DEMONSTRATIO MATHEMATICA
Vol. XL No 4 2007

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A NOTE ON MURTHY'S CONJECTURE

Abstract. In this paper, we consider a conjecture made by Murthy to the effect that a $C_0^f \cap Q_0$ matrix is positive semidefinite (PSD) and show that the conjecture is true for $n \times n$ matrices of rank 1 or 3×3 matrices that are not in Q. We also consider the class of P_1-matrices which is a subclass of Q_0 and obtain the following: for $A \in P_1$, if $A \in C_0^f$ and $A \notin Q$, then A is PSD; if $A \in C_0^f$ and $A \in Q$, then A is PSD for $n \leq 3$ and A isn't PSD for $n > 3$.

1. Introduction

The linear complementarity problem (LCP) with data $A \in R^{n\times n}$ and $q \in R^n$ involves finding a vector $z \in R^n$ such that

$$z \geq 0,\ q + Az \geq 0 \ \text{ and } \ z^T(q + Az) = 0.$$

LCP has numerous applications, both in theory and practice, treated by vast literature (see[1]). A number of matrix classes have been defined in connection with LCP. We shall briefly introduce the concepts and notation required for presentation of the results of this paper. For $A \in R^{n\times n}$, $q \in R^n$, let $F(q, A) = \{z \in R_+^n : Az+q \geq 0\}$, $S(q, A) = \{z \in F(q, A) : z^T(Az+q) = 0\}$. For most of other notion, we shall follow that used in [1].

For any positive integer n, write $\overline{n} = \{1, 2, \cdots, n\}$, and for any subset α of $\overline{n}$, write $\overline{\alpha} = \overline{n}\backslash\alpha$. Consider $A \in R^{n\times n}$. If $\alpha \subseteq \overline{n}$ such that $\det A_{\alpha\alpha} \neq 0$, then the matrix M defined by

$$M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1},\ M_{\alpha\overline{\alpha}} = -M_{\alpha\alpha}A_{\alpha\overline{\alpha}},$$
$$M_{\overline{\alpha}\alpha} = A_{\overline{\alpha}\alpha}M_{\alpha\alpha},\ \ M_{\overline{\alpha}\overline{\alpha}} = A_{\overline{\alpha}\overline{\alpha}} - M_{\overline{\alpha}\alpha}A_{\alpha\overline{\alpha}}$$

is known as the principal pivotal transform (PPT) of A with respect to α

This work is supported by YSF of Guangdong University of Technology(062058)
Key words and phrases: Linear complementarity problem; C_0^f-matrices; PSD-matrices.
CLC number: O151.22.
2000 *Mathematics Subject Classification*: 90C33.

and will be denoted by $\wp_\alpha(A)$. Note that a PPT is defined only with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. By convention, when $\alpha = \emptyset$, $\det A_{\alpha\alpha} = 1$ and $M = A$ (see[1]). Whenever we refer to PPTs, we mean the ones which are well defined.

DEFINITION 1. Let $A \in R^{n\times n}$. Then A is called:

(1) a Q-matrix, if $S(q, A) \neq \emptyset$ for all $q \in R^n$;

(2) a Q_0-matrix, if for all $q \in R^n$, $F(q, A) \neq \emptyset \Rightarrow S(q, A) \neq \emptyset$;

(3) a $\overline{Q}_0$-matrix , if $A_{\alpha\alpha}$ is a Q_0-matrix for all $\alpha \subseteq \overline{n}$;

(4) a P_0-matrix, if $\det A_{\alpha\alpha} \geq 0$ for all $\alpha \subseteq \overline{n}$;

(5) a P_1-matrix, if the principal minors of A are nonnegative, exactly one of which is zero;

(6) a PSD-matrix (positive semidefinite matrix), if $x^T Ax \geq 0$ for every $x \in R^n$;

(7) a C_0-matrix, if $x^T Ax \geq 0$ for every $x \geq 0$;

(8) a C_0^f-matrix, if every PPT of A is a C_0 matrix.

By $\boldsymbol{\Omega}$ we denote the set of all Ω-matrices, where Ω denote a class of matrix in Definition 1.

REMARK 1 (see [1], [2]). If A belongs to any of C_0^f, Q_0, Q, PSD, P_1, then every PPT of A is also in the same class.

Murthy (see [2]) introduced the class of C_0^f-matrices and obtained that $C_0^f \cap Q_0$-matrices are P_0-matrices which contain PSD-matices. In [3, 4], Murthy proved that bisymmetric $C_0^f \cap Q_0$-matrices as well as 2×2 $C_0^f \cap Q_0$-matrices are PSD and raised the following conjecture.

Murthy's conjecture: Suppose $A \in C_0^f \cap Q_0$, then A is PSD.

In [5], Murthy showed that $C_0^f \cap Q_0$-matrices are sufficient. This belongs to a series of results showing that $C_0^f \cap Q_0$-matrices and PSD matrices have many properties in common. So they raised the conjecture as an open problem. In [6], Mohan presented a counterexample to this conjecture. In this article, we present some sufficient conditions to this conjecture, settle the open problem.

2. Main results

In [3], Murthy proved the conjecture is true when $n = 2$. In this article, we assume $n \geq 3$. For the sake of completeness, we state relevant portions of some known theorems.

LEMMA 1 (see [3]). *Suppose $A \in R^{n\times n} \cap C_0^f$, then the following conditions are equivalent:*

(a) $A \in Q_0$;

(*b*) *for every PPT M of A, $m_{ii} = 0 \Longrightarrow m_{ij} + m_{ji} = 0$ $\forall i, j \in \overline{n}$;*
(*c*) $A \in \overline{Q_0}$.

LEMMA 2 (see [7]). *Suppose $A \in R^{n\times n} \cap P_1$. The following statements hold:*
(*a*) *There exists a unique α such that $\wp_\alpha(A) \in P_1$ and $det\wp_\alpha(A) = 0$.*
(*b*) *A is in Q_0.*

LEMMA 3 (see [1]). *Let $A \in R^{n\times n} \cap C_0$, if there exists a vector $x > 0$ such that $x^T Ax = 0$, then A is PSD.*

LEMMA 4 (see [5]). *If $A \in P_0$, then $A \in Q$ if and only if $A \in R_0$, where $A \in R_0$ if and only if $S(0, A)=\{0\}$.*

Now we prove the following theorem.

THEOREM 1. *Suppose $A \in R^{n\times n} \cap Q_0$, rank$A$=1, then A is PSD if and only if $A \in C_0^f$.*

Proof. The 'only if' part is obvious. We shall prove the 'if' part. If $a_{ii} = 0$, $\forall i \in \{1, 2, \cdots, n\}$, then from Lemma 1 it follows that $a_{ij} + a_{ji} = 0$, $\forall i, j \in \{1, 2, \cdots, n\}$. Thus, rank$A$=0 or rank$A \geq 2$. This contradicts the assumption that rank$A$=1. There then exists an index i, $1 \leq i \leq n$, such that $a_{ii} > 0$. Suppose $A = ab^T$, where $a = (a_1, a_2, \cdots, a_n)^T$, $b^T = (b_1, b_2, \cdots, b_n)$, then there must exist an index i such that $a_i \neq 0$ and $b_i \neq 0$. Without loss of generality, we may assume $i = 1$, then $a_{11} = a_1 b_1 > 0$. Take $\alpha = \{1\}$, $M = \wp_\alpha(A)$. Then

$$M = \begin{pmatrix} \frac{1}{a_1 b_1} & -\frac{b_2}{b_1} & \cdots & -\frac{b_n}{b_1} \\ \frac{a_2}{a_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_n}{a_1} & 0 & \cdots & 0 \end{pmatrix}.$$

From $A \in C_0^f \cap Q_0$, $M \in C_0^f \cap Q_0$. It follows from Lemma 1 that $-\frac{b_i}{b_1} + \frac{a_i}{a_1} = 0$ for all $i = 2, 3, \cdots, n$. For an arbitrary $x \in R^n$, $x^T Mx = \frac{1}{a_1 b_1} x_1^2 \geq 0$. So M is PSD. It follows that A is PSD. □

THEOREM 2. *Suppose $A \in R^{3\times 3} \cap Q_0$ and $A \notin Q$, then A is PSD if and only if $A \in C_0^f$.*

Proof. It suffices to prove the 'if' part. Here we consider two cases: A has a zero diagonal entry or A has no zero diagonal entry.

Case (1): A has a zero diagonal entry. Without loss of generality, we may assume that $i = 1$ is such that $a_{ii} = 0$. From Lemma 1 it follows that $a_{1j} + a_{j1} = 0$ for $j \neq 1$. Take $\alpha = \overline{\{1\}}$. Then $A_{\alpha\alpha} \in R^{2\times 2}$ and $A_{\alpha\alpha} \in C_0^f \cap Q_0$, from Theorem 4.9 in [3], $A_{\alpha\alpha}$is PSD. For an arbitrary $x \in R^3$, $x^T Ax = x_\alpha^T A_{\alpha\alpha} x_\alpha \geq 0$, so A is PSD.

Case (2): all diagonal entries of A are positive. Here we consider two subcases: (a) all principal minors of order two are positive, (b) A has a zero principal minor of order two.

Consider case (a): Since diagonal entries of A and its principal minors of order two are positive, then $\det A=0$(otherwise if $\det A > 0$, then $A \in P \subseteq Q$. It contradicts to that $A \notin Q$). So $A \in P_1 \setminus Q$ and $detA = 0$, there exists a vector $p > 0$ such that $Ap = 0$ (it follows from Theorem 3.1 of [8] and Lemma 4.1 of [9]). Thus we have $p^T Ap = 0, p > 0$. Moreover, $A \in C_0^f \subset C_0$. From Lemma 3, it follows that A is PSD.

Consider case (b): Without loss of generality, we may assume $\alpha = \{1, 2\}$ such that $\det A_{\alpha\alpha} = 0$, i.e. $a_{11}a_{22} = a_{12}a_{21}$. Let $\beta = \{1\}$, $M = \wp_\beta(A)$. Then

$$M = \begin{pmatrix} \frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ \frac{a_{21}}{a_{11}} & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{21}a_{13}}{a_{11}} \\ \frac{a_{31}}{a_{11}} & a_{32} - \frac{a_{12}a_{31}}{a_{11}} & a_{33} - \frac{a_{31}a_{13}}{a_{11}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ \frac{a_{21}}{a_{11}} & 0 & a_{23} - \frac{a_{21}a_{13}}{a_{11}} \\ \frac{a_{31}}{a_{11}} & a_{32} - \frac{a_{12}a_{31}}{a_{11}} & a_{33} - \frac{a_{31}a_{13}}{a_{11}} \end{pmatrix}.$$

So M has a zero diagonal entry $m_{22} = 0$, and $M \in R^{3\times 3} \cap C_0^f \cap Q_0 \setminus Q$. From the proof of case (1) , we conclude that M is PSD. Hence A is PSD. □

REMARK 2. Theorem 2 is not true for $n > 3$. Consider the example

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

It is easy to verify that $A \in P_0$ and $A \notin R_0$ (because $(1, 0, 0, 0)^T \in S(0, A)$). From Lemma 4, we know that $A \notin Q$.

We now show that $A \in C_0^f \cap Q_0$.

Note that there are two distinct PPTs of A. One is the matrix A itself. It is the PPT of A corresponding to

$$\alpha = \emptyset, \{2\}, \{3\}, \{2, 3\}.$$

Another PPT is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}.$$

It is the PPT of A corresponding to

$$\alpha = \{4\}, \{2,4\}, \{3,4\}, \{2,3,4\}.$$

The copositivity of matrices A and M can be demonstrated by an analysis of the corresponding quadratic forms which can be rewritten as follows:

(1) $x^T Ax = x_2^2 + x_2x_4 + (x_3 - x_4)^2$,

(2) $x^T Mx = (x_2 - \frac{1}{2}x_4)^2 + x_3^2 + 2x_3x_4 + \frac{3}{4}x_4^2$.

Hence $A \in C_0^f \cap Q_0$ and $A \notin Q$. But selecting $x^T = (1, 1, -2, -2)$, we have $x^T Ax = -1 < 0$, so A isn't PSD.

Now, considering the class of P_1-matrices which is a subclass of Q_0 (from Lemma 2). We have:

THEOREM 3. *If $A \in R^{3\times 3} \cap P_1$, then $A \in PSD$ if and only if $A \in C_0^f$.*

Proof. The 'only if'part is obvious. We shall prove the 'if' part. Here we consider two cases: $A \notin Q$ or $A \in Q$.

Case (1): $A \notin Q$. Since $A \in P_1 \subseteq Q_0$, from the proof of Theorem 2, it concludes that A is PSD.

Case (2): $A \in Q$. Here we consider three subcases: (a) A has a zero diagonal, (b) A is sigular , and (c) A is nonsigular with positive diagonal entries .

Consider case (a). Without loss of generality, we may assume that $i = 1$ is such that $a_{ii} = 0$. Since $A \in C_0^f \cap Q \subseteq C_0^f \cap Q_0$. From Lemma 1, it follows that $a_{1j} + a_{j1} = 0$ for $j \neq 1$. Take $\alpha = \overline{\{1\}}$. From $A \in P_1$, $A_{\alpha\alpha} \in P \subset Q$. So $A_{\alpha\alpha} \in R^{2\times 2}$ and $A_{\alpha\alpha} \in C_0^f \cap Q \subseteq C_0^f \cap Q_0$, from Theorem 4.9 in [3], $A_{\alpha\alpha}$is PSD. For an arbitrary $x \in R^3$, $x^T Ax = x_\alpha^T A_{\alpha\alpha} x_\alpha \geq 0$. So A is PSD .

Next, consider case (b): Take $\alpha = \{1,2\}$. Since $A \in P_1$, $detA_{\alpha\alpha} \neq 0$. Let $M = \wp_\alpha(A)$, then

$$M = \begin{pmatrix} (A_{\alpha\alpha})^{-1} & -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1} & p \end{pmatrix},$$

where $p = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$ is a number. Since $\det A$=$\det A_{\alpha\alpha}\det p$=0, $p = 0$. So M has a zero diagonal entry $m_{33} = 0$, and $M \in R^{3\times 3} \cap C_0^f \cap P_1 \cap Q$. From the proof of case (a) , we conclude that M is PSD. Hence A is PSD.

Last, consider case(c): From Lemma 2, there exists a unique α such that $\wp_\alpha(A) \in P_1$ and $\det\wp_\alpha(A)$=0. Then $\wp_\alpha(A) \in R^{3\times 3} \cap C_0^f \cap P_1 \cap Q$, and $det\wp_\alpha(A) = 0$. From the proof of case (b), it follows that $\wp_\alpha(A)$ is PSD. Therefore A is PSD. □

THEOREM 4. *Let $A \in R^{n\times n}$, $n > 3$. If $A \in P_1$ and $A \notin Q$, then $A \in PSD$ if and only if $A \in C_0^f$.*

Proof. It suffices to prove the 'if' part. Here we consider two subcases: (a) A is sigular, and (b) A is nonsigular.

Consider case (a): Since $A \in P_1 \setminus Q$ and $detA = 0$, there exists a vector $p > 0$ such that $Ap = 0$ (it follows from Theorem 3.1 of [8] and Lemma 4.1 of [9]). Thus we have $p^T Ap = 0$, $p > 0$. And $A \in C_0^f \subset C_0$. From Lemma 3, it follows A is PSD.

Next, consider case (b): From $A \in P_1$ and Lemma 2, there exists a unique α such that $\wp_\alpha(A) \in P_1$ and $\det\wp_\alpha(A)$=0. Since $A \notin Q$, $\wp_\alpha(A) \notin Q$. So $\wp_\alpha(A) \in P_1 \setminus Q$ and $det\wp_\alpha(A) = 0$. From the proof of case (a), it follows that $\wp_\alpha(A)$ is PSD. Therefore A is PSD. □

REMARK 3. When $n > 3$, if $A \in Q$, then Theorem 4 is not true. Consider the example

$$A = \begin{pmatrix} 0 & 1 & 1 & -2 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 1 \end{pmatrix}.$$

It is easy to verify $A \in P_1 \cap R_0$, and $P_1 \subseteq P_0$. From Lemma 4, we know that $A \in Q$.

We only need to prove $A \in C_0^f$.

$x^T Ax = x_2^2 + x_2x_4 + (x_3 - x_4)^2$.

So $A \in C_0$. For other fourteen PPTs of A, we can verify them in turn and obtain that every PPT of A is a C_0-matrix. So $A \in C_0^f \cap P_1 \cap Q$. But selecting $x^T = (1, 1, -2, -2)$, then $x^T Ax = -1 < 0$. A isn't PSD.

References

[1] R. W. Cottle, J. S. Pang, and R. E. Stone, *The Linear Complementarity Problem[M]*, New York, Academic Press, 1992.

[2] G. S. R. Murthy and T. Parthasarathy, *Some properties of fully semimonotone Q_0-matrices[J]*, SIAM J. Matrix Anal. Appl. 16 (1995), 1268–1286.

[3] G. S. R. Murthy and T. Parthasarathy, *Fully copositive matrices[J]*, Math. Programming 82 (1998), 401–411.

[4] G. S. R. Murthy, T. Parthasarathy and B. Sriparna, *Some recent results on the linear complementarity problem[J]*, SIAM J. Matrix Anal. Appl. 19 (1998), 898–905.
[5] G. S. R. Murthy, T. Parthasarathy and B. Sriparna, *On the solution sets of linear complementarity problems[J]*, SIAM J. Matrix Anal. Appl. 21 (2000), 1229–1235.
[6] S. R. Mohan, S. K. Neogy and A. K. Das, *On the classes of fully copositive and fully seminotone matrices[J]*, Linear Algebra Appl. 323 (2001), 87–97.
[7] G. S. R. Murthy, S. R. Neogy and F. Thuijsman, *A note on P_1 and Lipschitzian matrices[J]*, SIAM J. Matrix Anal. Appl. 21 (1999), 636–641.
[8] R. A. Danao, *A note on E'-matrices[J]*, Linear Algebra Appl. 259 (1997), 299–305.
[9] R. A. Danao, *On a class of semimonotone Q_0-matrices in the linear complementarity problem[J]*, Operations Research Letters 13 (1993), 121–125.

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Received September 3, 2006; revised version June 18, 2007.