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Rajendra Prasad, Amit Kumar Rai, S. S. Shukla, M. M. Tripathi

A NOTE ON TRANSVERSAL HYPERSURFACES OF LORENTZIAN ALMOST PARACONTACT MANIFOLDS

Abstract. Transversal hypersurfaces of Lorentzian almost paracontact manifolds are studied. It is proved that transversal hypersurfaces of Lorentzian almost paracontact manifolds admit an almost product structure and almost product Lorentzian metric structure.We shall also derive a formula which gives the relation between the connection with respect to the Lorentzian metric g and that with respect to induced Lorentzian metric G. Some properties of Lorentzian (f, g, u, v, λ)-structure, transversal hypersurfaces of Lorentzian cosymplectic manifolds and Lorentzian paracontact Sasakian manifolds are also studied.

1. Introduction

Let $\overline{M}$ be a Lorentzian almost paracontact manifold (cf. [1], [2]) with a Lorentzian almost paracontact structure (ϕ, ξ, η, g),that is, ϕ is a (1, 1) tensor field, ξ is a (time-like) vector field, η is a 1-form and g is a Lorentzian metric on $\overline{M}$ such that

$$\phi^2 = I + \eta \otimes \xi, \qquad \eta(\xi) = -1, \qquad \phi\xi = 0,$$

$$\eta \circ \phi = 0, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2}$$

$$\Phi(X, Y) \equiv g(\phi X, Y) = g(X, \phi Y) = \Phi(Y, X), g(X, \xi) = \eta(X), \tag{3}$$

for all $X, Y \in T\overline{M}$.

A Lorentzian almost paracontact manifold is called (cf. Matsumoto [1]): Lorentzian paracontact manifold if

$$\Phi(X, Y) = \frac{1}{2}\left(\left(\overline{\nabla}_X \eta\right) Y + \left(\overline{\nabla}_Y \eta\right) X\right), \qquad X, Y \in T\overline{M}. \tag{4}$$

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Lorentzian para-Sasakian (in brief, LP-Sasakian [2]) manifold if

$$(\overline{\nabla}_X\phi)\,Y = g(\phi X, \phi Y)\,\xi + \eta(Y)\,\phi^2 X, \qquad X, Y \in T\overline{M}. \tag{5}$$

Lorentzian special paracontact (in brief, LSP-Sasakian) manifold, if

$$\Phi(X, Y) = eg(\phi X, \phi Y), \qquad e^2 = 1, \qquad X, Y \in T\overline{M}. \tag{6}$$

Here $\overline{\nabla}$ is the covariant differentiation with respect to g.

Let M be a hypersurface of a Lorentzian almost paracontact manifold $\overline{M}$ with Lorentzian almost paracontact structure(ϕ, ξ, η, g). Let the induced metric on M also be denoted by g.

2. Transversal hypersurface

Let M be a hypersurface of Lorentzian almost paracontact manifold$\overline{M}$ equipped with an Lorentzian almost paracontact structure (ϕ, ξ, η). We assume that the structure vector field ξ never belongs to the tangent hyperplane of the hypersurface M. Such a hypersurface is called a transversal hypersurface of a Lorentzian almost paracontact manifold. Transversal hypersurfaces never contain the structure vector field ξ of the defining Lorentzian almost contact structure. In this case the structure vector field ξ can be taken as an affine normal to the hypersurface. Since the vector field $X \in TM$ and ξ are linearly independent, therefore we may write

$$\phi X = FX + \omega(X)\,\xi, \qquad X \in TM, \tag{7}$$

where F is a (1, 1) tensor field and ω is a 1-form on M.

From (7) we have

$$\phi\xi = F\xi + \omega(\xi)\,\xi,$$

or

$$0 = F\xi + \omega(\xi)\,\xi, \tag{8}$$

$$\phi^2 X = F(\phi X) + \omega(\phi X)\,\xi,$$

$$X + \eta(X)\,\xi = F^2 X + (\omega \circ F)(X)\,\xi. \tag{9}$$

Taking account of equation (9) we get

$$F^2 = I, \tag{10}$$

$$\eta(X) = \omega \circ F(X), \qquad X \in TM. \tag{11}$$

In view of (10), we have

THEOREM 2.1. *Each transversal hypersurface of a Lorentzian almost paracontact manifold admits an almost product structure F and a 1-form ω.*

From (10) and (11), it follows that

$$\begin{aligned} \eta &= \omega \circ F, \\ \eta(FX) &= (\omega \circ F) FX, \\ \eta(FX) &= \omega\left(F^2 X\right), \\ (\eta \circ F) X &= \omega(X), \\ \omega &= \eta \circ F. \end{aligned} \tag{12}$$

Now, we assume that $\overline{M}$ admits a Lorentzian almost paracontact metric structure (ϕ, ξ, η, g).

We denote by g the induced metric on M also. Then for all $X, Y \in TM$, we obtain

$$g(FX, FY) = g(X, Y) + \eta(X)\eta(Y) - \omega(X)\omega(Y). \tag{13}$$

We define a new metric G on the transversal hypersurface given by

$$G(X, Y) = g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y). \tag{14}$$

So,

$$G(FX, FY) = g(FX, FY) + \eta(FX)\eta(FY)$$

using (12), we have

$$\begin{aligned} G(FX, FY) &= g(X, Y) + \eta(X)\eta(Y) - \omega(X)\omega(Y) \\ &\quad + (\eta \circ F)(X)(\eta \circ F)(Y) \\ &= g(X, Y) + \eta(X)\eta(Y) - \omega(X)\omega(Y) + \omega(X)\omega(Y) \\ &= g(X, Y) + \eta(X)\eta(Y) = G(X, Y), \end{aligned} \tag{15}$$

where equations (10), (12), (13) and (14) are used,then we get

$$G(FX, FY) = G(X, Y), \qquad X, Y \in TM.$$

Then G is Lorentzian metric on M that is (F, G) is an almost product Lorentzian structure on the transversal hypersurface M of $\overline{M}$. Thus, we are able to state the following

THEOREM 2.2. *Each transversal hypersurface of Lorentzian almost paracontact manifold admits an almost product Lorentzian structure.*

We now assume that M is orientable and choose a unit vector field N of $\overline{M}$, normal to M. Then Gauss and Weingarten formulae are given respectively by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y) N, \qquad X, Y \in TM, \\ \overline{\nabla}_X N &= -HX, \qquad X \in TM, \qquad N \in T^{\perp} M, \end{aligned} \tag{16}$$

where $\overline{\nabla}$ and ∇ are respectively the Levi-Civita connections in $\overline{M}$ and M, and h is the second fundamental form related to H by

$$h(X,Y) = g(HX,Y). \tag{17}$$

For any vector field X tangent to M, defining

$$\phi X = fX + u(X)N, \tag{18}$$

$$\phi N = -U, \tag{19}$$

$$\xi = V + \lambda N, \tag{20}$$

$$\eta(X) = v(X),$$

$$\lambda = \eta(N) = g(\xi, N). \tag{21}$$

For $X \in TM$ we get an induced Lorentzian (f,g,u,v,λ)-structure on the transversal hypersurface such that

$$f^2 = I + u \otimes V + v \otimes U, \tag{22}$$

$$fU = -\lambda V, \qquad fV = \lambda U, \tag{23}$$

$$u \circ f = \lambda v, \qquad v \circ f = -\lambda u, \tag{24}$$

$$u(U) = -1 - \lambda^2, \qquad u(V) = v(U) = 0, \qquad v(V) = -1 - \lambda^2, \tag{25}$$

$$g(fX, fY) = g(X,Y) + u(X)u(Y) + v(X)v(Y), \tag{26}$$

$$g(X, fY) = g(fX, Y), \qquad g(X,U) = u(X), \qquad g(X,V) = v(X), \tag{27}$$

for all $X, Y \in TM$ where $\lambda = \eta(N)$.

Thus, we see that every transversal hypersurface of an Lorentzian almost paracontact metric manifold also admits a Lorentzian (f,g,u,v,λ)-structure.

Next we find relation between the induced almost product metric structure (F,G) and the induced Lorentzian (f,g,u,v,λ)-structure on the transversal hypersurface of an Lorentzian almost paracontact metric manifold.

In fact, we have the following

THEOREM 2.3. *Let M be a transversal hypersurface of an Lorentzian almost para contact manifold $\overline{M}$ equipped with Lorentzian almost para contact structure (ϕ, ξ, η, g) and induced almost product metric structure (F, G).*

Then we have

$$\lambda\omega = u, \tag{28}$$

$$F = f - \frac{1}{\lambda} u \otimes V, \tag{29}$$

$$FU = \frac{1}{\lambda} V, \tag{30}$$

$$u \circ F = u \circ f = \lambda v, \tag{31}$$

$$FV = fV = \lambda U, \tag{32}$$

$$v \circ F = \frac{1}{\lambda} u. \tag{33}$$

Proof. We have

$$\phi X = FX + \omega(X)\xi,$$

$$\xi = V + \lambda N,$$

$$\phi X = FX + \omega(X)V + \lambda\omega(X)N, \tag{34}$$

$$\phi X = fX + u(X)N. \tag{35}$$

From equation (34) and (35) we have

$$\lambda\omega X = u(X),$$

or

$$\lambda\omega = u, \omega = \frac{1}{\lambda} u,$$

which is equation (28)

$$fX = FX + \omega(X)V,$$

$$f = F + \omega \otimes V,$$

$$f = F + \frac{1}{\lambda} u \otimes V,$$

or,

$$F = f - \frac{1}{\lambda} u \otimes V,$$

which is equation (29)

$$(uoF)(X) = (uof)(X) - \frac{1}{\lambda} u(X)\, u(V) = uof, \qquad u(V) = 0$$

$uoF = uof = \lambda v$ which gives equation (31)

$$FU = fU - \frac{1}{\lambda} u(U)\, V = -\lambda V - \frac{\left(-1-\lambda^2\right)V}{\lambda} = \frac{1}{\lambda} V$$

which gives equation (30)

$$(voF)(X) = (vof)(X) - \frac{1}{\lambda}u(X)\,v(V) = (vof)(X) - \frac{1}{\lambda}u(X)\left(-1-\lambda^2\right)$$
$$= -\lambda u(X) - \frac{1}{\lambda}u(X)\left(-1-\lambda^2\right) = \frac{1}{\lambda}u(X)$$
$$voF = \frac{1}{\lambda}u,$$

which is equation (33)

$$FV = fV - \frac{1}{\lambda}u(V)\,V = fV = \lambda U,$$

which is equation (32). Here equations (18) to (29) are used.

LEMMA 2.4. *Let M be a transversal hypersurface with Lorentzian (f,g,u,v,λ)-structure of a Lorentzian almost para contact manifold $\overline{M}$. Then*

$$(36)\qquad \left(\overline{\nabla}_X\phi\right)Y = \left((\nabla_X f)\,Y - u(Y)\,HX + h(X,Y)\,U\right) + \left(\left(\overline{\nabla}_X u\right)Y + h(X,fY)\,N\right),$$

$$(37)\qquad \overline{\nabla}_X\xi = (\nabla_X V - \lambda HX) + (h(X,V) + X\lambda)\,N,$$

$$(38)\qquad \left(\overline{\nabla}_X\phi\right)N = (-\nabla_X U + fHX),$$

$$(39)\qquad \left(\overline{\nabla}_X\eta\right)Y = (\nabla_X v - \lambda h(X,Y)),$$

for all $X, Y \in TM$.

The proof is straight forward and hence ommited.

THEOREM 2.5. *Let M be a transversal hypersurfaces with Lorentzian (f,g,u,v,λ)-structure of a Lorentzian cosymplectic manifold $\overline{M}$.*

Then

$$(40)\qquad (\nabla_X f)\,Y = u(Y)\,HX - h(X,Y)\,U,$$

$$(41)\qquad (\nabla_X u)\,Y = -h(X,fY),$$

$$(42)\qquad \nabla_X V = \lambda HX,$$

$$(43)\qquad h(X,V) = -X\lambda,$$

$$(44)\qquad \nabla_X U = fHX,$$

$$(45)\qquad (\nabla_X v) = \lambda h(X,Y),$$

for all $X, Y \in TM$.

Proof. Using (36), we obtain

$$((\nabla_X f)\,Y - u(Y)\,HX + h(X,Y)\,U) + ((\nabla_X u)\,Y + h(X,fY))\,N = 0.$$

Equating tangential and normal parts in the above equation, we get (40) and (41) respectively. Using (37), we have

$$(\nabla_X V - \lambda HX) + (h(X,V) + X\lambda) N = 0.$$

Equating tangential and normal parts we get (42) and (43) respectively. Using (36) and equating tangential, we get (44). In the last (45) follows from (39). ∎

THEOREM 2.6. *If M be a transversal hypersurface with Lorentzian (f,g,u,v,λ)-structure of a Lorentzian cosymplectic manifold,then Ω on M is given by*

$$\Omega(X,Y) \equiv g(X,fY),$$

satisfying

$$(\nabla_X\Omega)(Y,Z) + (\nabla_Y\Omega)(Z,X) + (\nabla_Z\Omega)(X,Y) = 0.$$

Proof. From (40) we get

$$(\nabla_X\Omega)(Y,Z) = h(X,Y)u(Z) - h(X,Z)u(Y),$$

which gives

$$(\nabla_X\Omega)(Y,Z) + (\nabla_Y\Omega)(Z,X) + (\nabla_Z\Omega)(X,Y) = 0.$$

Hence the theorem is proved. ∎

THEOREM 2.7. *Let M be a transversal hypersurface with Lorentzian (f,g,u,v,λ)-structure of a Lorentzian para Sasakian manifold $\overline{M}$. Then*

$$(\nabla_X f)Y = (g(X,Y) + 2v(X)v(Y))V, \\ +v(Y)X + u(Y)HX - h(X,Y)U, \tag{46}$$

$$(\nabla_X u)Y = \lambda(g(X,Y) + 2v(X)v(Y)) - h(X,fY), \tag{47}$$

$$\nabla_X V = fX + \lambda HX, \tag{48}$$

$$h(X,V) = u(X) - X\lambda, \tag{49}$$

$$\nabla_X U = u(fX) + u(X)V + \lambda v(X) + fHX - \lambda X. \tag{50}$$

Proof. Using equations (1), (2), (5), (20) and (36), we obtain

$$\begin{aligned}(\overline{\nabla}_X\phi)Y &= g(\phi X,\phi Y)\xi + \eta(Y)\phi^2 X \\ &= g(X,Y)\xi + \eta(X)\eta(Y)\xi + \eta(Y)X + \eta(X)\xi\eta(Y),\end{aligned}$$

$$\begin{aligned}((\nabla_X f)Y - u(Y)HX + h(X,Y)U) + ((\nabla_X u)Y + h(X,fY)N) \\ = g(X,Y)(V+\lambda N) + \eta(X)\eta(Y)(V+\lambda N) \\ + \eta(Y)X + \eta(X)(V+\lambda N)\eta(Y),\end{aligned}$$

$$\begin{aligned}((\nabla_X f)Y - u(Y)HX + h(X,Y)U) + ((\nabla_X u)Y + h(X,fY)N) \\ = g(X,Y)V + v(X)v(Y)V + v(Y)X + (v(X)v(Y))V \\ + \lambda g(X,Y)N + \lambda v(X)v(Y)N + \lambda v(X)v(Y)N.\end{aligned}$$

Equating tangential and normal parts in the above equation, we get

$$\begin{aligned}(\nabla_X f)Y = (g(X,Y) + 2v(X)v(Y))V \\ +v(Y)X + u(Y)HX - h(X,Y)U,\end{aligned}$$

$$(\nabla_X u)Y = \lambda(g(X,Y) + 2v(X)v(Y)) - h(X,fY),$$

$$\begin{aligned}\left(\overline{\nabla}_X\phi\right)\xi &= \overline{\nabla}_X(\phi\xi) - \phi\left(\overline{\nabla}_X\xi\right), \\ -\phi\left(\overline{\nabla}_X\xi\right) &= \eta(\xi)\phi^2X, \\ -\phi\left(\overline{\nabla}_X\xi\right) &= -\phi^2X, \\ \left(\overline{\nabla}_X\xi\right) &= \phi X,\end{aligned}$$

$$(\nabla_X V - \lambda HX) + (h(X,V) + X\lambda)N = fX + u(X)N,$$

comparing tangential and normal parts we get

$$\begin{aligned}\nabla_X V &= fX + \lambda HX, \\ h(X,V) &= u(X) - X\lambda.\end{aligned}$$

Hence equations (42) and (43) are obtained. Here equations (18), (20) and (37) are used.

Using (39) and definition of Lorentzian para Sasakian manifold, we obtain

$$\nabla_X U = u(fX) + u(X)V + \lambda(v)X + fHX - \lambda X.$$

THEOREM 2.8. *If M is transversal hypersurface with Lorentzian (f,g,u,v,λ)-structure of a Lorentzian para Sasakian manifold $\overline{M}$. Then Φ on M given by*

$$\Phi(X,Y) = g(X,fY)$$

satisfying

$$\begin{aligned}(\nabla_X\Phi)(Y,Z) + (\nabla_Y\Phi)(Z,X) + (\nabla_Z\Phi)(X,Y) \\ = g(X,Z)v(Y) + g(Y,X)v(Z) \\ + g(Z,Y)v(X) + 6v(X)v(Y)v(Z) \\ + v(X)g(Z,Y) + v(Y)g(X,Z) + v(Z)g(Y,X).\end{aligned}$$

Proof. We have

$$(51)\qquad \begin{aligned}(\nabla_X\Phi)(Y,Z) &= g(Y,(\nabla_X f)Z)\\ &= g(Y,(g(X,Z)+2v(X)v(Z)V\\ &\quad +v(Z)X+u(Z)HX-h(X,Z)U)\\ &= g(X,Z)g(Y,V)+2v(X)v(Z)g(Y,V)\\ &\quad +v(Z)g(Y,X)+u(Z)g(Y,HX)\\ &\quad -h(X,Z)g(Y,U)\\ &= g(X,Z)v(Y)+2v(X)v(Z)v(Y)\\ &\quad +v(Z)g(Y,X)+u(Z)h(X,Y)\\ &\quad -h(X,Z)u(Y).\end{aligned}$$

Similarly we obtain,

$$(52)\qquad \begin{aligned}(\nabla_Y\Phi)(Z,X) &= g(Y,X)v(Z)+2v(Y)v(X)v(Z)+v(X)g(Z,Y)\\ &\quad +u(X)h(Y,Z)-h(Y,X)u(Z),\end{aligned}$$

$$(53)\qquad \begin{aligned}(\nabla_Z\Phi)(X,Y) &= g(Z,Y)v(X)+2v(Z)v(Y)v(X)+v(Y)g(X,Z)\\ &\quad +u(Y)h(Z,X)-h(Z,Y)u(X).\end{aligned}$$

Adding equations (51), (52) and (53), we obtain

$$\begin{aligned}&(\nabla_X\Phi)(Y,Z)+(\nabla_Y\Phi)(Z,X)+(\nabla_Z\Phi)(X,Y)\\ &\quad = g(X,Z)v(Y)+g(Y,X)v(Z)+g(Z,Y)v(X)\\ &\qquad +6v(X)v(Y)v(Z)+v(X)g(Z,Y)+v(Y)g(X,Z)\\ &\qquad +v(Z)g(Y,X)+u(X)h(Y,Z)+u(Y)h(Z,X)\\ &\qquad +u(Z)h(X,Y)-h(Z,Y)u(X)-h(X,Z)u(Y)\\ &\qquad -h(Y,X)u(Z),\end{aligned}$$

$$\begin{aligned}&(\nabla_X\Phi)(Y,Z)+(\nabla_Y\Phi)(Z,X)+(\nabla_Z\Phi)(X,Y)\\ &\quad = g(X,Z)v(Y)+g(Y,X)v(Z)+g(Z,Y)v(X)\\ &\qquad +6v(X)v(Y)v(Z)+v(X)g(Z,Y)\\ &\qquad +v(Y)g(X,Z)+v(Z)g(Y,X).\end{aligned}$$ ∎

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Rajendra Prasad, Amit Kumar Rai and S. S. Shukla
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALLAHABAD
ALLAHABAD -211 002, INDIA
E-mail: amit_patwadh@rediffmail.com

M. M. Tripathi
DEPARTMENT OF MATHEMATICS AND ASTRONOMY
UNIVERSITY OF LUCKNOW
LUCKNOW-226 01, INDIA

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