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A RESIDUAL SKEWAFFINE PLANE OF A MÖBIUS OR MINKOWSKI PLANE

Abstract. For Möbius and Minkowski planes of characteristic different from 2 a residual skewaffine plane associated with any point p is constructed. Following the construction given by Andre (cf. [1]) we obtain the residual plane as the group space of some normally transitive group of automorphisms fixing p. This is a skewaffine plane without straight lines in the Möbius case and with two families of straight lines in the Minkowski case.

Introduction

There is a well known construction of a derived affine plane associated with a fixed point p of any Benz plane. The lines are induced by circles through p and generators (not through p) in the case of Minkowski and Laguerre planes. Therefore Benz geometries are partially characterized by properties of affine planes. However, in this construction we loose the set of circles not passing through p. This set (possibly extended by generators not through p) to the points joinable[1] to p defines the so called *residual plane* at p. It is a natural idea to characterize it by some linear structure. A convenient tool is the notion of a *skewaffine plane* introduced by J. Andre (cf. [1]). It is a noncommutative linear structure. In [16] H. Wilbrink presented some conditions for Minkowski planes to define a so called *residual nearaffine plane* which is a special case of skewaffine plane with two families of straight lines (associated with generators). In [13] the present author gave a construction of a residual skewaffine plane in a Laguerre plane. This result used the general construction of the so called *group space* given by J. Andre in [1].

In this note we adopt the J. Andre construction to obtain a residual skewaffine plane at any point p of a miquelian Möbius or Minkowski plane of characteristic different from 2 as a group space (cf. Theorem 3.1). The

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[1]A point x is *joinable* to p if there exist a circle through x, p.

base of the construction is the group of *displacements* associated with p, i.e. compositions of two symmetries with respect to circles through p. If the circles are tangent or have two common points the displacement is called a *translation* or a *rotation* respectively. Following the construction of a group space any circle not through p is the orbit of a point y under the subgroup of rotations with center x joinable to x and forms together with x the line of the residual skewaffine plane with basepoint x (denoted by $x \sqcup y$) (cf. Proposition 3.1). In a Minkowski plane we additionally get straight lines in case y is joinable to x. A class of parallel lines of the residual skewaffine plane is the orbit of a line under the subgroup of translations. In the last section we define the harmonic relation and give some applications of the residual skewaffine plane to characterizing pencils of circles of Möbius and Minkowski planes.

1. Preliminaries

Let $\mathcal{P}$ be a nonempty set elements of which are called points. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ be disjoint subfamilies of $\mathcal{P}$, called generators, such that the following axioms hold:

(N1) For every point p and every $i \in I := \{1, \ldots, k\}$ there exists at most one generator E with $x \in E \in \mathcal{G}_i$.

(N2) If $i, j \in I$, $i \neq j$, $E \in \mathcal{G}_i$, $F \in \mathcal{G}_j$, then E meets F in a unique point.

We set $G := \bigcup_{i \in J} \mathcal{G}_i$, $[p]_i = E$ with $x \in E \in \mathcal{G}_i$ and $[p] := \bigcup_{i \in J} [p]_i$.

A subset $M \subset \mathcal{P}$ is called *joinable* if $\forall X \in \mathcal{G} : |M \cap X| \leq 1$. A maximal joinable set of points is called a *circle*. It is easy to check that a subset $C \subset \mathcal{P}$ is a circle iff the following condition is satisfied:

(N3) Every element $X \in \mathcal{G}$ intersects C in exactly one point.

Let $(\mathcal{P}, \mathcal{G})$ be any structure satisfying axioms (N1), (N2) and provided with a family $\mathcal{C}$ of circles. The structure $\mathcal{M} = (\mathcal{P}, \mathcal{C}, \mathcal{G}, \in)$ is a *circle plane* if $\mathcal{G}$ contains at most two families of generators and the following axioms hold:

(C1) Through three distinct joinable points p, q, r there is a unique circle K with $p, q, r \in K$ (notation $K := (p, q, r)^\circ$).

(C2) For any circle K and joinable points p, q such that $p \in K$, $q \notin K$ there is a unique circle L such that $q \in L$ and $K \cap L = \{p\}$.

(C3) There is a circle R with at least three points such that $\mathcal{P} \setminus R \neq \emptyset$.

A circle plane is called a *Möbius plane* or a *Minkowski plane* if it has 0 or 2 families of generators, respectively.

For any circle K and points x, y of a Minkowski plane we write $xy := [x]_1 \cap [y]_2$, $xK = [x]_1 \cap K$, $Kx := [x]_2 \cap K$. Additionally we put $[p] := \{p\}$ for any point p of a Möbius plane.

Let p be any point of a Möbius or Minkowski plane. We define $\mathbb{A}_p := (\mathcal{P}^p, \mathcal{C}^p, \in)$ where $\mathcal{P}^p = \mathcal{P} \setminus [p]$ and

$$\mathcal{C}^p = \begin{cases} \{K \cap \mathcal{P}^p \mid K \in \mathcal{P}, p \in K\} \text{ for a Möbius plane,} \\ \{K \cap \mathcal{P}^p \mid K \in \mathcal{P}, p \in K\} \cup \{E \cap \mathcal{P}^p \mid E \in \mathcal{G}, p \notin E\} \\ \hfill \text{for a Minkowski plane.} \end{cases}$$

Then $\mathbb{A}_p$ is an affine plane called the *derived plane* at the point p. We denote its projective extension by $\overline{\mathbb{A}_p}$. The order of $\mathbb{A}_p$ at any point is the same and is called the order of the Möbius (or Minkowski) plane, denoted by $\operatorname{ord}\mathcal{M}$.

An *automorphism* of a Möbius or Minkowski plane is a permutation of the point set which maps circles to circles and generators to generators. A *symmetry* with respect to a circle K is an involutory automorphism which fixes K pointwise. For any circle K there exists at most one symmetry denoted by S_K. If $p \in K$ then S_K induces a homology of $\overline{\mathbb{A}_p}$ with improper axis, hence there exists a circle L passing through p such that $\mathrm{S}_K(M) = M$ for any circle M tangent to L at p.

In the case of a plane for which every symmetry exists we can introduce the notion of the characteristic of the plane. It is the characteristic of $\overline{\mathbb{A}_p}$ at any point p (cf. [5], [14]).

A circle K of a Möbius or Minkowski plane is called *orthogonal* to a circle L, in symbols $K \perp L$, if $\mathrm{S}_K(L) = L$. This condition defines a symmetric relation on the set $\mathcal{C}$ (cf. [4]). In particular, if a plane is of characteristic different from 2 and $K \perp L$, then K is nontangent to L.

For joinable points p, q and a circle K passing through p we define $\langle p, K\rangle := \{L \in \mathcal{C} \mid L \cap K = \{p\}\} \cup \{K\}$, $\langle p, q\rangle := \{L \in \mathcal{C} \mid p, q \in L\}$ and call them the *parabolic pencil with vertex p and direction K* and the *hyperbolic pencil with vertices p, q* respectively. For any set $\mathcal{S} \in \mathcal{C}$ we define $\mathcal{S}^{\perp} := \{L \in \mathcal{C} \mid \forall M \in \mathcal{S} : \; L \perp M\}$. We have $\langle p, K\rangle^{\perp} = \langle p, L\rangle$ where $L \perp K$, $p \in K \cap L$ and $\langle p, q\rangle^{\perp} = \langle pq, qp\rangle$ for a Minkowski plane. The set $\{K, L\}^{\perp}$ will be called an *elliptic pencil* if $K \cap L = \emptyset$ for Minkowski planes, and if $|K \cap L| = 2$ for Möbius planes (cf. [17], [3]).

A *translation* or a *homothety* of a Minkowski or Möbius plane is an automorphism φ which has at least one fixed point p and φ induces a translation or a homothety resp. of the derived affine plane $\mathbb{A}_p$. For Möbius and Minkowski planes there exist translations with fixed pencil $\langle p, K\rangle$ for some $p \in \mathcal{P}$, $K \in \mathcal{C}$, $p \in K$. In the case of a Minkowski plane there is one more type of automorphism: translations with one pointwise fixed generator.

A homothety fixes two distinct joinable points p, q and the hyperbolic pencil $\langle p, q\rangle$. In the case of a Minkowski plane $\mathcal{M}$ with $\operatorname{ord}\mathcal{M} > 3$ any homothety preserves the sets $\mathcal{G}_1$ and $\mathcal{G}_2$.

The Miquel Theorem describes the classical class of Möbius and Minkowski planes:

(M) For any eight different and pairwise joinable points a, b, c, d, e, f, g, h if the quadruples $\{a,c,b,d\}$, $\{a,e,b,h\}$, $\{a,g,d,h\}$, $\{b,f,c,e\}$, $\{c,g,d,f\}$ are joinable then the quadruple $\{e,g,f,h\}$ is joinable.

Miquelian Möbius and Minkowski planes satisfy the so called *three reflection theorem* in any parabolic and hyperbolic pencil (cf. [11], [14]):

THEOREM 1.1. *If* $L_1, L_2, L_3 \in \langle p, K\rangle$ *or* $L_1, L_2, L_3 \in \langle p, q\rangle$ *where* p, q, K *are fixed joinable points and circle resp. with* $p \in K$*, then there exists a circle* $M \in \langle p, K\rangle$ *(*$M \in \langle p, q\rangle$ *resp.) such that* $\mathrm{S}_{L_3} \circ \mathrm{S}_{L_2} \circ \mathrm{S}_{L_1} = \mathrm{S}_M$.

In the remainder of the paper we will consider only miquelian Möbius and Minkowski planes of characteristic different from 2. Such planes are isomorphic to chain geometries $\Sigma(\mathbb{K},\mathbb{L})$ where $\mathbb{K}$ is a field ($\operatorname{char}\mathbb{K} \neq 2$) and $\mathbb{L} = \mathbb{K}(i)$ is an extension of $\mathbb{K}$ to a commutative algebra such that $\dim_{\mathbb{K}} \mathbb{L} = 2$ (cf. [3]). In the case of a Möbius plane, $\mathbb{L}$ is a field and the element i satisfies the condition $i^2 = -k$ for some $k \in \mathbb{K}$ ($k \neq 0, -1$). In the case of a Minkowski plane, $i^2 = 1$ and $\mathbb{L}$ is a commutative algebra with two maximal ideals.

THEOREM 1.2. *For any circles* K, L *of a parabolic pencil there exist exactly one circle* M *of this pencil such that* $\mathrm{S}_M(K) = L$.

We remark that this is not true for hyperbolic and elliptic pencils: some stronger restrictions on the field are needed.

According to the general description of a geometry $\Sigma(\mathbb{K},\mathbb{L})$ points of a miquelian Minkowski and Möbius geometry are elements of $\mathbf{P}(\mathbb{L})$ and $\mathcal{C} = \{\mathbf{P}(\mathbb{K})^\gamma \mid \gamma \in \mathbf{PGL}(2,\mathbb{L})\}$. It can be proved that any circle of a Möbius (Minkowski resp.) plane is the set

$$\left\{ [z_1, z_2] \;\middle|\; [z_1, z_2]\ \mathrm{M} \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \end{bmatrix} = 0 \right\}$$

for some $\mathrm{M} \in \mathcal{M}_{2\times 2}(\mathbb{L})$ such that $\overline{\mathrm{M}}^{\mathrm{T}} = \mathrm{M}$ and $\det \mathrm{M} \in -\mathrm{D}$ in the Möbius case (resp. $\det \mathrm{M}$ is invertible in the Minkowski case). The bar denotes the involutory automorphism of $\mathbb{L}$ such that $\overline{x+iy} = x - iy$ ($x, y \in \mathbb{K}$) and $\mathrm{D} := \{x^2 + ky^2 \mid x, y \in \mathbb{K}, (x,y) \neq (0,0)\}$. In a standard way we define

$$\mathrm{N}(x+iy) := (x+iy)(x-iy) = \begin{cases} x^2 + ky^2 & \text{for a Möbius plane,} \\ x^2 - y^2 & \text{for a Minkowski plane.} \end{cases}$$

For any $a, b \in \mathbb{L}$ we have $\mathrm{N}(a \cdot b) = \mathrm{N}(a) \cdot \mathrm{N}(b)$ and hence the set D is closed under multiplication.

The following analytical description of the derived plane at a point $p = [1, 0]$ is convenient. The points have representation $[x+iy, 1]$ where $x, y \in \mathbb{K}$. The lines are sets of points of equations $mx + ny + t = 0$ $(m, n, t \in \mathbb{K}, (m, n) \neq (0, 0))$ (and generators $x = n$, $y = k$ in the case of a Minkowski plane).

Because we will describe the structure of circles not passing through the point $p = [1, 0]$ we will use their explicit equations. The points $[x + iy, 1]$ of a circle satisfy the equation

$$(1.1) \qquad x^2 + ky^2 + mx + ny + d = 0 \quad (m, n, d \in \mathbb{K}, 4d - m^2 - \tfrac{n^2}{k} - \mathrm{D})$$

in the Möbius case, and

$$(1.2) \qquad x^2 - y^2 + mx + ny + d = 0 \quad (m, n, d \in \mathbb{K}, 4d - m^2 + n^2 \neq 0)$$

in the Minkowski case.

Any automorphism of a Möbius or Minkowski plane can be determined by a bijection Φ^{τ}_{M} of the form $\Phi^{\tau}_{\mathrm{M}} : [z_1, z_2] \to [z_1^{\tau}, z_2^{\tau}]\mathrm{M}$ where $\tau \in \mathrm{Aut}(\mathbb{K})$ and $M \in \mathbf{PGL}(2, \mathbb{L})$. In particular, $\Gamma(\mathbb{L}) := \{\Phi_{\mathrm{M}} \mid \mathrm{M} \in \mathbf{PGL}(2, \mathbb{L})\} \trianglelefteq \mathrm{Aut}(\mathcal{M})$, where $\Phi_{\mathrm{M}} := \Phi^{id}_{\mathrm{M}}$.

Symmetries with respect to a circle have the following analytic representation:

$$(1.3) \qquad [z_1, z_2] \to [\overline{z_1}, \overline{z_2}] \begin{bmatrix} a & ri \\ si & \overline{a} \end{bmatrix}$$

where $r, s \in \mathbb{K}$, $a \in \mathbb{L}$ and $\mathrm{N}(a) + krs \neq 0$ in the Möbius case (resp. $\mathrm{N}(a) - rs \neq 0$ in the Minkowski case).

A *skewaffine space* (cf. [1]) is an incidence structure $\mathbb{S} = (X, \sqcup, \parallel)$, where X is a nonempty set of points, denoted by small Latin letters, and

$$\sqcup : \{(x, y) \in X^2 \mid x \neq y\} \to 2^X$$

is a function. The sets of the form $x \sqcup y$ $(x \neq y)$ are called lines. They will be denoted by capital Latin letters. The symbol $\parallel$ denotes an equivalence relation among the lines. The following axioms must be satisfied:

(L1) $x, y \in x \sqcup y$,

(L2) $z \in x \sqcup y \setminus \{x\}$ implies $x \sqcup y = x \sqcup z$ (exchange condition),

(P1) given any line L and any point x there exists exactly one line $x \sqcup y$ parallel to L (Euclid's axiom),

(P2) $\forall x, x', y, y' : (x \neq y, x' \neq y' \wedge x \sqcup y \parallel x' \sqcup y') \to y \sqcup x \parallel y' \sqcup x'$ (symmetry condition),

(T) if x, y, z are pairwise different points such that $x \sqcup y \parallel x' \sqcup y'$, then there exists a point z' such that $x \sqcup z \parallel x' \sqcup z'$ and $y \sqcup z \parallel y' \sqcup z'$ (Tamaschke's condition).

If we assume $x = x'$ in axiom (T), then the axiom is called the affine Veblen condition (V).

We will consider additional conditions for a skewaffine space:

(Pgm) $\forall x, y, z \in X, \{x, y, z\}_{\neq} \exists w \in X \colon x \sqcup y \parallel z \sqcup w \wedge x \sqcup z \parallel y \sqcup w,$
(Des) $\forall u, x, y, z \in X, \{u, x, y, z\}_{\neq} : \; x' \in u \sqcup x \setminus \{u\} \to \exists y' \in u \sqcup y \setminus \{u\},$
$z' \in u \sqcup z \setminus \{u\}$ with $x \sqcup y \parallel x' \sqcup y', x \sqcup z \parallel x' \sqcup z', y \sqcup z \parallel y' \sqcup z'.$

A skewaffine plane satisfying the condition (Des) is called *desarguesian.*

If a line L has the form $x \sqcup y$ then the point x is called a *basepoint* of L. It is a simple consequence of the axioms that any line has either exactly one basepoint or all its points are basepoints (cf. [15]). A line all of whose points are basepoints is called a *straight line.* A line which is not straight (and hence has exactly one basepoint) is called a proper line.

A group $\mathbf{G}$ acting on a set X is called *normally transitive* if $\mathbf{G}$ is transitive and $\mathbf{G}_x \setminus \mathbf{G}_y \neq \emptyset$ for any $x, y \in X$ with $x \neq y$ ($\mathbf{G}_x$ denotes the stabilizer of the point x with respect to $\mathbf{G}$). For any group acting on a set X one can construct a *group space* $\mathbf{V}(\mathbf{G}) = (X, \sqcup, \parallel)$ with

- $x \sqcup y = \mathbf{G}_x\{x, y\} = \{x\} \cup \mathbf{G}_x y,$
- for any lines L, L', $L \parallel L'$ if there exists $g \in \mathbf{G}$ such that $gL = L'$.

The following theorem will be the basis of our construction ([1, p. 5], cf. also [15, Proposition 6.5, p. 94]).

THEOREM 1.3. *The group space* $\mathbf{V}(\mathbf{G})$ *with respect to a normally transitive group* $\mathbf{G}$ *is a desarguesian skewaffine space.*

A more detailed discussion of the properties of the group space $\mathbf{V}(\mathbf{G})$ can be found in [15].

A bijection $\gamma : X \to X$ is called an *automorphism* of a skewaffine plane $\mathbb{S} = (X, \sqcup, \parallel)$ with the set $\mathcal{L}$ of lines if the following axioms are satisfied:

(A1) $\forall x, y \in X, x \neq y : \; \gamma(x \sqcup y) = \gamma(x) \sqcup \gamma(y),$
(A2) $\forall L, L' \in \mathcal{L} : \; L \parallel L' \Rightarrow \gamma(L) \parallel \gamma(L').$

An automorphism γ is called a *dilatation* if additionally

(D) $\forall L \in \mathcal{L} : \; L \parallel \gamma(L).$

PROPOSITION 1.1. *For every* $g \in \mathbf{G}$ *the map* $\gamma_g : X \to X$, $\gamma_g(x) = gx$, *is a dilatation of the skewaffine plane* $\mathbf{V}(\mathbf{G})$ (cf. [1]).

The group of all dilatations of a skewaffine plane $\mathbb{S}$ will be denoted by $\operatorname{Dil} \mathbb{S}$.

2. The group of displacements associated with a point of a Möbius or Minkowski plane

Let p be a fixed point of a Möbius or Minkowski plane.

DEFINITION 2.1. An automorphism $\varphi \in (\mathrm{Aut}\,\mathcal{M})_p$ is called a *displacement* associated with p if φ is the composition of two symmetries with respect to circles containing p.

From Theorem 1.1 it follows that the set $\mathbf{G} := \{\mathrm{S}_K \circ \mathrm{S}_L \mid K, L \in \mathcal{C},\ p \in K \cap L\}$ of all displacements associated with p forms a subgroup of the group $(\mathrm{Aut}\,\mathcal{M})_p$. The group $\mathbf{G}$ contains only automorphisms of the following two kinds. If $K \cap L = \{p\}$ then $\mathrm{S}_K \circ \mathrm{S}_L(M) = M$ for any circle M such that $p \in M$ and $M \perp K$. It is a translation in the direction of the circle M. If $K \cap L = \{p, q\}$ for some joinable points p, q then $\mathrm{S}_K \circ \mathrm{S}_L$ has the fixed pencil $\langle p, q\rangle^{\perp}$. We will call this automorphism a *rotation*. The group $\mathbf{T} := \{\mathrm{S}_K \circ \mathrm{S}_L \mid K, L \in \mathcal{C},\ K \cap L = \{p\}\}$ is a transitive (on $\mathcal{P} \setminus [p]$) normal subgroup of $\mathbf{G}$ by Theorem 1.1. Hence we obtain

LEMMA 2.1. *For any point p of a Möbius or Minkowski plane the group $\mathbf{G}$ is normally transitive. The elements of $\mathbf{G}$ without fixed points (on $\mathcal{P} \setminus [p]$) are translations. They form a transitive normal subgroup $\mathbf{T} \trianglelefteq \mathbf{G}$.*

We remark that the group $\mathbf{G}$ is of type IIIA in the Hering classification of automorphism groups of Möbius planes (cf. [7]) and of type 17 of the analogous Klein-Kroll classification for Minkowski planes (cf. [9], [10]).

To get the analytical representation of the group $\mathbf{G}$ we assume that $p = [1, 0]$. Any symmetry with respect to a circle passing through p has the matrix $\begin{bmatrix} m & 0 \\ c & \overline{m} \end{bmatrix}$ where $m \in \mathbb{L}^*$, $c \in \mathbb{K}$. Hence $\mathbf{G} = \{\Phi_{\mathrm{M}} \in \Gamma(\mathbb{L}) \mid \begin{bmatrix} p & 0 \\ c & 1 \end{bmatrix},\ p \in \mathbb{L}^*, c \in \mathbb{K}, \mathrm{N}(p) = 1\}$ and $\mathbf{T} = \{\Phi_{\mathrm{M}} \in \Gamma(\mathbb{L}) \mid \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix},\ c \in \mathbb{L}\}$. We get $\mathbf{G}_{[0,1]} = \{\Phi_{\mathrm{M}} \mid \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix},\ \mathrm{N}(p) = 1\}$.

Direct calculations show that the condition $\mathrm{N}(p) = 1$ implies $p = \frac{v^2 - ku^2}{v^2 + ku^2} - \frac{2uv}{v^2 + ku^2} i$ for some $(u, v) \neq (0, 0)$ in the Möbius case and $p = -\frac{v^2 + u^2}{v^2 - u^2} - \frac{2uv}{v^2 - u^2} i$ for some u, v with $u \neq \pm v$ in the Minkowski case.

One can present any element of the group $\mathbf{G}_{[0,1]}$ as the composition of the symmetry with respect to the line $ux + vy = 0$ and the symmetry with respect to the line $x = 0$. Thus we obtain an interpretation of the parameters u, v in the representation of elements of $\mathbf{G}_{[0,1]}$ (cf. [3]).

In the case $\mathbb{K} = \mathbb{R}$ we get the groups of Euclidean and pseudoeuclidean rotations.

3. Residual plane of a Möbius or Minkowski plane

In the previous section we defined the normally transitive group $\mathbf{G}$ associated with any point p of a Möbius or Minkowski plane as a subgroup of $(\operatorname{Aut}\mathcal{M})_p$. Now we will investigate the group space $\mathbf{V}(\mathbf{G})$.

PROPOSITION 3.1. *For any circle K of a Möbius or Minkowski plane with $p \notin K$ there exists a point q such that $\mathbf{G}_p(x) = K \cap (\mathcal{P} \setminus [p])$ for any $x \in K \setminus ([p] \cup [q])$. Dually for any $q \in \mathcal{P} \setminus [p]$ and $x \in \mathcal{P} \setminus ([p] \cup [q])$, $\mathbf{G}_p(x) = K \cap (\mathcal{P} \setminus [p])$ for some $K \in \mathcal{C}$, $p \notin K$.*

If $x \in [q]_i \setminus [p]$ then $\mathbf{G}_q(x) = [q]_i \setminus [q]$ ($i = 1, 2$ for a Minkowski plane).

Proof. Let K be any circle such that $p \notin K$ and $q := \mathrm{S}(p)$. In the Minkowski case assume that $x, x' \in K$ and $L \in \langle p, q\rangle$, $x, x' \notin L$. If $a := (xL)(Lx)$ then the points $p, q, ax', x'a$ are concyclic. If $M := (p, q, ax', x'a)^\circ$ then $\mathrm{S}_m \circ \mathrm{S}_K(x) = x'$. □

According to Lemma 2.1 and Theorem 1.3 the group space $\mathbf{V}(\mathbf{G})$ is a skewaffine plane. It will be called the *residual plane* at the point p of a Möbius or Minkowski plane and denoted by $\mathcal{M}^p$ (cf. [16]). By the definition and Proposition 3.1, $\mathcal{M}^p = (\mathcal{P}^p, \mathcal{L}^p, \sqcup, \|, \in)$ where

- $\mathcal{P}^p := \mathcal{P} \setminus [p]$, $\mathcal{L}^p := \{(\{S_K(p)\} \cup K) \cap \mathcal{P}^p \mid K \in \mathcal{C}, p \notin K, \} \cup \{E \cap \mathcal{P}^p \mid E \in \mathcal{G}, p \notin E\}$,
- $x \sqcup y := \{\varphi(y) \mid \varphi \in \mathbf{G}_x\} \cup \{x\}$ for any $x, y \in \mathcal{P}^p$, $x \neq y$,
- $L_1 \parallel L_2 \Leftrightarrow \exists \varphi \in \mathbf{G} : \varphi(L_1) = L_2$.

From the construction it follows that any circle K, not through p, induces a line $x \sqcup y$ of $\mathcal{M}^p$ where $x = \mathrm{S}_K(p)$ and y is any point of $K \setminus [p]$. The point x will also be called the *basepoint of the circle K*.

We get

THEOREM 3.1. *For any miquelian Möbius or Minkowski plane $\mathcal{M}$ with $\operatorname{char}\mathcal{M} \neq 2$ and $\operatorname{ord}\mathcal{M} > 3$ and a point $p \in \mathcal{P}$ the residual plane $\mathcal{M}^p$ is a desarguesian skewaffine plane. $\mathcal{M}^p$ does not contain any straight lines in the case of a Möbius plane and contains two classes of parallel straight lines in the case of a Minkowski plane. Classes of parallel lines are orbits of the subgroup of translations of the group $\mathbf{G}$.*

Proof. It is sufficient to prove the last statement because the rest follows from Theorem 1.3, Lemma 2.1 and Proposition 3.1.

Let $\varphi \in \mathbf{G}_p$ and $L := a \sqcup b$ for some $a, b \in \mathcal{P}^p$. Then $\varphi(L) = \varphi(a) \sqcup \varphi(b)$. If $\tau \in \mathbf{G}$ is a translation such that $\tau(a) = \varphi(a)$ then $\varphi^{-1}\tau \in \mathbf{G}_a$, hence

$\varphi^{-1}\tau(L) = L$ and $\tau(L) = \varphi(L)$. This shows that any line parallel to L is of the form $\tau(L)$ for some translation τ. □

4. Harmonic quadruples and properties of pencils

LEMMA 4.1. *For any distinct joinable points a, b, c there exists exactly one circle K through c such that $\mathrm{S}_K(a) = b$.*

Proof. Let L, M be circles orthogonal to $N := (a, b, c)^\circ$ through a, c and b, c respectively. Then L, M are tangent at c. By Theorem 1.2, there exists exactly one circle K' such that $\mathrm{S}_{K'}(L) = M$. We also have $\mathrm{S}_{K'}(N) = N$, hence $\mathrm{S}_{K'}(a) = b$. If $\mathrm{S}_K(a) = b$ and $\mathrm{S}_K(c) = c$ then S_K interchanges the circles of the pencil $\langle c, N\rangle^\perp$ through a, b resp., hence $K = K'$. □

DEFINITION 4.1. We call distinct points c, d *harmonic conjugate* with respect to distinct points a, b if there exist pairwise orthogonal circles K, L, M such that $a, b, c, d \in K$, $a, b \in L$, $c, d \in M$.

From the definition it follows immediately that c, d are harmonic conjugate to a, b iff a, b are harmonic conjugate to c, d. From Lemma 4.1 it follows that there exists exactly one point harmonic conjugate to c with respect to a, b for any distinct joinable points a, b, c.

LEMMA 4.2. *Let q be a point of $\mathcal{M}^p$ and let $\mathbb{E}$ be the set of circles with basepoint q.*

(i) *If $\mathcal{M}$ is a Minkowski plane then $\mathbb{E} = \langle p, q\rangle^\perp$.*
(ii) *If $\mathcal{M}$ is a Möbius plane then $\mathbb{E}$ is an elliptic pencil.*

Proof. (i) By the construction of a basepoint for any circle K with basepoint q, $\mathrm{S}_K(p) = q$. Hence $pq, qp \in K$ and $\mathbb{E} = \langle pq, qp\rangle = \langle p, q\rangle^\perp$.

(ii) follows from the fact that $\langle p, q\rangle^\perp$ is an elliptic pencil. □

COROLLARY 4.1.

(i) *A rotation of a Minkowski plane is a homothety.*
(ii) *A rotation of a Möbius plane is a homothety iff it is an involution.*

Proof. Let $K \cap L = \{p, q\}$.

(i) The rotation $\mathrm{S}_K \circ \mathrm{S}_L$ fixes the circles of the pencil $\langle pq, qp\rangle$.

(ii) " $\Rightarrow$ " If the rotation $\mathrm{S}_K \circ \mathrm{S}_L$ is a homothety then it fixes the circles of the pencil $\langle p, q\rangle$. Let $x \neq p, q$, $M := (p, q, x)^\circ$ and N be the circle through x orthogonal to the circles K, L. Then $N \perp M$ and N, M have another common point x' by the assumption that the characteristic is different from 2. It follows that $\mathrm{S}_K \circ \mathrm{S}_L(x) = x'$ and $\mathrm{S}_K \circ \mathrm{S}_L(x') = x$ since it fixes M, N.

" $\Leftarrow$ " If $\mathrm{S}_K \circ \mathrm{S}_L$ is an involution then $K \perp L$ and the assertion follows from [8, Lemma 4.22, p. 101]. □

According to Proposition 1.1 every translation and rotation of a Möbius or Minkowski plane induces a dilatation of the residual plane at any fixed point. We prove that the group $\operatorname{Dil}\mathbf{V}(\mathbf{G})$ is larger.

LEMMA 4.3. *Every symmetry with respect to a circle passing through p induces a dilatation of the residual plane $\mathcal{M}^p$.*

Proof. Any automorphism φ of the plane $\mathcal{M}$ fixing p induces an automorphism of $\mathcal{M}^p$ since according to Proposition 3.1 it maps a pair $(\mathrm{S}_L(p), L)$ where $p \notin L \in \mathcal{C}$ to a pair $(\mathrm{S}_{\varphi(L)}(p), \varphi(L))$. If additionally $\varphi = \mathrm{S}_K$ where $p \in K \in \mathcal{C}$ we have

$$\mathrm{S}_K(L) = \mathrm{S}_M \circ \mathrm{S}_K(L) \quad \text{and} \quad \mathrm{S}_K(\mathrm{S}_L(p)) = \mathrm{S}_M \circ \mathrm{S}_K(\mathrm{S}_L(p)),$$

where $M := (p, \mathrm{S}_L(p), \mathrm{S}_K(\mathrm{S}_L(p))^\circ$, hence the displacement $\mathrm{S}_M \circ \mathrm{S}_K$ realizes parallelity of the lines associated with the circles $L, \mathrm{S}_K(L)$. Thus S_K induces a dilatation. □

PROPOSITION 4.1. *Circles of any hyperbolic and parabolic pencil of a Möbius or Minkowski plane have basepoints on a circle passing through p.*

Proof. Let $\langle a, b\rangle$ be any hyperbolic pencil of a Möbius or Minkowski plane where a, b are joinable to p and let q be harmonic conjugate to p with respect to a, b. If L denotes the unique circle of the pencil $\langle p, q\rangle$ orthogonal to $(a, b, p)^\circ$, then S_L fixes the circles of $\langle a, b\rangle$. By Proposition 4.1, S_L induces an automorphism of the residual plane $\mathcal{M}^p$. Hence S_L fixes the basepoints of the circles of the pencil $\langle a, b\rangle$. It follows that they belong to the circle L.

The proof in the case of a parabolic pencil is similar. □

Any elliptic pencil of a Möbius plane determines one of the two different possibilities for a residual plane. If $\mathbb{E} = \langle a, p\rangle^\perp$ then in $\mathcal{M}^p$ the pencil $\mathbb{E}$ determines the set of lines with basepoint a. The circles of $\mathbb{E}$ are disjoint and form a partition of the set $\mathcal{P}^p \setminus \{b\}$. This is a generalization of the set of concentric Euclidean circles. The second case is obtained if $p \neq a, b$.

PROPOSITION 4.2. *Let $\mathbb{E} = \langle a, b\rangle^\perp$ be an elliptic pencil of a Möbius plane and $p \neq a, b$. The set of basepoints in $\mathcal{M}^p$ of the circles of $\mathbb{E}$ is the circle $(p, a, b)^\circ$.*

Proof. Let $L := (a, b, p)^\circ$. By Proposition 4.1, S_L induces an automorphism of $\mathcal{M}^p$. It fixes the circles of $\mathbb{E}$, hence preserves their basepoints. Thus the basepoints of the circles of $\mathbb{E}$ belong to L. □

PROPOSITION 4.3. *The basepoints of parallel lines associated with circles tangent to a circle with basepoint q form a line after adding q as the basepoint.*

Proof. Let K, L be tangent circles with basepoints q, q' respectively. By Proposition 3.1 and Theorem 3.1, the orbit of q' under the group of rotations with center q is a circle with basepoint q. The orbit of L under this group is the set of circles tangent to K associated with parallel lines of $\mathcal{M}^p$. □

Finally, we give a generalization of the classical theorem about bisectrices for the configuration of circles of a miquelian Möbius or Minkowski plane of characteristic different from 2. It can be treated as a construction of the basepoint of a line of the residual plane $\mathcal{M}^p$ by means of symmetries with respect to lines of the derived plane $\mathcal{A}^p$.

PROPOSITION 4.4. *Assume that the points a, b, c are joinable; $p \notin K := (a,b,c)^\circ$, $A := (p,b,c)^\circ$, $B := (p,a,c)^\circ$, $C := (p,a,b)^\circ$; the points a', b', c' are harmonic to p with respect to (b,c), (a,c), (a,b) resp.; and the circles A', B', C' are the circles of the pencils $\langle p, a'\rangle$, $\langle p, b'\rangle$, $\langle p, c'\rangle$ orthogonal to A, B, C resp. Then A', B', C' belong to one hyperbolic pencil. The second point of intersection of these circles is symmetric to p with respect to K.*

Proof. The circles A', B' cannot be tangent, since otherwise B' is orthogonal to A, hence $\mathrm{S}_{B'}(c) = a \in A$, a contradiction with the assumption $p \notin K$. Let q be the second common point of the circles A', B', $Q := (p,q,c)^\circ$. By Theorem 1.1 there exists a circle R such that $\mathrm{S}_{A'} \circ \mathrm{S}_R \circ \mathrm{S}_{B'} = \mathrm{S}_R$. It is easy to check that $\mathrm{S}_R(a) = b$. By Lemma 4.1, $R = C'$. □

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