-1)$-dimensional projective space. Every independent set X_0 with $|X_0| = n$ of points of $\mathfrak{P}$ can be completed to a closed configuration $\mathfrak{D}$ in $\mathfrak{P}$ such that $\mathfrak{D} \cong \mathbf{D}_n$.*

Proof. From assumption, $\mathfrak{P}$ can be considered as a projective completion of an affine space $\mathfrak{A}$, defined over a vector space $\mathbb{V}$ with $\dim(\mathbb{V}) = m \geq n$. Without loss of generality we can assume that $X_0 = \{e_0, e_1, \ldots, e_n\}$, where e_0 is the zero vector of $\mathbb{V}$, and thus $X_0 \setminus \{e_0\}$ is an independent set of vectors of $\mathbb{V}$. Let us consider a map $f' \colon X \longrightarrow X_0$ defined by $f'(i) = e_i$. Clearly, no two lines of $\mathfrak{A}$ determined by points in X_0 are mutually parallel. For $i,j \in X$, $i \neq j$ let $f''(\{i,j\}^\infty)$ be the direction of the line of $\mathfrak{A}$ which joins $f'(i)$ and $f'(j)$. The map f' embeds K_n into $\mathfrak{A}$, and $f = f' \cup f''$ is an embedding of $\mathbf{D}(K_n)$ onto a closed configuration in $\mathfrak{P}$. □

PROPOSITION 2.4.

(i) *Let $\mathfrak{G}$ be an arbitrary graph on n vertices. Then $\mathfrak{G}$ is a subgraph of K_n.*

(ii) *If $\mathfrak{G}_1$ is a subgraph of $\mathfrak{G}_2$, then $\mathfrak{D}_1 = \mathbf{D}(\mathfrak{G}_1)$ is a subspace of $\mathfrak{D}_2 = \mathbf{D}(\mathfrak{G}_2)$, and the horizon of $\mathfrak{G}_1$ is a subspace of the horizon of $\mathfrak{G}_2$.*

(iii) *Let $n \geq 4$. Then the $(n-1)$-dimensional Desargues space $\mathbf{D}_n$ can be embedded into the projective $(n-1)$-dimensional space over $GF(2)$, and the horizon of K_n, isomorphic to $\mathbf{D}_{n-1}$ can be embedded into the projective $(n-2)$-space over $GF(2)$.*

(iv) *Let $\mathfrak{G}$ be as in* (i). *Then the horizon of $\mathfrak{G}$ is a partial Steiner triple system embedable into the projective $(n-2)$-space over $GF(2)$.*

Proof. (i) and (ii) are evident.
(iii): By 2.3, $\mathbf{D}_n$ is isomorphic to a closed configuration in the $(n-1)$-dimensional projective space $\mathfrak{P}$ over $GF(2)$. Since lines of $\mathfrak{P}$ have the same size as lines of $\mathbf{D}_n$, we obtain a required embedding.

To close the proof of this statement it suffices to recall that the map

$$i \longmapsto \{0,i\}^{\infty} \longmapsto i \quad \text{for } i \in X \setminus \{0\}; \qquad 0 \longmapsto 0$$

determines an automorphism of $\mathbf{D}_n$, which maps the set of improper points of $\mathbf{D}(K_n)$ onto $\mathbf{D}(\mathfrak{G}')$, where $\mathfrak{G}'$ is the complete graph on the vertices $X \setminus \{0\}$.
(iv) is an immediate consequence of (i), (ii), and (iii). □

The observations of 2.4, of a pure theoretical importance (comp. e.g. problems on embedability examined in [1]), can not be directly used to determine the geometry of the Desarguesian closure of particular graphs and their horizons.

3. Desarguesian horizon

Note the following

PROPOSITION 3.1. *Let $\mathfrak{G} = \langle S, \mathcal{E} \rangle$ be a connected graph with constant point degree. Then the following conditions are equivalent.*

(i) *All points of $\mathbf{D}(\mathfrak{G})$ have the same degree.*
(ii) *$\mathfrak{G} \cong K_n$ for some n, and thus $\mathbf{D}(\mathfrak{G}) \cong \mathbf{D}_n$ is a Desargues space.*

Proof. The implication (ii) $\Longrightarrow$ (i) is evident. Assume that (i) holds. Let $a = \{x,y\} \in \mathcal{E}$ and $q = a^{\infty}$. In accordance with 1.2, $\boldsymbol{r}_{\mathbf{D}(\mathfrak{G})}(q) = \boldsymbol{r}^{\infty}_{\mathfrak{G}}(a) + 1 = \boldsymbol{r}_{\mathfrak{G}}(x)$, which means that $\{y,t\} \in \mathcal{E}$ whenever $\{x,t\} \in \mathcal{E}$ and $t \neq y$. Thus $\mathfrak{G}$ is determined by a transitive relation, which proves that $\mathcal{E} = \wp_2(S)$. □

In view of 3.1, in general, the geometry of the Desarguesian closure of a graph and the geometry of its horizon may differ. Therefore, in the sequel, we shall be mainly concerned with Desarguesian horizons of graphs.

To get an idea, how the horizon of a graph looks like let us note the following observation. Let $\mathfrak{G} = \langle S, \mathcal{E} \rangle$ be a graph and $\mathcal{K}$ be the family of the maximal cliques in $\mathfrak{G}$. Every $K \in \mathcal{K}$ determines in $\mathfrak{G}$ a complete subgraph $\cong K_s$ for $s = |K|$, and, by 2.4, K determines a subspace $K^{\infty} = \{a^{\infty} : a \in \wp_2(K)\}$ of the horizon $\mathbf{H}(\mathfrak{G})$, which is isomorphic to $\mathbf{D}_{s-1}$. Let us write $\mathcal{K}^\infty = \{K^\infty : K \in \mathcal{K}\}$. Since every edge of $\mathfrak{S}$ is contained in at least one clique we get that $\mathcal{K}^\infty$ is a covering of $\mathbf{H}(\mathfrak{S})$ by a family of Desargues spaces. Therefore, to characterize the geometry of the horizon of $\mathfrak{S}$ it suffices to determine the structure of cliques in $\mathfrak{S}$.

To this aim the following observation is useful.

FACT 3.2. *Let $\mathfrak{S}$ be an arbitrary graph and $(\mathcal{K}_i : i = 1, \ldots, s)$ be a partition of the family $\mathcal{K}$ of all the maximal cliques of $\mathfrak{S}$ such that*

(i) if $K_1, K_2 \in \mathcal{K}_i$ for some i, then $|K_1 \cap K_2| \le 1$;
(ii) if $K_r \in \mathcal{K}_{i_r}$ for $r = 1, 2$ and $i_1 \ne i_2$, then $K_1 \cap K_2$ is at most an edge of $\mathfrak{S}$.

Then the family $(\mathcal{K}_i{}^\infty : i = 1, \ldots, s)$ has the following properties:

(iii) *Let $q \in M_1, M_2 \in \mathcal{K}_i{}^\infty$ for some i. Then $M_1 = M_2$.*
(iv) *Let $M_r \in \mathcal{K}_{i_r}{}^\infty$ for $r = 1, 2$ and $i_1 \ne i_2$. Then $|M_1 \cap M_2| \le 1$.*
(v) *Let $q \in M_r \in \mathcal{K}_{i_r}{}^\infty$ for $r = 1, 2$ and $i_1 \ne i_2$, and $q_i \in M_i \setminus M_{3-i}$. Then q_1 and q_2 are not collinear in $\mathbf{H}(\mathfrak{S})$.*

Proof. The statement (iii) is an immediate consequence of (i), and (iv) follows by (ii).

Let $M_r = K_r{}^\infty$ and $K_r \in \mathcal{K}_{i_r}$, and let $q = a^\infty$ with $a = \{x, y\} \in \wp_2(K_r)$; by (ii), $a = K_1 \cap K_2$. Consider $q_r = a_r{}^\infty$, where $a_r = \{x_r, y_r\} \in \wp_2(K_r)$; then $q_r \in M_r$. Suppose that q_1 and q_2 are collinear in $\mathbf{H}(\mathfrak{S})$, then a_1, a_2 are sides of a triangle in $\mathfrak{S}$ so, without loss of generality, we can assume that $x_1 = x_2$ and $c = \{y_1, y_2\}$ is an edge of $\mathfrak{S}$. Since $x_1 \in K_1 \cap K_2$ we infer that $x_1 \in a$, say: $x_1 = x$. Since q, q_1, q_2 are pairwise distinct, the edges a, a_1, a_2 are pairwise distinct as well. The set $Z = \{x, y, y_1, y_2\}$ is a clique in $\mathfrak{S}$ so, it can be extended to a maximal clique K_0. Note that $|K_0 \cap K_r| \ge 3$ so, by (i),(ii), $K_0 = K_1 = K_2$, which is impossible. This proves (v). □

The covering $\mathcal{K}^\infty$ is determined entirely by the geometry of $\mathbf{H}(\mathfrak{S})$. Namely, we have the following

FACT 3.3. *Let $\mathfrak{S} = \langle S, \mathcal{E} \rangle$ be an arbitrary graph, we write $\mathfrak{J} = \mathbf{J}(\mathbf{H}(\mathfrak{S}))$ for the graph of collinearity of points of $\mathbf{H}(\mathfrak{S})$.*

(i) *Every line of $\mathbf{H}(\mathfrak{S})$ is a maximal clique in $\mathfrak{J}$.*

(ii) *Let H be a maximal clique in $\mathfrak{J}$. Then there is a clique K in $\mathfrak{S}$ such that K^∞ is the subspace of $\mathbf{H}(\mathfrak{S})$ spanned by H. If no three points of H are collinear in $\mathbf{H}(\mathfrak{S})$, then K is maximal.*

(iii) *Let K be a maximal clique in $\mathfrak{S}$. Then there is a maximal clique H in $\mathfrak{J}$ such that K^∞ is the subspace of $\mathbf{H}(\mathfrak{S})$ spanned by H.*

Proof. Note that, directly from definition, if $q_i = a_i{}^\infty$ for $a_i \in \mathcal{E}$ $(i = 1, 2)$, $q_1 \ne q_2$, and q_1, q_2 are collinear in $\mathbf{H}(\mathfrak{S})$, then $a_1 \cap a_2 \ne \emptyset$ and $a_1 \cup a_2 \in \mathcal{T}_{\mathfrak{S}}$.

(i): Let L be a line in $\mathbf{H}(\mathfrak{S})$, then $L = T^\infty$, where T is a triangle in $\mathfrak{S}$. Clearly, L is a clique in $\mathfrak{J}$. Let $q = a^\infty$ with $a \in \mathcal{E}$ and suppose that q is collinear with all points on L. Then $a \cap c \neq \emptyset$ for every side $c \in \wp_2(T)$, which yields $a \subset T$ so, $q \in L$.

(ii): If H is a line of $\mathbf{H}(\mathfrak{S})$, the claim is evident (cf. (i)). Assume that no three points in H are collinear and let $q_1, q_2 \in H$. Then there are vertices y, x_1, x_2 of $\mathfrak{S}$ such that $\{y, x_1, x_2\} \in \mathcal{T}_\mathfrak{S}$ and $q_i = a_i{}^\infty$ with $a_i = \{y, x_i\}$ for $i = 1, 2$. We set $K = \bigcup\{a : y \in a \in \mathcal{E},\ a^\infty \in H\}$. Evidently, K is a clique in $\mathfrak{S}$. Suppose that $K_0 = K \cup \{z\}$ is a clique and $z \notin K$. Then $\{y, z\}^\infty$ is collinear with all the points in H, which contradicts the maximality of H. Clearly, K^∞ is a (desarguesian) subspace of the horizon of $\mathfrak{S}$, and $H \subseteq K^\infty$; it is seen that it is the smallest subspace of $\mathbf{H}(\mathfrak{S})$ that contains H.

(iii): If K is a triangle, the claim follows directly from (i); therefore we assume $|K| > 3$. From 2.4(iii) we know, that K^∞ is a Desargues space, so it can be presented in the form $\mathbf{D}(\mathfrak{J}')$ for some complete subgraph $\mathfrak{J}'$ of $\mathfrak{J}$. The point is to show the vertices of $\mathfrak{J}'$ explicitly.
Let $y \in K$, we set $H = \{a^\infty : y \in a \in \wp_2(K)\}$. Clearly, H is a clique in $\mathfrak{J}$ and, as in the proof of 2.4, we prove that the elements of K^∞ can be identified with the points of the set $\mathbf{D}(H)$ which, in turn, is the subspace of $\mathbf{H}(\mathfrak{S})$ spanned by H. It remains to show that H is maximal. Let (as in (i)) $q = a^\infty$ be collinear with all the points in H. There are at least three edges of $\mathfrak{S}$ passing through y and contained in K; since a crosses them all, $y \in a$. Thus $a = \{y, z\}$ for some z. Now, the requirement that q is collinear with every $q' \in H$ yields that $\{z, z'\} \in \mathcal{E}$ for every $z' \in K \setminus \{z\}$ and, since K is a maximal clique, we get $z \in K$ so, finally, $q \in H$. □

Let $\mathfrak{S} = \langle X, \mathcal{E} \rangle$ be an arbitrary graph and $t \in X$. We write $\mathcal{E}_{(t)} = \{a \in \mathcal{E} : t \in a\}$. As we already noted in the proof of 1.1, the family $\mathcal{T}_{\mathfrak{S}^*}$ of triangles of $\mathfrak{S}^*$ can be divided into two sets

$$\mathcal{F}_{\mathfrak{S}^*} := \{\wp_2(Z) : Z \in \mathcal{T}_\mathfrak{S}\},$$
$$\mathcal{V}_{\mathfrak{S}^*} := \bigcup_{t \in X} \wp_3(\mathcal{E}_{(t)}),$$

and thus the family of lines of $\mathbf{H}(\mathfrak{S}^*)$ is divided into the following two:

$$\mathcal{F}^\infty_{\mathfrak{S}^*} := \{T^\infty : T \in \mathcal{F}_{\mathfrak{S}^*}\}, \quad \text{and} \quad \mathcal{V}^\infty_{\mathfrak{S}^*} := \{T^\infty : T \in \mathcal{V}_{\mathfrak{S}^*}\}.$$

Moreover, we have evident

FACT 3.4. *Let $\mathfrak{S} = \langle S, \mathcal{E} \rangle$ be an arbitrary graph and t be one of its vertices with $\mathbf{r}_\mathfrak{S}(t) \geq 3$. Then $\mathcal{E}_{(t)}$ is a maximal clique in $\mathfrak{S}^*$. Moreover, if $t_1 \neq t_2$, then $\mathcal{E}_{(t_1)} \neq \mathcal{E}_{(t_2)}$ (note:* Cx *is used here!).*

If K is a maximal clique in $\mathfrak{S}^*$, then either $K \in \mathcal{F}_{\mathfrak{S}^*}$ or $K = \mathcal{E}_{(t)}$ for some vertex t of $\mathfrak{S}$.

Let us write

$$\mathcal{H}_{\mathfrak{S}^*} := \{\mathcal{E}_{(t)} : t \in S,\ \boldsymbol{r}_{\mathfrak{S}}(t) \geq 2\}.$$

If $K_1, K_2 \in \mathcal{F}_{\mathfrak{S}^*}$ or $K_1, K_2 \in \mathcal{H}_{\mathfrak{S}^*}$, and $K_1 \neq K_2$, then $|K_1 \cap K_2| \leq 1$. If $K_1 \in \mathcal{F}_{\mathfrak{S}^*}$ and $K_2 \in \mathcal{H}_{\mathfrak{S}^*}$, then $K_1 \cap K_2$ is at most an edge of $\mathfrak{S}^*$. Therefore, $\mathfrak{S}^*$ satisfies conditions (i) *and* (ii) *of 3.2. Consequently, we have the following.*

Let $T \in \mathcal{V}_{\mathfrak{S}^*}$. Then T is contained in a (unique) clique K in $\mathfrak{S}^*$, and $K \in \mathcal{H}_{\mathfrak{S}^*}$. Therefore, lines contained in ${\mathcal{E}_{(t)}}^{\infty}$ are members of $\mathcal{V}^{\infty}_{\mathfrak{S}^*}$. Put $\mathcal{H}^{\infty}_{\mathfrak{S}^*} = \{M^{\infty} : M \in \mathcal{H}_{\mathfrak{S}^*}\}$. From previous considerations, elements of $\mathcal{H}^{\infty}_{\mathfrak{S}^*}$ are (Desargues) subspaces of $\mathbf{H}(\mathfrak{S}^*)$. Distinct lines in $\mathcal{F}^{\infty}_{\mathfrak{S}^*}$ are pairwise disjoint, and distinct subspaces M^{∞} with $M \in \mathcal{H}_{\mathfrak{S}^*}$ are pairwise disjoint as well. If $L \in \mathcal{F}^{\infty}_{\mathfrak{S}^*}$ and $M \in \mathcal{H}^{\infty}_{\mathfrak{S}^*}$ then $L \cap M$ is at most one element set. If, moreover, $q_1 \in L \setminus M$ and $q_2 \in M \setminus L$ then q_1, q_2 are not collinear in $\mathbf{H}(\mathfrak{S}^*)$.

As a consequence of 3.4 we note the following observation:

FACT 3.5. *Let $\mathfrak{S} = \langle S, \mathcal{E} \rangle$ be an arbitrary graph and t be one of its vertices. Let $\boldsymbol{r} = \boldsymbol{r}_{\mathfrak{S}}(t) \geq 4$. Consider a pair $C = \{a, b\}$ of edges of $\mathfrak{S}$ with $t = a \cap b$ and set $q = C^{\infty}$; then q is a point of $\mathfrak{H} = \mathbf{H}(\mathfrak{S}^*)$. If $\sigma(a, b) = 1$ then there is (exactly one) line of $\mathfrak{D}$ which passes through q and belongs to $\mathcal{F}^{\infty}_{\mathfrak{S}^*}$. Let $\mathcal{V}^{\infty}_q$ be the (remaining) lines through q; they are in $\mathcal{V}^{\infty}_{\mathfrak{S}^*}$. Then points of $\mathcal{V}^{\infty}_q$ span a $(r-2)$-dimensional Desargues subspace of $\mathfrak{H}$ which coincides with ${\mathcal{E}_{(t)}}^{\infty}$.*

Proof. The only non trivial part of the statement consists in the fact that points of $\mathcal{V}^{\infty}_q$ span ${\mathcal{E}_{(t)}}^{\infty}$.

Let y, z be vertices of $\mathfrak{S}$ with $a = \{t, y\}$, $b = \{t, z\}$, and let $\{x_1, \ldots, x_{r-2}\}$ be the set of the remaining vertices joinable with t; we put $d_i = \{t, x_i\}$ and then $\mathcal{D} := \{d_1, \ldots, d_{r-2}\} \subseteq \mathcal{E}_{(t)}$; in particular, every two edges in $\mathcal{D}$ are neighbor. Then every line L_i of $\mathfrak{H}$ through q which belongs to $\mathcal{V}^{\infty}_{\mathfrak{S}^*}$ is determined by a triangle (a, b, d_i) of $\mathfrak{S}^*$ and consists of the points q, $u_i = A_i^{\infty}$, and $v_i = B_i^{\infty}$, where $A_i = \{a, d_i\}$, $B_i = \{b, d_i\}$. Every two points u_i, u_j are on a line of $\mathfrak{H}$ determined by the triangle (a, d_i, d_j), the third point of this line is $\omega_{i,j} = \{d_i, d_j\}^{\infty}$, points $v_i, v_j, \omega_{i,j}$ are on the line determined by the triangle (b, d_i, d_j), and for every triple (i, j, k) the points $\omega_{i,j}$, $\omega_{j,k}$, and $\omega_{i,k}$ are on a line of $\mathfrak{H}$ determined by the triangle (d_i, d_j, d_k). This proves that the set $\{q, \omega_{i,j}, u_i, v_j : i, j = 1, \ldots, r-2,\ i \neq j\}$ is a subspace of $\mathfrak{H}$, spanned by $\mathcal{D}^{\infty}$, and equal to ${\mathcal{E}_{(t)}}^{\infty}$. □

Directly from 3.5 we obtain

COROLLARY 3.6. *Let $\mathfrak{S}$ be a graph with the degree of every point at least 4, let $\mathfrak{H}$ be the horizon of $\mathfrak{S}^*$ and let g be a line of $\mathfrak{H}$. We write $\mathcal{V}^{\infty} = \mathcal{V}^{\infty}_{\mathfrak{S}^*}$,*

$\mathcal{H}^\infty = \mathcal{H}^\infty_{\mathfrak{S}^*}$, *and* $\mathcal{F}^\infty = \mathcal{F}^\infty_{\mathfrak{S}^*}$. *The following conditions are equivalent:*

(i) $g \in \mathcal{V}^\infty$;

(ii) for every (equivalently: for some) point $q \in g$ *there are lines* g_1, g_2 *of* $\mathfrak{H}$ *such that* $q \in g_1$ *and* (g, g_1, g_2) *are sides of a triangle in* $\mathfrak{H}$.

If the condition (ii) *is not satisfied, then* $g \in \mathcal{F}^\infty$.

Consequently, the families $\mathcal{F}^\infty$ *and* $\mathcal{V}^\infty$ *are intrinsically distinguishable. Elements of* $\mathcal{H}^\infty$ *are at least 2-dimensional Desargues spaces. The families* $\mathcal{F}^\infty$ *and* $\mathcal{H}^\infty$ *yield in* $\mathfrak{H}$ *a partial net.*

In particular, we obtain

COROLLARY 3.7. *Let* $\mathfrak{S} = K_n$ *and* $\mathfrak{H}$ *be the horizon of* $\mathfrak{S}^*$.

(i) *Let* $n = 4$. *The families* $\mathcal{F}^\infty$ *and* $\mathcal{V}^\infty$ *of lines of* $\mathfrak{H}$ *yield a partial net.*

(ii) *Let* $n > 4$. *The horizon* $\mathfrak{H}$ *can be considered as a union of pairwise disjoint* $(n-3)$*-dimensional Desargues spaces formed by the lines from* $\mathcal{V}^\infty$. *The family* $\mathcal{H}^\infty$ *of these configurations together with the family* $\mathcal{F}^\infty$ *yield a partial net.*

4. Automorphisms

With every automorphism f of a graph $\mathfrak{S} = \langle S, \mathcal{E}\rangle$ we associate the map $f^*\colon \mathcal{E} \longrightarrow \mathcal{E}$ defined by the condition $f^*(\{x,y\}) = \{f(x), f(y)\}$ and the map f^∞ defined on the set S^∞ by the condition $f^\infty(a^\infty) = f^*(a)^\infty$ for $a \in \mathcal{E}$.

PROPOSITION 4.1. *The two maps*

$$\mathrm{Aut}(\mathfrak{S}) \ni f \longmapsto f^* \in \mathrm{Aut}(\mathfrak{S}^*) \text{ and } \mathrm{Aut}(\mathfrak{S}) \ni f \longmapsto f^\infty \in \mathrm{Aut}(\mathbf{H}(\mathfrak{S}))$$

are group monomorphisms.

Proof. It is evident that $f^* \in \mathrm{Aut}(\mathfrak{S}^*)$ and $f^\infty \in \mathrm{Aut}(\mathbf{H}(\mathfrak{S}))$ for every $f \in \mathrm{Aut}(\mathfrak{S})$. Suppose that $f^*(a) = a$ for every $a \in \mathcal{E}$; take $a = \{x, y\}$. Let S_0 be the connected component of x, we can assume that $r_{\mathfrak{S}}(x) > 1$. From assumption, either

a) $f(x) = x$ and $f(y) = y$, or
b) $f(x) = y$ and $f(y) = x$.

In the case (a) we obtain that $f(z) = z$ for every z joinable with x and therefore, f is the identity on S_0. In the case (b) we consider any z such that $a \neq b = \{x, z\} \in \mathcal{E}$. Since $f^*(b) = b$ we get $f(z) = x$ and $z = y$. Similarly, x is the only point joinable with y. Finally, $S_0 = a$, which contradicts Cx. Now, let us assume that $f^\infty = \mathrm{id}$, i.e. $f^*(a)^\infty = f^\infty(a^\infty) = a^\infty$ for every edge a of $\mathfrak{S}$. This yields $f^* = \mathrm{id}$ and, by the above, $f = \mathrm{id}$. □

Proposition 4.1 enables us to associate with every $f \in S_n = \mathrm{Aut}(K_n)$ an automorphism $f^{(n)}$ of $K_n^{(m)}$ defined inductively: $f^{(0)} = f$, $f^{(m+1)} = f^{(m)*}$. Recall that $\mathrm{Aut}(K_n) \cong S_n$ and $\mathrm{Aut}(\mathbf{D}(K_n)) \cong S_{n+1}$ (comp. [7]).

We are going to prove that, with one exception (cf. 4.6) every automorphism of $\mathrm{Aut}(\mathbf{H}(K_n^{(m)}))$ is determined by a permutation in S_n. Note, first, an evident observation. Let $\mathfrak{G}$ be an arbitrary graph. If $f \in \mathrm{Aut}(\mathfrak{G})$ then f^* leaves invariant the families $\mathcal{F}_{\mathfrak{G}^*}$ and $\mathcal{V}_{\mathfrak{G}^*}$ of triangles of $\mathfrak{G}^*$. Consequently, $f^{*\infty}$ leaves invariant the families of lines $\mathcal{F}^{\infty}_{\mathfrak{G}^*}$ and $\mathcal{V}^{\infty}_{\mathfrak{G}^*}$ of the horizon $\mathbf{H}(\mathfrak{G}^*)$.

To determine the automorphism group of the horizon of $K_n{}^*$ for $n > 4$ we shall prove a more general fact.

PROPOSITION 4.2.

(i) *Let* $\mathfrak{G}$ *be a graph such that the degree of each of its vertices is at least* 3. *Then the graph* $\mathfrak{G}^*$ *satisfies the following:*

(a) the degree of each of vertices of this graph is at least 4.

(b) every edge of the graph is a side of a triangle of it.

(ii) *Let* $\mathfrak{G}$ *satisfy* (a) *and* (b), *and let* F *be an automorphism of the horizon* $\mathfrak{H}$ *of* $\mathfrak{G}^*$. *Then* $F = f^{*\infty}$ *for some* $f \in \mathrm{Aut}(\mathfrak{G})$. *Consequently, from 4.1 we get that* $\mathrm{Aut}(\mathfrak{H}) \cong \mathrm{Aut}(\mathfrak{G})$.

Proof. (i): From 1.1(iii) and assumptions, $\mathfrak{G}^*$ satisfies (a). Now, let A be an edge of $\mathfrak{G}^*$, then $A \subset \mathcal{E}_{(t)}$ for some point t of $\mathfrak{G}$. Since $r_{\mathfrak{G}}(t) \geq 3$, from 3.4 we can complete A to a triangle in $\mathfrak{G}^*$, which proves that this graph satisfies (b).

(ii): In view of 3.6, every automorphism F of $\mathfrak{H}$ must preserve the families $\mathcal{F}^{\infty} = \mathcal{F}^{\infty}_{\mathfrak{G}^*}$ and $\mathcal{V}^{\infty} = \mathcal{V}^{\infty}_{\mathfrak{G}^*}$. Moreover, F must preserve the partition of the set of points of $\mathfrak{H}$ into suitable subconfigurations ${\mathcal{E}_{(t)}}^{\infty} =: H_t$, t – a point of $\mathfrak{G}$, as defined in 3.4.

Thus F determines a permutation f of the vertices of $\mathfrak{G}$ such that F maps H_t onto $H_{f(t)}$. To close the proof we must show that $f \in \mathrm{Aut}(\mathfrak{G})$ and $F = f^{*\infty}$.

Let $a_0 = \{t_1, t_2\}$ be an edge of $\mathfrak{G}$; we complete it with t_0 to a triangle in $\mathfrak{G}$ and set $a_1 = \{t_2, t_0\}$, $a_2 = \{t_1, t_0\}$, $A_1 = \{a_0, a_2\}$, $A_2 = \{a_0, a_1\}$, $A_0 = \{a_1, a_2\}$, and $q_i = A_i{}^{\infty}$ for $i = 0, 1, 2$. Then points q_0, q_1, q_2 are one one line in $\mathcal{F}^{\infty}$. Under our assumptions, $q_i \in H_{t_i}$ is transformed onto $q'_i := F(q_i) \in H_{f(t_i)}$ for $i = 1, 2$, moreover the points q'_1 and q'_2 must lie on a line in $\mathcal{F}^{\infty}$. Therefore, there exists in $\mathfrak{G}$ a triangle t'_0, t'_1, t'_2 such that $q'_i = A'^{\infty}_i$, $A'_i = \{a'_j, a'_s\}$, $a'_i = \{t'_j, t'_s\}$ ($\{i, j, s\} = \{0, 1, 2\}$). Then $q'_i \in H_{t'_i}$ so, $t'_i = f(t_i)$ and thus $f(t_1)$ and $f(t_2)$ must be joinable in $\mathfrak{G}$. This proves that $f \in \mathrm{Aut}(\mathfrak{G})$.

Let $G = F^{-1} \circ f^{*\infty}$, then $G \in \mathrm{Aut}(\mathfrak{H})$ and G preserves every subconfiguration H_t. Let g be a line in $\mathcal{F}^\infty$, then g is determined by a triangle (x, y, z) of $\mathfrak{S}$ under the rule

$$g = \Big\{ \big\{\{x,y\},\{x,z\}\big\}^\infty, \big\{\{y,x\},\{y,z\}\big\}^\infty, \big\{\{z,x\},\{z,y\}\big\}^\infty \Big\}.$$

Note that g crosses H_x, H_y, and H_z. Then the image $G(g)$ must be in $\mathcal{F}^\infty$ and must cross the same subconfigurations H_x, H_y, and H_z. It is seen that this yields $G(g) = g$ i.e. G leaves every line in $\mathcal{F}^\infty$ invariant. Since through every point of $\mathfrak{H}$ there pass exactly one line in $\mathcal{F}^\infty$ and exactly one subconfiguration of the type H_t, we get $G = \mathrm{id}$, and thus $F = f^{*\infty}$, as required.
In view of (i) above, $\mathfrak{S}^*$ satisfies Cx as well. Applying 4.1 we get that the map $\mathrm{Aut}(\mathfrak{G}) \ni f \mapsto f^{*\infty} \in \mathrm{Aut}(\mathfrak{S})$ is an isomorphism. □

COROLLARY 4.3. *Let $\mathfrak{S}$ be a graph satisfying* (a) *and* (b) *of 4.2, and $F \in \mathrm{Aut}(\mathfrak{S}^*)$. Then $F = f^*$ for some $f \in \mathrm{Aut}(\mathfrak{S})$. Consequently,* $\mathrm{Aut}(\mathfrak{S}^*) \cong \mathrm{Aut}(\mathfrak{S})$.

Proof. We set $F' = F^\infty$. By 4.2(ii), there is $f \in \mathrm{Aut}(\mathfrak{S})$ such that $F^\infty = F' = f^{*\infty}$. From 4.1 we infer that $F = f^*$, which is our claim. □

Directly from 2.1 and 4.2(i) we obtain

COROLLARY 4.4. *Let, either $n > 4$ and $m \geq 0$, or $n = 4$ and $m > 0$. Then $K_n^{(m)}$ satisfies the conditions* (a) *and* (b) *of 4.2.*

One particular case is left: $\mathbf{D}(K_4{}^*)$. Its horizon has a slightly different geometry.

For arbitrary integer n we consider $X = \{1, \ldots, n\}$; let $S = X \times X$. Then we define the cartesian net on S:

$$l'_i = X \times \{i\}, \qquad l''_i = \{i\} \times X \qquad \text{for } i = 1, \ldots, n, \tag{1}$$

$$\mathcal{L}' = \{l'_i \colon i = 1, \ldots, n\}, \mathcal{L}'' = \{l''_i \colon i = 1, \ldots, n\}, \mathcal{L} = \mathcal{L}' \cup \mathcal{L}''. \tag{2}$$

Finally, from the incidence structure $\langle S, \mathcal{L} \rangle$ we remove the diagonal $\Delta = \{(i,j) \in S \colon i = j\}$, and the obtained structure $\langle S \setminus \Delta, \mathcal{L} \rangle$ we denote by $\mathbf{N}_n^\circ$.

PROPOSITION 4.5. *Let n be an integer with $n \geq 3$.*

(i) *The map μ defined by $\mu(i,j) = (j,i)$ is an involutory automorphism of $\mathbf{N}_n^\circ$, which interchanges families $\mathcal{L}'$ and $\mathcal{L}''$.*

(ii) *For every permutation $\alpha \in S_n$ there is exactly one $F_\alpha \in \mathrm{Aut}(\mathbf{N}_n^\circ)$ such that $F_\alpha(l''_i) = l''_{\alpha(i)}$ for every $i = 1, \ldots, n$. Moreover, $F_\alpha \circ \mu = \mu \circ F_\alpha$.*

(iii) *If $f \in \mathrm{Aut}(\mathbf{N}_n^\circ)$, then there is $\alpha \in S_n$ such that $f = F_\alpha$ or $f = \mu \circ F_\alpha$. Consequently,* $\mathrm{Aut}(\mathbf{N}_n^\circ) \cong C_2 \oplus S_n$.

Proof. (i) is evident, since μ is a reflection in the diagonal Δ of the structure defined by (1), (2).

(ii) Let $\alpha \in S_n$. We set

$$F_\alpha(i,j) = (\alpha(i), \alpha(j)). \tag{3}$$

It is seen that F_α is an automorphism of the cartesian net defined by (1), (2), and F_α preserves Δ so, $F_\alpha \in \mathrm{Aut}(\mathbf{N}_n^\circ)$. Evidently, $F_\alpha(l_i'') = l_{\alpha(i)}''$, and F_α and μ commute.

To close the proof, we consider $f \in \mathrm{Aut}(\mathbf{N}_n^\circ)$ such that $f(l_i'') = l_i''$ for every $i = 1, \ldots, n$. Note that for every i there is exactly one j such that l_i'' and l_j' have no point in common, we simply take $j = i$. Therefore, f leaves invariant every line l_j', and thus $f = \mathrm{id}$.

(iii) Let $f \in \mathrm{Aut}(\mathbf{N}_n^\circ)$, and let $q = (1,n) \in S \setminus \Delta$. If f maps l_n' (passing through q) onto a line in $\mathcal{L}'$, we set $f' = f$; if f maps l_n' onto a line in $\mathcal{L}''$, then we set $f' = \mu \circ f$. Let $p = (i_0, j_0) = f'(q) \in S \setminus \Delta$. Therefore, $i_0 \neq j_0$ and thus there exists $\alpha \in S_n$ with $\alpha(1) = i_0$, $\alpha(n) = j_0$. From (3) we obtain $F_\alpha(q) = p$; let $G = (F_\alpha)^{-1} \circ f'$. Then $G \in \mathrm{Aut}(\mathbf{N}_n^\circ)$, $G(q) = q$, and $G(l_n') \in \mathcal{L}'$ so, $G(l_n') = l_n'$ and $G(l_1'') = l_1''$. The lines crossed by l_n' are l_j'' with $j \neq 1$, and thus G must leave this family invariant; similarly we note that G leaves invariant the family l_j' with $j \neq n$ and, finally, G leaves invariant families $\mathcal{L}'$ and $\mathcal{L}''$. Thus G determines a permutation $\beta \in S_n$ such that $G(l_i'') = l_{\beta(i)}''$ (note: $\beta(1) = 1$ and $\beta(n) = n$), From (ii) we obtain $F_\beta = G$, which closes the proof. □

Note, as an interesting fact, that $\mathbf{N}_3^\circ$ is a hexagon C_6 – it is trivial that $\mathrm{Aut}(\mathbf{N}_3^\circ) = C_2 \oplus S_3$ (see any standard textbook, e.g. [3]). More important observation is (cf. 3.7(i)) that the horizon of $K_4{}^*$ is $\mathbf{N}_4^\circ$.

To justify this observation more directly we give two tables. The first of them shows the incidence matrix of K_4; points are $1, 2, 3, 4$, and lines (edges) are a, b, c, d, e, f. Then we present how the points of $\mathbf{H}(K_4{}^*)$ are grouped into lines (T_i is obtained from the triangle without the point i; L_i is obtained from the edges passing through i, for $i = 1, 2, 3, 4$):

	a	b	c	d	e	f
1	•	•		•		
2	•		•		•	
3		•	•			•
4				•	•	•

	T_1 =	T_2 =	T_3 =	T_4 =
$L_1 =$	$\{$	$\{a,b\}^\infty$,	$\{b,d\}^\infty$,	$\{a,d\}^\infty\}$
$L_2 =$	$\{\{e,f\}^\infty$,		$\{f,d\}^\infty$,	$\{d,e\}^\infty\}$
$L_3 =$	$\{\{e,c\}^\infty$,	$\{c,a\}^\infty$,		$\{a,e\}^\infty\}$
$L_4 =$	$\{\{f,c\}^\infty$,	$\{c,b\}^\infty$,	$\{b,f\}^\infty$,	$\}$

Now it is seen that $\mathbf{H}(K_4) \cong \mathbf{N}_4^\circ$.

As an immediate consequence of 4.5 we get

COROLLARY 4.6. *Let $\mathfrak{H}$ be the horizon of $K_4{}^*$. Then* $\mathrm{Aut}(\mathfrak{H}) \cong C_2 \oplus S_4$.

COROLLARY 4.7. *Let $n > 4$ and $m \geq 0$, and let $\mathfrak{H}$ be the horizon of $\mathbf{D}(K_n^{(m)})$. Then $\mathrm{Aut}(K_n^{(m)}) \cong S_n$ and, consequently, $\mathrm{Aut}(\mathfrak{H}) \cong S_n$. If $m > 0$, then $\mathrm{Aut}(\mathbf{H}(K_4^{(m)})) \cong \mathrm{Aut}(K_4^{(m)}) \cong \mathrm{Aut}(K_4{}^*)$, where the latter was described in 4.6.*

Proof. By 4.2 and 4.4, $\mathrm{Aut}(\mathfrak{H}) \cong \mathrm{Aut}(K_n^{(m-1)})$, which, by 4.3 closes the proof. □

A. Iteration of the construction of line graphs

In this appendix we shall apply the results of previous sections to compute explicit formulas for basic parameters of the Desarguesian closures and of Desarguesian horizons of iterated line graphs. Let us begin with the following general observation, which follows immediately from 1.1.

FACT A.1. *Let $\mathfrak{G}$ be a graph with constant point degree. We denote $\boldsymbol{v}^{(m)} = \boldsymbol{v}_{\mathfrak{G}^{(m)}}$, $\boldsymbol{b}^{(m)} = \boldsymbol{b}_{\mathfrak{G}^{(m)}}$, $\boldsymbol{r}^{(m)} = \boldsymbol{r}_{\mathfrak{G}^{(m)}}$, and $\boldsymbol{\tau}^{(m)} = \boldsymbol{\tau}_{\mathfrak{G}^{(m)}}$. Then the following recursive formulas are satisfied:*

$$\begin{aligned} \boldsymbol{r}^{(m+1)} &= 2 \cdot \boldsymbol{r}^{(m)} - 2, \\ \boldsymbol{v}^{(m+1)} &= \boldsymbol{b}^{(m)}, \\ \boldsymbol{b}^{(m+1)} &= \boldsymbol{v}^{(m)} \cdot \tbinom{\boldsymbol{r}^{(m)}}{2}, \\ \boldsymbol{\tau}^{(m+1)} &= \boldsymbol{\tau}^{(m)} + \boldsymbol{v}^{(m)} \cdot \tbinom{\boldsymbol{r}^{(m)}}{3}. \end{aligned} \tag{4}$$

In particular, all points of $\mathfrak{G}^{(m)}$ have the same degree.

As an immediate corollary we can note

PROPOSITION A.2. *The only connected graph $\mathfrak{G}$ which satisfies $\boldsymbol{r}_{\mathfrak{G}} = \boldsymbol{r}_{\mathfrak{G}^*}$ is a cycle, i.e. a closed polygon C_n with $n = \boldsymbol{v}_{\mathfrak{G}}$. Consequently, $\mathfrak{G} \cong \mathfrak{G}^*$ iff $\mathfrak{G} \cong C_n$ for some integer n.*

Proof. It suffices to note that the only solution of $\boldsymbol{r}^{(m)} = \boldsymbol{r}^{(m+1)} = 2 \cdot \boldsymbol{r}^{(m)} - 2$ is $\boldsymbol{r}^{(m)} = 2$. □

Solving the system (4) we obtain

COROLLARY A.3. *Under assumptions of A.1 we have*

$$\boldsymbol{r}^{(m)} = 2^m \cdot \boldsymbol{r}^{(0)} - \sum_{i=1}^{m} 2^i = 2^m \cdot \boldsymbol{r}^{(0)} - 2(2^m - 1) = 2^m(\boldsymbol{r}^{(0)} - 2) + 2. \tag{5}$$

$$\boldsymbol{b}^{(m+2)} = \boldsymbol{b}^{(m)} \cdot \binom{\boldsymbol{r}^{(m+1)}}{2}; \text{ so} \tag{6}$$

$$\boldsymbol{b}^{(m)} = \begin{cases} \boldsymbol{b}^{(0)} \cdot \prod_{i=1}^{k} \binom{\boldsymbol{r}^{(2i-1)}}{2} & \text{for } m = 2k \\ \boldsymbol{v}^{(0)} \cdot \prod_{i=0}^{k} \binom{\boldsymbol{r}^{(2i)}}{2} & \text{for } m = 2k+1 \end{cases},$$

$$\boldsymbol{\tau}^{(m)} = \boldsymbol{\tau}^{(0)} + \sum_{i=0}^{m-1} \boldsymbol{v}^{(i)} \binom{\boldsymbol{r}^{(i)}}{3}. \tag{7}$$

Note that iteration of the construction of the line graph leads to graphs such that their Desarguesian horizon may contain points with different degrees. Let us begin with somehow evident

FACT A.4. *Let $\mathfrak{G}$ be an arbitrary graph and $h = \{A, B\}$ be an edge of $\mathfrak{G}^{(2)}$. Then $A = \{a, b\}$, $B = \{a, c\}$ for some edges a, b, c of $\mathfrak{G}$ and one of the following holds:*

(i) *a, b, c yield a triangle in $\mathfrak{G}$, i.e. there are points x, y, z of $\mathfrak{G}$ such that $a = \{x, y\}$, $b = \{y, z\}$, and $c = \{z, x\}$;*

(ii) *a, b, c have one common edge so, $a = \{x, t\}$, $b = \{y, t\}$, and $c = \{z, t\}$ for some points x, y, z, t of $\mathfrak{G}$;*

(iii) *b, a, c are consecutive sides of a path in $\mathfrak{G}$, i.e. $b = \{u, x\}$, $a = \{x, y\}$, and $c = \{y, z\}$ for a path u, x, y, z with pairwise distinct vertices.*

In the cases (i) *and* (ii) *it holds $\sigma(A, B) = 1$, and $\sigma(A, B) = 0$ in the case* (iii).

Consequently, the horizon $\mathbf{H}(\mathfrak{G}^{(2)})$ has $\boldsymbol{\tau}_{\mathfrak{G}^*} = \boldsymbol{\tau}_{\mathfrak{G}} + \sum_{x \in S} \binom{\boldsymbol{r}_{\mathfrak{G}}(x)}{3}$ points $\{A, B\}^\infty$ with $\sigma(A, B) = 1$.

Proof. In the cases (i) and (ii) we set $C = \{b, c\}$; then C is an edge of $\mathfrak{G}^{(1)}$ which completes A, B to a triangle. If there were such a completion in the case (iii) then $C = \{b, c\}$ would be an edge of $\mathfrak{G}^{(1)}$ and then $b \cap c \neq \emptyset$, which is impossible. □

From A.4, as a consequence of 1.3(iii) and A.3 we obtain

LEMMA A.5. *Let $\mathfrak{G}$ be a graph with the following properties:*

(i) *every point of $\mathfrak{G}$ has the same degree ≥ 3;*

(ii) *$\mathfrak{G}$ contains a not closed path of length 4.*

Then $\mathfrak{D} = \mathfrak{G}^{(2)\infty}$ has points p, q with $\boldsymbol{r}_{\mathfrak{D}}(p) \neq \boldsymbol{r}_{\mathfrak{D}}(q)$.

The horizon of $\mathfrak{D}$ contains $\boldsymbol{\tau}_{\mathfrak{G}^*} = \boldsymbol{\tau}_{\mathfrak{G}} + \boldsymbol{v}_{\mathfrak{G}} \cdot \binom{\boldsymbol{r}_{\mathfrak{G}}}{3}$ points $\{A, B\}^\infty$ with $\sigma(A, B) = 1$. Consequently, the horizon of $\mathfrak{G}$ has $\boldsymbol{\tau}_{\mathfrak{G}} + \boldsymbol{v}_{\mathfrak{G}} \cdot \binom{\boldsymbol{r}_{\mathfrak{G}}}{3}$ points with degree $2\boldsymbol{r}_{\mathfrak{G}} - 3$, and the remaining points have degree $2\boldsymbol{r}_{\mathfrak{G}} - 4$.

Proof. [Computation] Let $q = h^\infty$, $h = \{A, B\}$, where A, B are points of $\mathfrak{G}^{(2)}$ i.e. edges of $\mathfrak{G}^*$. Let $a = A \cap B$ and $a = \{x, y\}$. Then $\boldsymbol{r}_{\mathfrak{D}}(q) = \boldsymbol{r}_{\mathfrak{G}^{(2)}}{}^\infty(h) = \boldsymbol{r}_{\mathfrak{G}^{(1)}}(a) - 2 + \sigma(A, B) = (\boldsymbol{r}_{\mathfrak{G}}(x) + \boldsymbol{r}_{\mathfrak{G}}(y) - 2) - 2 + \sigma(A, B)$. □

As a consequence of A.3 we obtain

COROLLARY A.6. *Let us substitute $\mathfrak{G} = K_n$ in A.1 and let us write $\mathsf{x}_n^{(m)} = \mathsf{x}_{K_n^{(m)}}$ for $\mathsf{x} = \boldsymbol{v}, \boldsymbol{b}, \boldsymbol{r}, \boldsymbol{\tau}$. For every integer n with $n \geq 3$ and arbitrary m it* *holds*

$$\boldsymbol{b}_n^{(m)} = \begin{cases} \binom{n}{2} \cdot \prod_{i=1}^{k} \binom{2^{2i-1}(n-3)+2}{2} & \textit{for } m = 2k \\ \binom{n}{1} \cdot \prod_{i=0}^{k} \binom{2^{2i}(n-3)+2}{2} & \textit{for } m = 2k+1 \end{cases},$$

$$\boldsymbol{\tau}_n^{(m)} = \binom{n}{3} + \sum_{i=0}^{m-1} \boldsymbol{v}_n^{(i)} \cdot \binom{2^i(n-3)+2}{3},$$

$$\boldsymbol{v}_n^{(m)} = \boldsymbol{b}_n^{(m-1)}, \quad \boldsymbol{r}_n^{(m)} = 2^m(n-3)+2.$$

COROLLARY A.7. *Let $\mathfrak{D} = \mathbf{D}(K_n^{(m)})$ for $m \geq 2$ and $H_n^{(m)}$ be the horizon of $\mathfrak{D}$. Then the parameters of $H_n^{(m)}$ are as follows:*

$$\boldsymbol{v}_{H_n^{(m)}} = \boldsymbol{v}_n^{(m)\infty} = \boldsymbol{b}_n^{(m)}, \quad \boldsymbol{b}_{H_n^{(m)}} = \boldsymbol{b}_n^{(m)\infty} = \boldsymbol{\tau}_n^{(m)},$$

$H_n^{(m)}$ *has*

$$\begin{cases} \boldsymbol{\tau}_n^{(m-1)} & \textit{points with degree } 2\boldsymbol{r}_n^{(m-2)} - 3, \\ \boldsymbol{b}_n^{(m)} - \boldsymbol{\tau}_n^{(m-1)} & \textit{points with degree } 2\boldsymbol{r}_n^{(m-2)} - 4. \end{cases}$$

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