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PERSPECTIVE CASE OF THE PAPPUS THEOREM IN THE n-DIMENSIONAL PROJECTIVE SPACE

Abstract. A generalisation to the n-dimensional projective space $P_n(F)$ of the perspective case of the Pappus theorem is given. It is shown, additionally, that in such a case the ground field F might be non-commutative.

The well-known theorem of Pappus is one of the most important theorems of projective geometry. It is known ([5], p. 205) that a necessary and sufficient condition that Pappus' theorem be universally true in $P_n(F)$ is that F be commutative. On the other hand ([5], p. 275), a necessary and sufficient condition that a projectivity between two ranges be uniquely determined by the assignment of three pairs of corresponding points is that Pappus' theorem be true. That is why this theorem was a subject of many investigations. We may mention here for instance the works [1], [2], [3] and [4], where this theorem was generalised to n-dimensional projective space $P_n(F)$, where the ground field F is supposed to be commutative. In particular, the generalisation from [1] concerns two sets of points $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_n\}$ on two hyperplanes H_1 and H_2, respectively. The theorem says that the dimension of the join of subspaces (points in general) $S_0, \ldots, S_n$ is not greater than $n-1$ ($S_j = \bigcap_{i=0,\, i\neq j}^{n} S_{ij}$, where $S_{ij} = J(b_i, A \setminus \{a_i, a_j\})$, $i \neq j$ (the symbol $J(P_1, \ldots, P_m)$ denotes the join of subspaces $P_1, \ldots, P_m$)). Points $a_0, \ldots, a_n$ as well as $b_0, \ldots, b_n$ are assumed to be in a general position, i.e. no n of them are in an $(n-2)$-dimensional subspace. Obviously, when $n = 2$, this is the usual plane Pappus theorem.

In this work we present a much more general theorem than that from [4]. It concerns, as in [4], the perspective case of the theorem of Pappus. We do not assume here that the dimension of the space is even, and we consider an arbitrary integer k instead of $k = n/2$. However, the most important

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assumption is that the ground field F is arbitrary, not necessarily commutative.

REMARK. It is known that in such a case for $n = 2$ the Pappus line passes through the point $H_1 \cap H_2$. We shall show that an analogous property holds in our generalisation.

Throughout the paper we investigate two sets of points $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_n\}$ contained in two hyperplanes H_1 and H_2, respectively. Next we assume that there is a point S such that points S, a_i, b_i are collinear for $i = 0, \ldots, n$, and that points $a_0, \ldots, a_n$ are in a general position. Let k be an integer satisfying the condition $1 \leq k \leq n-1$. Let $I = \{0, \ldots, n\} \setminus \{i\}$, $J = \{j, \ldots, j+k-1\}$, $J^* = I \setminus J$, where $j \in I$, and I is supposed to be cyclically ordered (consequently J is cyclically ordered). Let

$$S_{ij} = J(\ldots, a_r, \ldots, b_s, \ldots), \qquad i \neq j,$$

where r and s run throughout sets J and J^*, respectively. Finally, we define

$$S_i = \bigcap_{j \in I} S_{ij}, \qquad j = 0, \ldots, n,$$

and

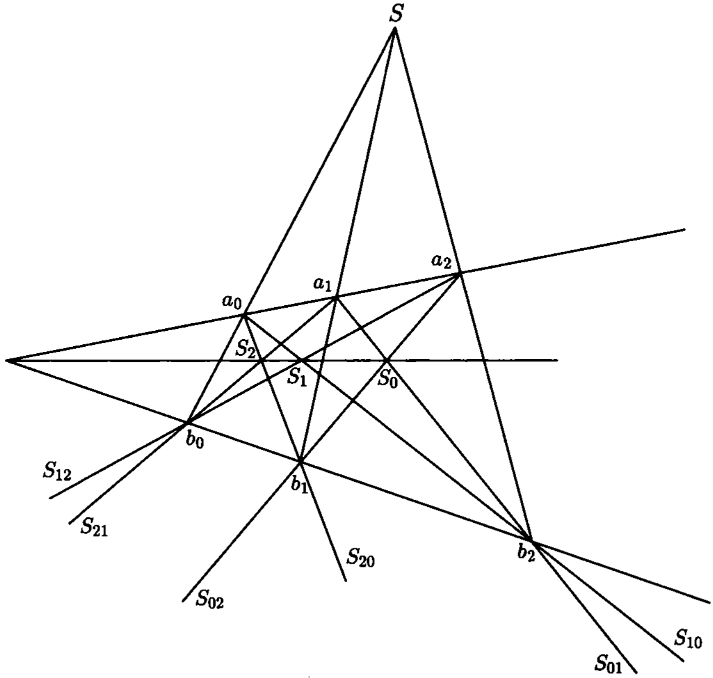
$$H = J(S_0, \ldots, S_n).$$

THEOREM. *The dimension of subspace H is not greater than $n-1$.*

First of all, notice that this is the classical Pappus' theorem (perspective case) when $n = 2$. In fact, we have then two straight lines H_1 and H_2, seven points $a_0, a_1, a_2 \in H_1$, $b_0, b_1, b_2 \in H_2$, S and six straight lines $S_{01} = J(a_1, b_2)$, $S_{02} = J(a_2, b_1)$, $S_{10} = J(a_0, b_2)$, $S_{12} = J(a_2, b_0)$, $S_{20} = J(a_0, b_1)$, $S_{21} = J(a_1, b_0)$ (of course $k = 1$). Theorem says that the dimension of the join of points $S_0 = S_{01} \cap S_{02}$, $S_1 = S_{10} \cap S_{12}$, $S_2 = S_{20} \cap S_{21}$ is not greater than 1, i.e. they are on a line (Fig.).

Proof of Theorem. Suppose, in addition, that some of points a_i, say $a_0, \ldots, a_{l-1}$, are in $H_1 \cap H_2$. Then $a_i = b_i$ for $i = 0, \ldots, l-1$. With respect to the assumptions it must be $l \leq n-1$. On the other hand, $0 \leq l$ (the equality $l = 0$ means that none of points a_i is in $H_1 \cap H_2$). By L we denote the set $\{0, \ldots, l-1\}$. Of course, $L = \emptyset$ when $l = 0$.

Choose in $P_n(F)$ an allowable coordinate system in such a way that the j-th coordinate of point a_i equals $c_i \delta_i^j$, the j-th coordinate of S equals δ_n^j and the equation of H_2 is $\sum_{i=l}^{n-1} x_i - x_n = 0$, where $i = 0, \ldots, n-1$, $j = 0, \ldots, n$, $c_i \neq 0$, and δ_i^j is the Kronecker δ. Hence the j-th coordinate of point b_i is equal to $c_i \delta_i^j$, for $i = 0, \ldots, l-1$, $j = 0, \ldots, n$. Similarly, for $i = l, \ldots, n-1$ the j-th coordinate of b_i equals $c_i \delta_i^j$, $j = 0, \ldots, n-1$, and the n-th coordinate of this point is equal to c_i. We may suppose, without loss of generality, that



Figure

homogeneous coordinates of points a_i are chosen in such a way that the i-th coordinate of point a_n equals c_i, $i = 0, \ldots, n-1$. Then the i-th coordinate of b_n is equal to c_i, $i = 0, \ldots, n-1$, and the n-th coordinate of this point equals $\sum_{i=l}^{n-1} c_i$.

It can be easily checked that hyperplanes S_{ij} have the following equations:

$$S_{ij} : \; c_i\Big(\sum_{t \in J^* \setminus (\{n\} \cup L)} x_t - x_n \Big) + x_i \sum_{t \in (J \cup \{i\}) \setminus L} c_t = 0 \text{ for } n \notin J \text{ and } i \leq n-1;$$

$$S_{ij} : \; c_i\Big(\sum_{t \in J^* \setminus L} x_t - x_n \Big) - x_i \sum_{t \in J^* \setminus L} c_t = 0 \quad \text{for } n \in J \text{ and } i \leq n-1;$$

$$S_{nj} : \; \sum_{t \in J^* \setminus L} x_t - x_n = 0.$$

Convention: $\sum_{t \in \emptyset} x_t = \sum_{t \in \emptyset} c_t = 0$.

Adding all equations of S_{ij} with fixed i ($i \leq n-1$) we obtain

$$c_i \Big[(n-k) \sum_{t=0}^{n-1} x_t - n x_n \Big] = 0.$$

Similarly, adding the equations of S_{nj} we have

$$(n-k)\sum_{t=0}^{n-1} x_t - nx_n = 0.$$

It means that all subspaces S_i are contained in the same hyperplane H_p : $(n-k)\sum_{t=0}^{n-1} x_t - nx_n = 0$.

REMARK 1. As it was expected hyperplane H_p contains $H_1 \cap H_2$.

REMARK 2. Theorem is not true without the assumption concerning the perspectivity of systems $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$.

REMARK 3. As it can be easily observed the property of commutativity of F was not used in the proof of Theorem. It shows the difference between the general case and perspective case of Pappus' theorem.

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