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Kuddusi Kayaduman, Hüsamettin Çoşkun

ON THE $\sigma^{(A)}$-SUMMABILITY AND $\sigma^{(A)}$-CORE

Abstract. In [6] and [9], the concepts of σ-core and statistical core of a bounded number sequence x have been introduced and also some inequalities which are analogues of Knopp's core theorem have been proved. In this paper, using the concept of $\sigma^{(A)}$-summability introduced by Savaş, we characterize the matrices of the classes $(m, V_{\sigma^{(A)}}), (c, V_{\sigma^{(A)}}), (V_\sigma, V_{\sigma^{(A)}})$ and $(S \cap m, V_{\sigma^{(A)}})_{reg}$ and determine necessary and sufficient conditions for a matrix B to satisfy $\sigma^{(A)}-core(Bx) \subseteq K-core(x), \sigma^{(A)}-core(Bx) \subseteq \sigma - core(x)$ and $\sigma^{(A)} - core(Bx) \subseteq st - core(x)$, for all $x \in m$.

1. Introduction

Let K be a subset of $\mathbb{N}$, the set of positive integers. The natural density δ of K is defined by

$$\delta(K) = \lim_n \frac{1}{n}|k \leq n : k \in K|,$$

where the vertical bars indicate the number of elements in the enclosed set. The number sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every ε, $\delta(\{k : |x_k - \ell| \geq \varepsilon\}) = 0$ (see [6]). In this case, we write $st - \lim x = \ell$. We shall also write S and S_0 to denote the sets of all statistically convergent sequences and of all sequences statistically convergent to zero. The statistically convergent sequences were studied by several authors (see [1], [6] and others).

Let m and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual supremum norm. Let σ be a one-to-one mapping from $\mathbb{N}$ into itself. A continuous linear functional ϕ on m said to be an invariant mean or a σ-mean if (i) $\Phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k, (ii) $\Phi(e) = 1$, where $e = (1, 1, 1, \ldots)$, (iii) $\Phi((x_{\sigma(k)})) = \Phi(x)$ for all $x \in m$.

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Throughout this paper we consider the mapping σ such that $\sigma^p(k) \neq k$ for all positive integers $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ is the pth iterate of σ at k. Thus, a σ-mean extends the limit functional on c in the sense that $\Phi(x) = \lim x$ for all $x \in c$ (see [10]). Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-means are equal.

In case $\sigma(k) = k + 1$, a σ-mean is often called a Banach limit and V_σ is the set of almost convergent sequences, introduced by Lorentz (see [7]). If $x = (x_n)$, write $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown [13] that

$$V_\sigma = \{x \in m : \lim_p t_{pn}(x) = s \text{ uniformly in } n,\ s = \sigma - \lim x\}$$

where

$$t_{pn}(x) = (x_n + Tx_n + \cdots + T^p x_n)/(p+1), \quad t_{-1,n}(x) = 0.$$

We say that a bounded sequence $x = (x_k)$ is σ-convergent if $x \in V_\sigma$. By Z, we denote the set of σ-convergent sequences with σ-limit zero. It is well known [12] that $x \in m$ if and only if $Tx - x \in Z$.

A matrix A called Cesaro matrix if $a_{nk} = 1/n$ if $1 \leq k \leq n$; $a_{nk} = 0$ if $k > n$ (see [8]).

It is known that [14], a bounded sequence x is said to be $\sigma^{(A)}$-convergent (or $\sigma^{(A)}$-summable) to ℓ if

$$\lim_m \sum_n a_{mn} x_{\sigma^n(p)} = \ell \text{ uniformly in } p.$$

The space of all $\sigma^{(A)}$- convergence and $\sigma^{(A)}$- convergence to zero sequence are denoted by $V_{\sigma^{(A)}}$ and $V_{0\sigma^{(A)}}$, respectively.

In case of A of matrix being taken into Cesaro matrix, the space $V_{\sigma^{(A)}}$ is reduced to the space V_σ.

Let A be an infinite matrix of real entries a_{nk} and $x = (x_k)$ be a real number sequence. Then $Ax = ((Ax)_n) = (\sum_k a_{nk} x_k)$ denotes the transformed sequence of x. If X and Y are two non-empty sequence spaces, then we use (X, Y) to denote the set of all matrices A such that Ax exists and $Ax \in Y$ for all $x \in X$. Throughout, $\sum_k$ will denote summation from $k = 1$ to ∞.

A matrix A is called (i) regular if $A \in (c, c)_{reg}$ and $\lim Ax = \lim x$, (ii) σ-regular if $A \in (c, V_\sigma)_{reg}$ and $\sigma - \lim Ax = \lim x$ for all $x \in c$, and (iii) σ-coercive if $A \in (m, V_\sigma)$. The necessary and sufficient conditions for A to be regular, σ-regular and σ-coercive are well-known [8] and [13].

For any real number λ we write $\lambda^- = max\{-\lambda, 0\}$, $\lambda^+ = max\{0, \lambda\}$. Then $\lambda = \lambda^+ - \lambda^-$. We recall (see [9]) that a matrix B is said to be σ-uniformly positive if

$$\lim_p \sum_k b^-(p,n,k) = 0 \quad \text{uniformly in } n$$

where

$$b(p,n,k) = \frac{1}{p+1} \sum_{i=0}^{p} b_{\sigma^i(n)}.$$

It is known [9] that a σ-regular matrix B is σ-uniformly positive if and only if

$$\lim_p \sum_k |b(p,n,k)| = 1 \quad \text{uniformly in } n.$$

Let us consider the following functional defined on m:

$$\ell(x) = \liminf x, \qquad L(x) = \limsup x, \qquad q_\sigma(x) = \limsup_p \sup_n t_{pn}(x),$$

$$L^*(x) = \limsup_p \sup_n \frac{1}{p+1} \sum_{i=0}^{p} x_{n+i}.$$

In [9], the σ-core of a real bounded number sequence x has been defined as the closed interval $[-q_\sigma(-x), q_\sigma(x)]$ and also the inequalities $q_\sigma(Ax) \leq L(x)$ (σ-core of $Ax \subseteq K$-core of x), $q_\sigma(Ax) \leq q_\sigma(x)$ (σ-core of $Ax \subseteq \sigma$-core of x), for all $x \in m$, have been studied. Here the K-core of x (or Knopp core of x) is interval $[\ell(x), L(x)]$ (see [2]).

When $\sigma(n) = n+1$, since $q_\sigma(x) = L^*(x)$, σ-core of x is reduced to the Banach core of x (B-core) defined by the interval $[-L^*(-x), L^*(x)]$ (see [11]).

The concepts of B-core and σ-core have been studied by many authors [4, 5, 9, 11].

Recently, Fridy and Orhan [6] have introduced the notions of statistical boundedness, statistical limit superior ($st - \limsup$) and inferior ($st - \liminf$), defined the statistical core (or briefly st-core) of a statistcally bounded sequence as the closed interval $[st - \liminf x, st - \limsup x]$ and also determined necessary and sufficient conditions for a matrix A to yield K-core$(Ax) \subseteq st$-core(x) for all $x \in m$.

DEFINITION 1.1. Let $x \in m$. Then, $\sigma^{(A)}$-core of x is defined by the closed interval $[-q_{\sigma^{(A)}}(-x), q_{\sigma^{(A)}}(x)]$, where

$$q_{\sigma^{(A)}}(x) = \limsup_m \sup_p \sum_n a_{mn} x_{\sigma^n(p)}.$$

From the definition, it is easy to see that $\sigma^{(A)}$- core $x = \{\ell\}$ if and only if $\sigma^{(A)} - \lim x = \ell$. In case of A matrix being taken into Cesaro matrix, since $q_{\sigma^{(A)}}(x) = q_\sigma(x)$, the $\sigma^{(A)}$-core of x is reduced to the σ-core of x.

2. Main results

The proofs of the following theorems are entirely analogous to the proof of Theorem 2.4. So, we omit the proofs.

THEOREM 2.1. *Let* $||A|| < \infty$. *Then* $B \in (m, V_{\sigma(A)})$ *if and only if*

$$||B|| = \sup_n \sum_k |b_{nk}| < \infty, \tag{2.1}$$

$$\lim_m \sum_n a_{mn} b_{\sigma^n(p),k} = \alpha_k \quad \text{uniformly in } p, \text{ for each } k, \tag{2.2}$$

$$\lim_m \sum_k \left| \sum_n a_{mn} b_{\sigma^n(p),k} - \alpha_k \right| = 0. \tag{2.3}$$

If the conditions (2.1)–(2.3) *hold, then* $\sigma^{(A)} - \lim B_n(x) = \sum_k \alpha_k x_k$ *for all* $x \in m$.

THEOREM 2.2. *Let* $||A|| < \infty$. *Then* $B \in (c, V_{\sigma(A)})$ *if and only if the conditions* (2.1) *and* (2.2) *hold, and*

$$\lim_m \sum_k \sum_n a_{mn} b_{\sigma^n(p),k} = \alpha \quad \text{uniformly in } p\ . \tag{2.4}$$

If the conditions (2.1), (2.2) *and* (2.4) *hold, then*

$$\sigma^{(A)} - \lim B_n(x) = \sum_k \alpha_k x_k + \ell(\alpha - \sum_k \alpha_k)$$

for all $x \in c$.

In the cases of matrix A being taken into Cesaro matrix Theorem 2.1 and Theorem 2.2, we respectively have Theorem 2, and Theorem 3 of Schaefer, [13].

THEOREM 2.3. *Let* $||A|| < \infty$. *Then* $B \in (V_\sigma, V_{\sigma(A)})$ *if and only if the conditions* (2.1) *and* (2.4) *hold and* $B(T - I) \in (m, V_{\sigma(A)})$.

THEOREM 2.4. *Let* $||A|| < \infty$. *Then,* $B \in (S \cap m, V_{\sigma(A)})_{reg}$ *if and only if* $B \in (c, V_{\sigma(A)})_{reg}$ *and*

$$\lim_m \sum_{k \in E} \left| \sum_n a_{mn} b_{\sigma^n(p),k} \right| = 0 \quad \text{uniformly in } p, \tag{2.5}$$

for every $E \subseteq \mathbb{N}$ *with* $\delta(E) = 0$.

Proof. First, suppose that $B \in (S \cap m, V_{\sigma(A)})_{reg}$. Then, $B \in (c, V_{\sigma(A)})_{reg}$ immediately follows from the fact that $c \subset S \cap m$. Now, define a sequence $t = (t_k)$ for $x \in m$ as

$$t_k = \begin{cases} x_k, & k \in E, \\ 0, & k \notin E, \end{cases}$$

where E is any subset of $\mathbb{N}$ with $\delta(E) = 0$. By our assumption, since $t \in S_0$, we have $Bt \in V_0\sigma^{(A)}$. On the other hand, since $Bt = \sum_{k\in E} b_{nk}x_k$, the matrix $D = (d_{nk})$ defined by

$$d_{nk} = \begin{cases} b_{nk}, & k \in E, \\ 0, & k \notin E, \end{cases}$$

for all n, must belong to the class $(m, V_{\sigma^{(A)}})$. Hence, necessity of (2.5) follows from Theorem 2.1.

Conversely, suppose that $B \in (c, V_{\sigma^{(A)}})_{reg}$ and (2.5) holds. Let x be any sequence in $S \cap m$ with $st - \lim x = \ell$. Write $E = \{k : |x_k - \ell| \geq \varepsilon\}$ for any given $\varepsilon > 0$, so that $\delta(E) = 0$. Since $B \in (c, V_{\sigma^{(A)}})_{reg}$ and $\sigma^{(A)} - \lim \sum_k b_{nk} = 1$, we have

$$\begin{aligned} \sigma^{(A)} - \lim(Bx) &= \sigma^{(A)} - \lim\Big(\sum_k b_{nk}(x_k - \ell) + \ell \sum_k b_{nk}\Big) \\ &= \sigma^{(A)} - \lim \sum_k b_{nk}(x_k - \ell) + \ell \\ &= \lim_m \sum_k \sum_n a_{mn} b_{\sigma^n(p),k}(x_k - \ell) + \ell. \end{aligned}$$

On the other hand, since

$$\Big|\sum_k \sum_n a_{mn} b_{\sigma^n(p)}(x_k - \ell)\Big| \leq ||x|| \sum_{k\in E} \Big|\sum_n a_{mn} b_{\sigma^n(p),k}\Big| + \varepsilon ||A||\,||B||,$$

the condition (2.5) implies that

$$\lim_m \sum_k \sum_n a_{mn} b_{\sigma^n(p)}(x_k - \ell) = 0 \quad \text{uniformly in } p.$$

Hence, $\sigma^{(A)} - \lim(Bx) = st - \lim x$; that is, $B \in (S \cap m, V_{\sigma^{(A)}})_{reg}$, which completes the proof. ∎

In case of A matrix being taken into Cesaro matrix, Theorem 2.4 is reduced to following theorem:

THEOREM 2.5 ([3]). *$B \in (S \cap m, V_\sigma)_{reg}$ if and only if B is σ-regular and*

$$\lim_p \sum_{k\in E} \left|\frac{1}{p+1} \sum_{i=0}^{p} b_{\sigma^i(n)}\right| = 0 \quad \textit{uniformly in } n,$$

for every $E \subseteq \mathbb{N}$ with natural density zero.

3. Core theorems for infinite matrices

We need the following lemma given by Das for the proof of next theorem:

LEMMA 3.1. *Let $||C = (c_{mk}(p))|| < \infty$ and $\lim_m \sup_p |c_{mk}(p)| = 0$. Then, there is a $y \in m$ such that $||y|| \leq 1$ and*

$$\lim_m \sup \sup_p \sum_k c_{mk}(p) y_k = \lim_m \sup \sup_p \sum_k |c_{mk}(p)|.$$

THEOREM 3.2. *Let $||A|| < \infty$. Then $\sigma^{(A)} - core(Bx) \subseteq K - core(x)$ for all $x \in m$ if and only if $B \in (c, V_{\sigma(A)})_{reg}$ and*

$$\lim_m \sup_p \sum_k \left| \sum_n a_{mn} b_{\sigma^n(p),k} \right| = 1. \tag{3.1}$$

Proof. Suppose first that $\sigma^{(A)} - core(Bx) \subseteq K - core(x)$ for all $x \in m$. If $x \in V_{\sigma(A)}$, then we have $q_{\sigma(A)}(Bx) = -q_{\sigma(A)}(-Bx)$. By hypothesis, we get

$$-L(-x) \leq -q_{\sigma(A)}(-Bx) \leq q_{\sigma(A)}(Bx) \leq L(x).$$

If $x \in c$, then $L(x) = -L(-x) = \lim x$. So we have $\sigma^{(A)} - \lim Bx = q_{\sigma(A)}(Bx) = -q_{\sigma(A)}(-Bx) = \lim x$ which implies that $B \in (c, V_{\sigma(A)})_{reg}$.

Now, let us consider the sequence $C = (c_{mk}(p))$ of infinite matrices defined by

$$c_{mk}(p) = \sum_n a_{mn} b_{\sigma^n(p),k} \quad \text{for all } n, k, p \in \mathbb{N}.$$

Then, it is easy to see that the conditions of the Lemma 3.1 are satisfied for the matrix sequence C. Thus, by using the hypothesis, we can write

$$1 \leq \lim_m \inf \sup_p \sum_k |c_{mk}(p)| \leq \lim_m \sup \sup_p \sum_k |c_{mk}(p)|$$
$$= \lim_m \sup \sup_p \sum_k c_{mk}(p) y_k = q_{\sigma(A)}(By) \leq L(y) \leq ||y|| \leq 1.$$

This gives the necessity of (3.1).

Conversely, assume $B \in (c, V_{\sigma(A)})_{reg}$ and (3.1) holds for all $x \in m$. Then, for any given $\varepsilon > 0$, there is a $k_0 \in \mathbb{N}$ such that $x_k < L(x) + \varepsilon$ for all $k > k_0$. Now, we can write

$$\sum_k c_{mk}(p) x_k = \sum_{k<k_0} c_{mk}(p) x_k + \sum_{k \geq k_0} (c_{mk}(p))^+ x_k - \sum_{k \geq k_0} (c_{mk}(p))^- x_k$$
$$\leq ||x|| \sum_{k<k_0} |c_{mk}(p)| + (L(x) + \varepsilon) \sum_k |c_{mk}(p)|$$
$$+ ||x|| \sum_k (|c_{mk}(p)| - c_{mk}(p)).$$

Thus, by applying the $\lim_m \sup \sup_p$ and using hypothesis, we have

$$q_{\sigma(A)}(Bx) \leq L(x) + \varepsilon.$$

This completes the proof since ε is arbitrary and for all $x \in m$. ∎

In case of A matrix being taken into Cesaro matrix, Theorem 3.2 is reduced to following theorem:

THEOREM 3.3 ([9]). *σ-Core of $Bx \subseteq K$-Core of x, i.e., $q_\sigma(Bx) \leq L(x)$ for all $x \in m$ if and only if B is σ-regular and σ-uniformly positive.*

THEOREM 3.4. *Let $||A|| < \infty$. Then $\sigma^{(A)} - core(Bx) \subseteq \sigma - core(x)$ for all $x \in m$ if and only if $B \in (V_\sigma, V_{\sigma^{(A)}})_{reg}$ and (3.1) holds.*

Proof. Let $\sigma^{(A)} - core(Bx) \subseteq \sigma - core(x)$ for all $x \in m$. Then, since $q_{\sigma^{(A)}}(Bx) \leq q_\sigma(x)$ and $q_\sigma(x) \leq L(x)$ for all $x \in m$, the necessity of (3.1) follows from Theorem 3.2.

One can also easily see that

$$-q_\sigma(-x) \leq -q_{\sigma^{(A)}}(-Bx) \leq q_{\sigma^{(A)}}(Bx) \leq q_\sigma(x),$$

i.e,

$$\sigma - \liminf x \leq -q_{\sigma^{(A)}}(-Bx) \leq q_{\sigma^{(A)}}(Bx) \leq \sigma - \limsup x.$$

If $x \in V_\sigma$, then $\sigma - \liminf x = \sigma - \limsup x = \sigma - \lim x$. Thus, the last inequality that $\sigma - \lim x = -q_{\sigma^{(A)}}(-Bx) = q_{\sigma^{(A)}}(Bx) = \sigma^{(A)} - \lim(Bx)$, that is, $A \in (V_\sigma, V_{\sigma^{(A)}})_{reg}$.

Conversely, suppose that (3.1) holds. In this case, since $c \subset V_\sigma$, by using Theorem 3.2, we have $q_{\sigma^{(A)}}(Bx) \leq L(x)$ for all $x \in m$. Thus, we write

$$\inf_{z \in V_{0\sigma}} q_{\sigma^{(A)}}(B(x+z)) \leq \inf_{z \in V_{0\sigma}} L(x+z) = w(x). \tag{3.2}$$

On the other hand, we have

$$\inf_{z \in V_{0\sigma}} q_{\sigma^{(A)}}(B(x+z)) \geq \inf_{z \in V_{0\sigma}} [q_{\sigma^{(A)}}(Bx) + (-q_\sigma(-Bz)] = q_{\sigma^{(A)}}(Bx), \tag{3.3}$$

since $-q_{\sigma^{(A)}}(-Bx) = q_{\sigma^{(A)}}(Bz) = 0$ for all $z \in V_{0\sigma^{(A)}}$. Now, combining (3.2) and (3.3), we obtain that $q_{\sigma^{(A)}}(Bx) \leq w(x)$ for all $x \in m$ which completes the proof, since $q_\sigma(x) = w(x)$, [9]. ∎

In case of A matrix being taken into Cesaro matrix, Theorem 3.4 is reduced to following theorem:

THEOREM 3.5 ([9]). *For an infinite matrix $B = (b_{nk})$, σ-Core of $Bx \subseteq \sigma$-Core of x, i.e., $q_\sigma(Bx) \subseteq \sigma(x)$ for all $x \in m$, if and only if B is strongly σ-regular and σ-uniformly positive.*

THEOREM 3.6. *Let $||A|| < \infty$. Then $\sigma^{(A)} - core(Bx) \subseteq st - core(x)$ for all $x \in m$ if and only if $B \in (S \cap m, V_{\sigma^{(A)}})_{reg}$ and (3.1) holds.*

Proof. Assume that $\sigma^{(A)} - core(Bx) \subseteq st - core(x)$ for all $x \in m$. Then $q_{\sigma^{(A)}}(Bx) \leq \beta(x)$ for all $x \in m$ where $\beta(x) = st - \limsup x$. Hence, since $\beta(x) = st - \limsup x \leq L(x)$ for all $x \in m$ (see [6]), we have (3.1) from

Theorem 3.2. Furthermore, one can also easily see that

$$-\beta(-x) \leq -q_{\sigma^{(A)}}(-Bx) \leq q_{\sigma^{(A)}}(Bx) \leq \beta(x),$$

i.e.,

$$st-\liminf x \leq -q_{\sigma^{(A)}}(-Bx) \leq q_{\sigma^{(A)}}(Bx) \leq st-\limsup x.$$

If $x \in S \cap m$, then $st-\liminf x = st-\limsup x = st-\lim x$. Thus, the last inequality implies that $st-\lim x = -q_{\sigma^{(A)}}(-Bx) = q_{\sigma^{(A)}}(Bx) = \sigma^{(A)}-\lim Bx$, that is, $B \in (S \cap m, V_{\sigma^{(A)}})_{reg}$.

Conversely, assume $A \in (S \cap m), V_{\sigma^{(A)}})_{reg}$ and (3.1). If $x \in m$, then $\beta(x)$ is finite. Let E be a subset of $\mathbb{N}$ defined by $E = \{k : x_k > \beta(x)+\varepsilon\}$ for a given $\varepsilon > 0$. Then it is obvious that $\delta(E) = 0$ and $x_k \leq \beta(x)+\varepsilon$ if $k \notin E$.

Now, we can write

$$\begin{aligned}
\sum_k c_{mk}(p)x_k &= \sum_{k<k_0} c_{mk}(p)x_k + \sum_{k\geq k_0} c_{mk}(p)x_k \\
&= \sum_{k<k_0} c_{mk}(p)x_k + \sum_{k\geq k_0} c^+_{mk}(p)x_k - \sum_{k\geq k_0} c^-_{mk}(p)x_k \\
&\leq \|x\| \sum_{k<k_0} |c_{mk}(p)| + \sum_{\substack{k\geq k_0\\ k\notin E}} c^+_{mk}(p)x_k + \sum_{\substack{k\geq k_0\\ k\in E}} c^+_{mk}(p)x_k \\
&\quad + \|x\| \sum_{k\geq k_0} (|c_{mk}(p)| - c_{mk}(p)) \leq \|x\| \sum_{k<k_0} |c_{mk}(p)| \\
&\quad + (\beta(x)+\varepsilon) \sum_{\substack{k\geq k_0\\ k\notin E}} |c_{mk}(p)| + \|x\| \sum_{\substack{k\geq k_0\\ k\in E}} |c_{mk}(p)| \\
&\quad + \|x\| \sum_{k\geq k_0} (|c_{mk}(p)| - c_{mk}(p)).
\end{aligned}$$

Applying the operator $\lim_m \sup \sup_p$ and using hypothesis, it follows that $q_{\sigma^{(A)}}(Bx) \leq \beta(x)+\varepsilon$. This completes the proof since ε is arbitrary. ∎

In the special case $A = (C, 1)$ of Theorem 3.6, we have the following theorem:

Theorem 3.7 ([3]). *$\sigma-core(Ax) \subseteq st-core(x)$ for all $x \in m$ if and only if $B \in (S \cap m, V_\sigma)_{reg}$ and B is σ-uniformly positive.*

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K. Kayaduman
GAZIANTEP ÜNIVERSITESI FEN EDEBIYAT FAKÜLTESI MATEMATIK BÖLÜMÜ
27310-GAZIANTEP, TURKEY
E-mail: kayaduman@gantep.edu.tr

H. Çoşkun
INÖNÜ ÜNIVERSITESI EGITIM FAKÜLTESI MATEMATIK BÖLÜMÜ
44100-MALATYA, TURKEY
E-mail: hcoskun@inonu.edu.tr

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Naseer Shahzad

A BEST PROXIMITY PAIR THEOREM

Abstract. The aim of this note is to obtain a best proximity pair theorem which contains a recent result of Kirk, Reich and Veeramani (*Numer. Funct. Anal. Optim.*, **24** (2003), 851–862) as a special case.

1. Introduction and preliminaries

Let A and B be nonempty closed convex subsets of a Hilbert space H. We denote by $K(B)$ the family of all nonempty compact subsets of H. The Hausdorff metric is defined by

$$D(C,G) = \max\left\{\sup_{c\in C} dist(c,G),\ \sup_{g\in G} dist(g,C)\right\}$$

for nonempty closed bounded subsets C and G of H, where $dist(c,G) = \inf_{g\in G} ||c-g||$. Throughout P_A will represent the nearest point projection of H onto A. It is well-known that P_A is nonexpansive. Let $f : A \to A$. A mapping $T : A \to K(B)$ is called f-Lipschitz if there exists $k \geq 0$ such that $D(Ta,Tb) \leq k||fa - fb||$ for any $a,b \in A$. If $0 \leq k < 1$ (resp. $k = 1$), then T is called an f-contraction (resp. f-nonexpansive mapping). A point $a \in A$ is called a coincidence point of f and T if $fa \in Ta$. The set of coincidence points of f and T is denoted by $C(f,T)$. A point $a \in A$ is called a coincidence point of f (resp. T) if $a = fa$ (resp. $a \in Ta$). The set of fixed points of f (resp. T) is represented by $F(f)$ (resp. $F(T)$). The mapping f is called weakly continuous if $\{a_n\}$ converges weakly to a_0 implies $\{fa_n\}$ converges weakly to fa_0. The notion of R-subweakly commuting multimaps was introduced by Shahzad [3]. Let $f : A \to A$ and $T : A \to K(A)$. Suppose $p \in F(f)$. Then the pair $\{f,T\}$ is said to be R-subweakly commuting if for all $a \in A$, fTa is a nonempty closed subset of A and there exists $R > 0$ such

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that $D(Tfa, fTa) \leq R\,dist(fa, [Ta, p])$, where $[Ta, p] = \{T_\lambda a : \lambda \in [0,1]\}$ and $T_\lambda x := \lambda Ta + (1-\lambda)p$. It is clear that commuting mappings on A are R-subweakly commuting. However, the converse is not true in general (see [3]).

We recall the following notations (see [1]). For any nonempty subsets A, B of E,

$$dist(A, B) = \inf\{||a - b|| : a \in A, b \in B\}$$
$$A_0 := \{a \in A : ||a - b|| = dist(A, B) \text{ for some } b \in B\}$$
$$B_0 := \{b \in B : ||a - b|| = dist(A, B) \text{ for some } a \in A\}.$$

A pair $(a_0, b_0) \in A_0 \times B_0$ is called a best proximity pair for A and B. In particular, $d(a_0, b_0) = dist(A, B)$. A mapping $T : A \to K(B)$ is said to have a best proximity pair solution if there exists a best proximity pair $(a_0, b_0) \in A_0 \times B_0$ such that $b_0 \in Ta_0$.

Fixed point theory is a useful tool for solving various types of operator equations and operator inclusions. The well-known approximation theorem of Fan yields the existence of approximate solutions; however, it does not give optimal solutions. On the other hand, best proximity pair theorems guarantee approximate solutions which are also optimal. Recently, Kirk, Reich and Veeramani [1] obtained the following remarkable best proximity pair theorem for nonexpansive mapping.

THEOREM 1.1. *Let H be a Hilbert space. Let A and B be nonempty closed convex subsets of H with A bounded. Let $T : A \to K(B)$ be such that*
(a) $T(A_0) \subseteq B_0$
(b) T is nonexpansive on A.
Then there exists $x_0 \in A$ such that

$$dist(a_0, Ta_0) = dist(A, B) = \inf\{dist(a, Ta) : a \in A\}.$$

More recently, O'Regan and Shahzad [2] extended Theorem 1.1 to f-nonexpansive mappings with an assumption that $\{f, P_A \circ T\}$ be R-subweakly commuting on A_0.

THEOREM 1.2. *Let H be a Hilbert space. Let A and B be nonempty closed convex subsets of H with A bounded. Let $f : A \to A$ be continuous and affine such that $f(A_0) = A_0$ and $T : A \to K(B)$ be such that*
(a) $T(A_0) \subseteq B_0$
(c) $\{f, P_A \circ T\}$ is R-subweakly commuting on A_0
(d) T is f-nonexpansive on A_0.
Then there exists $a_0 \in A$ such that

$$dist(fa_0, Ta_0) = dist(A, B) = \inf\{dist(fa, Ta) : a \in A\}.$$

In this note, we show that Theorem 1.2 remains valid if *R-subweak commutativity of the pair $\{f, P_A \circ T\}$ is dropped and also affineness and continuity of f are relaxed.* Thus we obtain a natural extension of Theorem 1.1 to f-nonexpansive mappings. For this we need the following results.

The following results are due to Kirk, Reich and Veeramani [1].

LEMMA 1.3. *Let A be a nonempty closed convex subset of a Hilbert space H. Then if B and C are nonempty closed bounded subsets of H,*

$$D(P_A(B), P_A(C)) \leq D(B, C).$$

LEMMA 1.4. *Let A be a nonempty closed bounded convex subset of a reflexive Banach space E and B a nonempty closed convex subset of E. Then A_0 and B_0 are nonempty and satisfy*

$$P_B(A_0) \subseteq B_0 \text{ and } P_A(B_0) \subseteq A_0.$$

The following lemma follows from a result in [4].

LEMMA 1.5. *Let $(X, ||.||)$ be a normed space, $f : X \to X$ and $T : X \to K(X)$ such that $T(X) \subset f(X)$. If $f(X)$ is complete and T is an f-contraction, then $C(f, T) \neq \emptyset$.*

2. Main results

LEMMA 2.1. *Let A be a nonempty closed bounded convex subset of a Hilbert space H and $f : A \to A$ be weakly continuous such that $f(A) = A$. Assume that $T : A \to K(A)$ is an f-nonexpansive map. Then $C(f, T) \cap A$ is nonempty.*

Proof. Choose $p \in A$ and a sequence $\{k_n\}$ with $0 < k_n < 1$ such that $k_n \to 1$ as $n \to \infty$. For each n, define T_n by

$$T_n a = (1 - k_n)p + k_n T a$$

for all $a \in A$. Then, for each n, $T_n : A \to K(A)$, $T_n(A) \subset A = f(A)$, and

$$\begin{aligned} D(T_n a, T_n b) &= k_n D(Ta, Tb) \\ &\leq k_n ||fa - fb|| \end{aligned}$$

for each $a, b \in A$. This implies that each T_n is an f-contraction. Notice that $f(A)$ is complete. Now Lemma 1.5 guarantees that, for each n, $C(f, T_n)$ is nonempty, i.e., there exists $a_n \in A$ such that $fa_n \in T_n a_n$. This implies that $fa_n - c_n = (1 - k_n)(p - c_n)$ for some $c_n \in Ta_n$. It then follows that $fa_n - c_n \to 0$ as $n \to \infty$. By weak compactness of A, we can find a subsequence $\{a_m\}$ of the sequence $\{a_n\}$ such that $\{a_m\}$ converges weakly to $a_0 \in A$ as $m \to \infty$. By weak continuity of f, $\{fa_m\}$ converges weakly to $fa_0 \in A$ as $m \to \infty$. Using the standard arguments, it can be shown that $0 \in (f - T)(a_0)$. Hence $C(f, T) \cap A$ is nonempty.

THEOREM 2.2. *Let H be a Hilbert space. Let A and B be nonempty closed convex subsets of H with A bounded. Let $f : A \to A$ be weakly continuous such that $f(A_0) = A_0$ and $T : A \to K(B)$ be such that*
(a) $T(A_0) \subseteq B_0$
(b) T is f-nonexpansive on A_0.
Then there exists $a_0 \in A$ such that

$$dist(fa_0, Ta_0) = dist(A, B) = \inf\{dist(fa, Ta) : a \in A\}.$$

Proof. By Lemma 1.4, A_0 is nonempty. Let $a \in A_0$. We claim that $P_A(Ta) \subseteq A_0$. Indeed, let $c \in P_A(Ta)$. Then $c \in P_A(b)$ for some $b \in Ta \subseteq B_0$ and so $||c - b|| = dist(b, A)$. Since $b \in B_0$, it follows that $||a - b|| = dist(A, B)$ for some $a \in A$. Thus $||c - b|| = dist(b, A) \leq ||a - b|| = dist(A, B)$. On the other hand, $dist(A, B) \leq ||c-b||$ for all $c \in A$ and $b \in B$. Consequently, $||c - b|| = dist(A, B)$. This proves our claim.
Since P_A is nonexpansive and so continuous, $P_A(Ta)$ is compact. As a result, $P_A \circ T : A_0 \to K(A_0)$. By Lemma 1.3, for any $a, c \in A_0$, we have

$$D(P_A(Ta), P_A(Tc)) \leq D(Ta, Tc) \leq ||fa - fc||.$$

Now Lemma 2.1 guarantees that $C(f, P_A \circ T) \cap A_0$ is nonempty, that is, there exists $a_0 \in A_0$ such that $fa_0 \in P_A(Ta_0)$. Consequently, for some $b \in Ta_0 \subseteq B_0$, we have

$$||fa_0 - b|| = dist(b, A).$$

Since $b \in B_0$, it follows that $||a - b|| = d(A, B)$ for some $a \in A$. This implies that

$$||fa_0 - b|| = dist(b, A) \leq ||a - b|| = dist(A, B)$$

and so

$$dist(fa_0, Ta_0) \leq ||fa_0 - y|| = dist(b, A) \leq dist(A, B).$$

Since $dist(A, B) \leq ||fa - b||$ for all $a \in A$ and $b \in Ta$, it follows that

$$dist(A, B) \leq \inf\{||fa - b|| : b \in Ta\} = dist(fa, Ta) \text{ for all } a \in A$$

and so

$$dist(A, B) \leq \inf\{dist(fa, Ta) : a \in A\}.$$

But

$$\inf\{dist(fa,Ta) : a \in A\} \leq dist(fa_0,Ta_0).$$

Therefore

$$dist(fa_0,Ta_0) \leq dist(A,B) \leq \inf\{dist(fa,Ta) : a \in A\} \leq dist(fa_0,Ta_0).$$

As a result,

$$dist(fa_0,Ta_0) = dist(A,B) = \inf\{dist(fa,Ta) : a \in A\}.$$

It is well-known that every continuous and affine mapping defined on a closed convex subset of a Hilbert space is weakly continuous. So we have the following corollary, which contains Theorem 1.1 (due to Kirk, Reich and Veeramani [1]) as a special case. It is worth mentioning that we do not require R-subweak commutativity of the pair $\{f, P_A \circ T\}$ as in [2].

COROLLARY 2.3. *Let H be a Hilbert space. Let A and B be nonempty closed convex subsets of H with A bounded. Let $f : A \to A$ be continuous and affine such that $f(A_0) = A_0$ and $T : A \to K(B)$ be such that*
(a) $T(A_0) \subseteq B_0$
(b) T is f-nonexpansive on A_0.
Then there exists $a_0 \in A$ such that

$$dist(fa_0,Ta_0) = dist(A,B) = \inf\{dist(fa,Ta) : a \in A\}.$$

EXAMPLE 2.4. Let $X = \mathbf{R}^2$ with the Euclidean norm, $A = \{(1,y) : 0 \leq y \leq 1\}$ and $B = \{(2,y) : 0 \leq y \leq 1\}$. Then $A_0 = A$ and $B_0 = B$ ([1]). Define $T : A \to B$ and $f : A \to A$ by

$$T(1,y) = (2, 1-y^2) \text{ and } f(1,y) = (1,y^2).$$

Then all hypotheses of Theorem 2.2 are satisfied. Note that $a_0 = (1, \frac{1}{\sqrt{2}})$ satisfies

$$||fa_0 - Ta_0|| = dist(A,B) = \inf\{||fa - Ta|| : a \in A\}.$$

Note also that Theorem 1.1 and Theorem 1.2 can not be used here.

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DEPARTMENT OF MATHEMATICS
KING ABDUL AZIZ UNIVERSITY
P. O. BOX 80203,
JEDDAH 21589, SAUDI ARABIA

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H. Gul Ince

A NOTE ON SOME SUBSPACES OF AN *FK*-SPACE

Abstract. The purpose of this paper is to give the properties of some distinguished FK spaces and to solve the problem of characterizing matrices A such that Y_A is Cesàro semiconservative space (for a given Y).

1. Introduction

In summability theory conservative spaces and matrices play a special role in its theory. However in [9], [11] Snyder and Wilansky shown that the results depend on a weaker assumption, that the spaces be semiconservative. First came conservative matrices, those for which $c_A \supset c$. When attention widened to FK spaces it was very natural to define one to be conservative if it includes c. Snyder and Wilansky studied the properties of any A matrix such that X_A is semiconservative space and shown that there is no FK space X such that X_A is semiconservative space if and only if $A \in (X, X)$.

In this paper we studied Cesàro semiconservative spaces which has weaker assumption then semiconservative space and shown that there is no FK space X such that X_A is Cesàro semiconservative space if and only if $A \in (X, X)$.

2. Notations and definitions

Let w denote the space of all real or complex-valued sequences. It can be topologized with the seminorms $p_i(x) = |x_i|$, $(i = 1, 2, \ldots)$, and any vector subspace of w is called a sequence space. A sequence space X, with a locally convex Hausdorff topology will be called a locally convex sequence space. A K space is a locally convex sequence space in which the inclusion mapping I: $X \to w$, $I(x) = x$ is continuous. An FK space is a Frechet K-space. An FK space whose topology is normable is called a BK space. The basic properties of such spaces can be found in [11], [12] and [13]. By m, c_0 we

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denote the spaces of all bounded sequences, null sequences, respectively. These are FK spaces under $\|x\| = \sup_n |x_n|$. By l we shall denote the space of all absolutely summable sequences. The sequences spaces

$$h = \left\{x \in w : \lim_j x_j = 0, \text{ and } \sum_{j=1}^{\infty} j\,|\Delta x_j| < \infty\right\},$$

$$q = \left\{x \in w : \sup_j |x_j| < \infty \text{ and } \sum_{j=1}^{\infty} j\,|\Delta^2 x_j| < \infty\right\},$$

$$\sigma b = \left\{x \in w : \sup_n \left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j\right| < \infty\right\},$$

$$\sigma s = \left\{x \in w : \lim_n \left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j\right| < \infty\right\}$$

and

$$\sigma_0 = \left\{x \in w : \lim_n \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right| = 0\right\}$$

are BK spaces with the norms

$$\|x\|_h = \sum_{j=1}^{\infty} j\,|\Delta x_j| + \sup_j |x_j|,$$

$$\|x\|_q = \sum_{j=1}^{\infty} j\,|\Delta^2 x_j| + \sup_j |x_j|,$$

$$\|x\|_{\sigma b} = \sup_n \left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j\right|$$

and

$$\|x\|_{\sigma_0} = \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right|$$

respectively, where $\Delta x_j = x_j - x_{j+1}$, $\Delta^2 x_j = \Delta x_j - \Delta x_{j+1}$. The space $q \cap c_0$ is denoted by q_0. Under the norm $\|.\|_q$, q_0 is a BK space ([1], [2]). In addition $bv = \{x \in w : \sum_{j=1}^{\infty} |x_j - x_{j+1}| < \infty\}$, $bv_0 := bv \cap c_0$.

Throughout the paper e denotes the sequence of ones, $(1, 1, \ldots, 1, \ldots)$; δ^j $(j = 1, 2, \ldots)$, the sequence $(0, 0, \ldots, 0, 1, 0, \ldots)$ with the one in the j-th position. Let $\phi := l.hull\left\{\delta^k : k \in N\right\}$ and $\phi_1 = \phi \cup \{e\}$. The topological dual of X is denoted by X'. The space X is said to have AD if ϕ is dense in X

and an FK space X is said to have AK or be an AK space, if $X \supset \phi$ and for each $x \in X$, $x^{(n)} \to x$ in X, where $x^{(n)} = \sum_{k=1}^{n} x_k \delta^k = (x_1, x_2, \ldots, x_n, 0, \ldots)$. In addition an FK space is said to have σK space if $X \supset \phi$ and for each $x \in X$, $\frac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x, (n \to \infty)$. Every AK space is a σK space. For example w, h, c_0 are AK spaces while q_0, σs are σK spaces ([1], [2], [8]). In addition, every σK space is an AD space.

Let X be an FK space containing ϕ. Then

$$X^f = \{\{f(\delta^k)\} : f \in X'\}.$$

In addition

$$X^\beta = \left\{x : \sum_{k=1}^{\infty} x_k y_k \text{ exists for every } y \in X\right\},$$

$$X^\sigma = \left\{x : \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j y_j \text{ exists for every } y \in X\right\},$$

$$X^{\sigma b} = \left\{x : \sup_n \frac{1}{n} \left|\sum_{k=1}^{n} \sum_{j=1}^{k} x_j y_j\right| < \infty \text{ for every } y \in X\right\}.$$

Let E, E_1 be sets of sequences. Then for $k = f, \beta, \sigma, \sigma_b$

$$(a)\ E \subset E^{kk}, \quad (b)\ E^{kkk} = E^k, \quad (c) \text{ if } E \subset E_1 \text{ then } E_1^k \subset E^k$$

holds ([4], [8]).

It is easy to prove that $X^\beta \subset X^\sigma \subset X^{\sigma b} \subset X^f$ and if X is σK space then $X^f = X^\sigma$ and if X is an AD space then $X^\sigma = X^{\sigma b}$.

Let $A = (a_{ij})$ be an infinite matrix. The matrix A may be considered as a linear transformation of sequences (x_k) by the formula $y = Ax$, where $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$, $(i = 1, 2, \ldots)$.

For an FK space (E, u) we consider the summability domain $E_A := \{x \in w : Ax \in E\}$. Then E_A is an FK space under the seminorms $p_i = |x_i|, (i = 1, 2, \ldots)$, $h_i(x) = \sup_m \left|\sum_{j=1}^{m} a_{ij} x_j\right|$, $(i = 1, 2, \ldots)$ and $(u \circ A)(x) = u(Ax)$ [11].

3. Some subspaces of an FK spaces

Let we recall some important subspaces of an FK space which introduced by Goes in [4].

DEFINITION 3.1. Let X be an FK space containing ϕ. Then

$$\sigma W := \sigma W(X) = \left\{x : \frac{1}{n}\sum_{k=1}^{n} x^{(k)} \to x \text{ (weakly) in } X\right\}$$

$$= \left\{x : f(x) = \lim_n \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j f(\delta^j), \text{ for all } f \in X'\right\},$$

$$\sigma S := \sigma S(X) = \left\{x : \frac{1}{n}\sum_{k=1}^{n} x^{(k)} \to x \text{ in } X\right\}$$

$$= \left\{x : x = \lim_n \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j \delta^j \right\},$$

$$\sigma B^+ := \sigma B^+(X) = \left\{x : \left\{\frac{1}{n}\sum_{k=1}^{n} x^{(k)}\right\} \text{ is bounded in } X\right\}$$

$$= \left\{x : \left\{x_n f(\delta^n)\right\} \in \sigma b \text{ for all } f \in X'\right\},$$

$$\sigma F^+ := \sigma F^+(X) = \left\{x : \frac{1}{n}\sum_{k=1}^{n} x^{(k)} \text{ is weakly Cauchy in } X\right\}$$

$$= \left\{x : \lim_n \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j f\left(\delta^j\right) \text{ exists for all } f \in X'\right\}$$

$$= \{x : \{x_n f(\delta^n)\} \in \sigma s \text{ for all } f \in X'\}.$$

Also $\sigma F = \sigma F^+ \cap X$, $\sigma B = \sigma B^+ \cap X$. An FK space is a $\sigma K-$space (respectively $S\sigma W-$space, $\sigma F-$space, $\sigma B-$space) if $X = \sigma S$ (respectively $X = \sigma W$, $X = \sigma F$, $X = \sigma B$) [1].

It is well known that for an FK space X

$$\phi \subset \sigma S \subset \sigma W \subset \sigma F \subset \sigma B \subset X.$$

THEOREM 3.2. *Let X be an FK space containing ϕ. Then $\sigma B^+(X) = X^{f\sigma b}$, $\sigma F^+(X) = X^{f\sigma}$* [4].

THEOREM 3.3. *Let X be an FK space containing ϕ. Then X has $F\sigma K$ if and only if $X^f = X^\sigma$* and *X has σB if and only if $X^f = X^{\sigma b}$* [4].

We note that subspaces σW are closely related to Cesàro conullity of the FK space X [6].

LEMMA 3.4. *Let X be an AD space, then X has $F\sigma K$ if and only if X has σB.*

Proof. Since X be an AD space then $X^{\sigma} = X^{\sigma b}$ and hence we get the proof by Theorem 3.3. □

Since $\sigma F \subset \sigma B$, $F\sigma K$ implies σB but not conversely:

EXAMPLE 3.5. *q and σb have σB but does not have $F\sigma K$.*

The proof is as follows:

$$\begin{aligned}\sigma B^{+}(q) &= q^{f\sigma b} = \sigma b^{\sigma b} = q \text{ then } \sigma B(q) = \sigma B^{+}(q) \cap q = q,\\ \sigma F^{+}(q) &= q^{f\sigma} = q_0 \text{ then } \sigma F(q) = \sigma F^{+}(q) \cap q = q_0,\\ \sigma B^{+}(\sigma b) &= \sigma b^{f\sigma b} = q^{\sigma b} = \sigma b \text{ then } \sigma B(\sigma b) = \sigma b,\\ \sigma F^{+}(\sigma b) &= \sigma b^{f\sigma} = q^{\sigma} = \sigma s \text{ then } \sigma F(\sigma b) = \sigma s \;\; ([1],[2]).\end{aligned}$$

In this section we give some results which are analogous those given in [11, Chapter 10].

THEOREM 3.6. *$\sigma_0 \subset X$ if and only if $\sigma_\infty \subset \sigma F^{+}$.*

Proof. Necessity: We have $\sigma F^{+}(\sigma_0) \subset \sigma F^{+}(X)$ holds by [5] and therefore $\sigma_0^{f\sigma} \subset \sigma F^{+}(X)$ by [4]. Since $\sigma_0^{f\sigma} = h^{\sigma} = \sigma_\infty$ [2] then $\sigma_\infty \subset \sigma F^{+}(X)$.

Sufficiency: We get $\sigma_\infty \subset X^{f\sigma}$ by [4] therefore $X^{f} \subset X^{f\sigma\sigma} \subset \sigma_\infty^{\sigma} \subset \sigma_\infty^{f} = \sigma_0^{f}$ [2]. Since σ_0 has AK [2] then $\sigma_0 \subset X$ [11, Theorem 8.6.1.]. Also we arrive at $\sigma F^{+}(\sigma_c) = \sigma F^{+}(\sigma_\infty) = \sigma F^{+}(\sigma_0) = \sigma_\infty$ by Theorem 3.3. □

THEOREM 3.7. *Let X be an FK space containing ϕ. Then $\overline{\sigma F} \subset \sigma B$ (closure in X) if and only if σF is closed in X. Thus the closedness of σB implies the closedness of σF.*

Proof. Sufficiency is trivial. Now suppose that $\overline{\sigma F} \subset \sigma B$. Fix $f \in X'$ and define $g : \overline{\sigma F} \to \sigma_\infty$ by $g(x) = \left\{f\left(x^{(n)}\right)\right\}$. Then $P_n \circ g : \overline{\sigma F} \to K$, where $P_n(x) = x_n$, given by $P_n(g(x)) = f\left(x^{(n)}\right) = \sum_{k=1}^{n} x_k f\left(\delta^k\right)$ is continuous, where $\overline{\sigma F}$ has the relative topology of X, and K is the scalar field R or C. Thus $g : \overline{\sigma F} \to \sigma_\infty$ is continuous [11, Theorem 4.2.3], hence since σ_c is closed in σ_∞ then $g^{-1}(\sigma_c)$ is closed in $\overline{\sigma F}$. Because of $\sigma F = \cap\left\{g^{-1}(\sigma_c) : f \in X'\right\}$, σF is closed in X.

If σB is closed then $\overline{\sigma F} \subset \sigma B$. □

4. Matrix domains

In this section we solve the problem of characterizing matrices A such that Y_A is Cesàro semiconservative space for given Y.

Before the following theorems we give the definition of Cesàro semiconservative space.

DEFINITION 4.1. An FK space X containig ϕ is called Cesàro semiconservative space if $X^{f} \subset \sigma s$ [7].

THEOREM 4.2. *Let Y be an FK space and A is a matrix. Then Y_A is Cesàro semiconservative space if and only if the columns of A are in Y and $g(a^k) \in \sigma s$ for each $g \in Y'$, where a^k is the kth column of A, $a^k_n = a_{nk}$.*

Proof. Necessity: The columns of A are in Y since $Y_A \supset \phi$ by definition of Cesàro semiconservative space. Given $g \in Y'$, let $f(x) = g(Ax)$ for $x \in Y_A$, so $f \in Y'_A$ by [11, Theorem 4.4.2.]. Then $f(\delta^k) = g(a^k)$ and the result follows since $Y^f_A \subset \sigma s$.

Sufficiency: We first note that each row of A belongs to σs since in the hyphotesis we may take $g = P_n$. Then$\{g(a^k)\} = \{P_n(a^k_n)\} = \{a_{nk}\} \in \sigma s, (k = 1, 2, 3, \dots)$. Hence $w_A \supset \sigma s^\beta$. Let $f \in Y'_A$.Then $f(x) = \sum_{k=1}^{\infty} \alpha_k x_k + g(Ax)$ by [11, Theorem 4.4.2.] with $g \in Y'$, $\alpha \in w^\beta_A \subset \sigma s^{\beta\beta} = \sigma s$. Thus $f(\delta^k) = \alpha_k + g(a^k)$. Since $\alpha_k \in \sigma s$ and $g(a^k) \in \sigma s$ then $\{f(\delta^k)\} \in \sigma s$. Thus $Y^f_A \subset \sigma s$ and Y_A is Cesàro semiconservative space. □

Given A, if there is any Y such that Y_A is Cesàro semiconservative space, then the rows of A belongs to σs. It is clear from Theorem 4.2. For giving an alternative proof; we assume that r is a row of A. Then $Y_A \subset w_A \subset r^\beta \subset r^\sigma$. Hence r^σ is Cesàro semiconservative space by [7, Theorem 3.6.].

Theorem 4.2 says that Y_A is Cesàro semiconservative space if $\left\{\frac{1}{n}\sum_{k=1}^{n} a^k\right\}$ is weakly Cauchy in Y. Also since Y_A is Cesàro semiconservative space then $Y_A \supset q_0$ by [7, Theorem 3.8.]. Hence if $\left\{\frac{1}{n}\sum_{k=1}^{n} a^k\right\}$ is weakly Cauchy then $Y_A \supset q_0$ i.e $A \in (q_0, Y)$.

THEOREM 4.3. *If Y_A is Cesàro semiconservative space then $A^T \in (Y^\beta, \sigma s)$, where A^T denotes transpose of matrix A.*

PROOF. Since $Y_A \supset q_0$ by [7, Theorem 3.8.] then $A \in (q_0, Y)$. Hence $A^T \in (Y^\beta, q^f_0) = (Y^\beta, \sigma b)$ by [11, Theorem 8.3.8.]. Let $z \in Y^\beta$ and define $g(y) = zy$, where $zy = \sum_{k=1}^{\infty} z_k y_k$, $y \in Y$. Then by Banach-Steinhaus Theorem $g \in Y'$. Let $f(x) = g(Ax)$ so that $f \in Y'_A$ by [11, Theorem 4.4.2.]. Hence $\{f(\delta^k)\} \in \sigma s$. $f(\delta^k) = \sum_{n=1}^{\infty} z_n a_{nk} = (A^T z)_k$ so $(A^T z) \in \sigma s$. □

COROLLARY 4.4. *Let Y be a BK space and suppose that Y_A is Cesàro semiconservative space. Then $A \in (q, Y^{\beta f})$.*

Proof. Its clear by using Theorem 4.3. and [11, Theorem 8.3.8.]. □

EXAMPLE 4.5. Let $A = I$, $Y = q$. Then

$A \in (q, q) = (q, q^{\beta f})$ but $Y_A = q$ is not Cesàro semiconservative space by [7]. Thus the converse of Corollary 4.4. is false. Also $A^T = I \in (\sigma s, \sigma s) = (Y^\beta, \sigma s)$ so the converse of Theorem 4.3. is false.

We can obtain a converse for Theorem 4.3. in the unimportant case in which Y has AK.

THEOREM 4.6. *Let Y be an FK space with AK. Then Y_A is Cesàro semiconservative space if and only if the columns of A belong to Y and $A^T \in (Y^\beta, \sigma s)$.*

Proof. Necessity is trivial by Theorem 4.3.

Sufficiency: Let $g \in Y', z_n = g(\delta^n)$. Then $z \in Y^f = Y^\beta$ by [11, Theorem 7.2.7.], so $A^T z \in \sigma s$. Hence we get

$$(A^T z)_k = \sum_{n=1}^{\infty} z_n a_{nk} = g\Big(\sum_{n=1}^{\infty} a_{nk}\delta^n\Big) = g(a^k) \in \sigma s.$$

Then Y_A is Cesàro semiconservative space by Theorem 4.2. □

LEMMA 4.7. *The following are eqivalent for an FK space X.*

(i) If $A \in (X, X)$ then X_A is Cesàro semiconservative space.

(ii) X is Cesàro semiconservative space.

Proof. $(i) \Rightarrow (ii)$: Take $A = I$.

$(ii) \Rightarrow (i)$: If $A \in (X, X)$ then $X_A \supset X$, hence X_A is Cesàro semiconservative space by [7]. □

LEMMA 4.8. *Suppose that an FK space X has the property: $(i)'$ If X_A is Cesàro semiconservative space then $A \in (X, X)$. Then $X \subset q$.*

Proof. We assume that $\sigma s \not\subset X^\sigma$. Because of $\sigma s \subset X^\sigma$ implies $X \subset X^{\sigma\sigma} \subset \sigma s^\sigma = q$. Let $z \in \sigma s \setminus X^\sigma$, $0 \neq v \in X$ and $a_{nk} = \frac{n-(k-1)}{n} v_n z_k$, if $1 \leqslant k \leqslant n$,and 0 otherwise. Since $(\mathrm{Ax})_n = \sum\limits_{k=1}^{\infty} a_{nk} x_k = \frac{1}{n} v_n \sum\limits_{k=1}^{n} \frac{n-(k-1)}{n} z_k x_k = \frac{1}{n} \sum\limits_{k=1}^{n} \sum\limits_{j=1}^{k} x_j z_j$ then $w_A = X_A = u^\sigma$. Since u^σ is Cesàro semiconservative space by [7] X_A is Cesàro semiconservative space. But $A \notin (X, X)$ since $X \not\subset w_A$. □

COROLLARY 4.9. *There exists no FK space X such that X_A is Cesàro semiconservative space if and only if $A \in (X, X)$.*

By Lemmas 4.7, 4.8. and [7, Theorem 3.5.], q would be Cesàro semiconservative space, contradicting [7, Example 3.9.].

However İnce (in [8]) has proved that X_A is strongly Cesàro conull (Cesàro conull) space if and only if $q \subset X_A$ and $A : q \to X$ (weakly) compact. In addition every (strongly) Cesàro conull space is Cesàro semiconservative space.

DEFINITION 4.10. A matrix A is called Cesàro semiconservative if c_A is Cesàro semiconservative space.

The reason for this definition is that summability theory deals with spaces of the form c_A and with FK spaces whose properties generalize those of such spaces. If we can extend theorems about conservative spaces to Cesàro semiconservative spaces so much better.

THEOREM 4.11. *A is Cesàro semiconservative if and only if*

(i) A has convergent columns, i.e. $c_A \supset \phi$,

(ii) $a \in \sigma s$ where $a = \{a_k\}, a_k = \lim_n a_{nk}$,

(iii) $A^T \in (l, \sigma s)$.

PROOF. Necessity:

(i) is clear by Definition 4.10. (ii): Since c_A is Cesàro semiconservative then we take $g := \lim$ in Theorem 4.2. Hence $a \in \sigma s$. (iii): By Theorem 4.3.

Sufficiency: Let $g \in c'$. Then $g(y) = \chi \lim y + \sum_{n=1}^{\infty} t_n y_n, t \in l$ by [11, 1.0.2.]. If we take $y = Ax$; $x = \delta^k$ in here we obtained $g(a^k) = \chi \lim a_{nk} + (tA)_k$ where $(tA)_k = \sum_{n=1}^{\infty} t_n a_{nk}$. Since $g(a^k) \in \sigma s$ from (ii) and (iii) then by Theorem 4.2. the result is obtained. □

In the following theorems we give simple conditions for the subspaces σF, σB in the FK space Y_A .The conditions will depend on the choice of the FK space Y and the matrix A. However the subspaces σS, σW are calculated in the FK space Y_A in [6].

THEOREM 4.12. *Let $z \in w$, Y be an FK space and A be a matrix such that $Y_A \supset \phi$ i.e. the columns of A belong to Y. Then the following conditions are equivalent:*

(i) $z \in \sigma B^+$,

(ii) $\left\{\frac{1}{r}\sum_{p=1}^{r} Az^{(p)}\right\}$ is bounded in Y,

(iii) $Y_{A.z} \supset q_0$ where the matrix $A.z$ is $(a_{nk}z_k)$,

(iv) $\{z_k g(a^k)\} \in \sigma b$ for each $g \in Y'$ where a^k is kth column of A.

Also these are equivalent: $z \in \sigma B$, $Y_{A.z} \supset q$, (ii) and $z \in Y_A$, (iv) and $z \in Y_A$.

Proof. $(i) \Leftrightarrow (ii)$: $z \in \sigma B^+ \Leftrightarrow z^{-1}.Y_A \supset q_0$, where $z^{-1}.Y_A = \{x : x.z \in Y_A\}$, $x.z = \{x_n z_n\} \Leftrightarrow Y_{A.z} \supset q_0$ by $z^{-1}.Y_A = Y_{A.z}$ and [7].

$(iii) \Leftrightarrow (iv)$: Since q_0 is an AD space and by hypothesis then $Y^f_{A.z} \subset q^f_0$ by [11, Theorem 8.6.1.]. Hence $f(\delta^k) = \alpha_k + g(a^k_n z_k)$ for each $f \in Y'_{A.z}$ with $\alpha \in w^\beta_{A.z}$, $g \in Y'$ by [11, Theorem 4.4.2.]. Since $\alpha \in w^\beta_{A.z} \subset Y^\beta_{A.z} \subset q^\beta_0 = \sigma b$ then $f(\delta^k) \in \sigma b$ if and only if $\{z_k g(a^k)\} \in \sigma b$ for each $g \in Y'$.

$(ii) \Leftrightarrow (iv)$: (iv) is true if and only if $\left\{g\left(\frac{1}{r}\sum_{p=1}^{r} Az^{(p)}\right)\right\}$ is bounded for each $g \in Y'$ by [11, Theorem 8.0.2.]. Here

$$g\left(\frac{1}{r}\sum_{p=1}^{r} Az^{(p)}\right) = g\left(\frac{1}{r}\sum_{p=1}^{r}\sum_{k=1}^{p} a_{nk}z_k\right) = \frac{1}{r}\sum_{p=1}^{r}\sum_{k=1}^{p} z_k g(a^k_n).$$

The second part is trivial because of $z \in Y_A$ if and only if $e \in Y_{A.z}$. □

THEOREM 4.13. *Let $z \in w$, Y be an FK space and A be a matrix such that $Y_A \supset \phi$ i.e. the columns of A belong to Y. Then the following conditions are equivalent:*

(i) $z \in \sigma F^+$.

(ii) $\left\{\frac{1}{r}\sum_{p=1}^{r} Az^{(p)}\right\}$ is weakly Cauchy in Y i.e. $\left\{g\left(\frac{1}{r}\sum_{p=1}^{r} Az^{(p)}\right)\right\}$ is convergent for each $g \in Y'$.

(iii) $Y_{A.z}$ is Cesàro semiconservative space.

(iv) $\{z_k g(a^k)\} \in \sigma s$ for each $g \in Y'$.

Also these are equivalent: $z \in \sigma F$, $Y_{A.z}$ is bounded convex Cesàro semiconservative space i.e. $Y_{A.z} \supset q$ and $Y_{A.z}$ Cesàro semiconservative space, (ii) and $z \in Y_A$, (iv) and $z \in Y_A$.

Proof. $(i) \Leftrightarrow (ii)$: $z \in \sigma F^+ \Leftrightarrow z^{-1}.Y_A$ is Cesàro semiconservative space$\Leftrightarrow$ $Y_{A.z}$ is Cesàro semiconservative space by [7, Theorem 4.2].

$(iii) \Leftrightarrow (ii)$: Since the kth column of $A.z$ is $z_k a^k$ and by Theorem 4.2. then, this equivalent is trivial.

$(iii) \Leftrightarrow (iv)$: By Theorem 4.2., since the kth column of $A.z$ is $z_k a^k$.

The second part is clear as in Theorem 4.12. □

THEOREM 4.14. *Let Y be an FK space such that weakly convergent sequences are convergent in the FK topology, let A be a row finite matrix such that $Y_A \supset \phi$. Then $\sigma S = \sigma W = \sigma F = \sigma F^+$ in Y_A.*

Proof. If $z \in \sigma F^+$, $\left\{\frac{1}{r}\sum_{p=1}^{r} Az^{(p)}\right\}$ is weakly Cauchy in Y by Theorem 4.13., hence Cauchy [11, Theorem 12.0.2.], hence convergent say $\frac{1}{r}\sum_{p=1}^{r} Az^{(p)} \longrightarrow y$. However $\frac{1}{r}\sum_{p=1}^{r} z^{(p)} \longrightarrow z$ in w_A since this a σK space because of w_A is an

AK space [11, 4.3.8]. Thus $\frac{1}{r}\sum_{p=1}^{r} Az^{(p)} \longrightarrow Az$ in w. But $\frac{1}{r}\sum_{p=1}^{r} Az^{(p)} \longrightarrow y$ in w since Y is an FK space hence $y = Az$ so $z \in \sigma S$ by [6].

Also take $A = I$ in Theorem 4.14. then $\sigma S = \sigma W = \sigma F = \sigma F^{+}$ *in* Y.

We take $Y = l, bv_0, bv$ in Theorem 4.14. □

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GAZI UNIVERSITY
FACULTY OF SCIENCES AND ARTS
DEPARTMENT OF MATHEMATICS
06500 TEKNIKOKULLAR, ANKARA - TURKEY
E-mail: ince@gazi.edu.tr

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Mehmet Şengönül

ON THE GENERALIZED NAKANO SEQUENCE SPACE

Abstract. The purpose of this note is to define and to investigate the generalized Nakano sequence space $\mathcal{A}(p)$ and to show that the sequence space $\mathcal{A}(p)$ equipped with the Luxemburg norm is rotund and posses property-H when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

1. Introduction

By w, we shall denote the space of all real or complex valued sequences. Each linear subspace of w is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i : w \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \dots\}$. A K- space λ is called an FK- space provided λ is a complete linear metric space. An FK- space whose topology is normable is called a BK- space [2, pp. 272-273]. A triangle is a lower triangular matrix with no zeros on the principal diagonal. A matrix A is called regular if A is limit preserving over c, where c denote the space of convergent sequences. For a Banach space λ, we denote by $S(\lambda)$ and $B(\lambda)$ the unit sphere and unit ball of λ, respectively. A point $x_0 \in S(\lambda)$ is called:

$a)$ an extreme point if for every $x, y \in S(\lambda)$ the equality $2x_0 = x + y$ implies $x = y$;

$b)$ an H-point if for any sequence (x_n) in λ such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x implies that $\|x_n - x\| \to 0$ as $n \to \infty$.

A Banach space λ is said to be rotund, if every point of $S(\lambda)$ is an extreme point. A Banach space λ is said to possess H-property provided every point of $S(\lambda)$ is an H-point.

Let λ be an arbitrary vector space over $\mathbb{C}$.

a) A functional $m : \lambda \to [0, \infty]$ is called modular if the following conditions hold:

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M1) $m(x) = 0 \Leftrightarrow x = 0$,

M2) $m(\alpha x) = m(x)$ for $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) with $|\alpha| = 1$, for all $x \in \lambda$,

M3) $m(\alpha x + \beta y) \leq m(x) + m(y)$ if $\alpha,\ \beta \geq 0,\ \alpha + \beta = 1$, for all $x, y \in \lambda$.

b) If *M3* is replaced by;

M4) $m(\alpha x + \beta y) = \alpha^s m(x) + \beta^s m(y)$ if $\alpha,\ \beta \geq 0,\ \alpha^s + \beta^s = 1$, with an $s \in [0,1]$ then the modular m is called an s-convex modular; and if $s = 1$, m is called a convex modular.

c) A modular m defines the corresponding modular space, i.e, the space λ_m given by

$$\lambda_m = \{x \in w : m(tx) \to 0 \ \ as \ \ t \to \infty\}.$$

Recall that for given any $\epsilon > 0$, a sequence (x_n) is said to be an ϵ-*separated sequence* if

$$\text{sep}(x_n) = \inf\{\|x_n - x_k\| : n \neq k\} > \epsilon.$$

We say Banach space λ has β-property if for every $\epsilon > 0$ such that, for each element $x \in B(\lambda)$ and each sequence $(x_n) \in B(\lambda)$ with $\text{sep}(x_n) \geq \epsilon$, there exists an index k such that

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

The Nakano sequence space $\ell(p)$ is defined by

$$\ell(p) = \{x = (x_k) \in w : m(tx) < \infty \quad \text{for some} \ \ t > 0\},$$

where $m(x) = \sum_k |x_k|^{p_k}$ and $p = (p_k)$ is a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The space $\ell(p)$ is a Banach space with the norm

$$\|x\| = \inf\left\{ t > 0 : m\left(\frac{x}{t}\right) \leq 1 \right\}.$$

If $p = (p_k)$ is bounded, we have

$$\ell(p) = \left\{ x \in w : \sum_k |x_k|^{p_k} < \infty \right\}.$$

Also, some geometric properties of $\ell(p)$ were studied in [1] and [3].

For $1 \leq p < \infty$, the Cesàro sequence space is defined by

$$ces_p = \left\{ x = (x_k) \in w : \left(\sum_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{\frac{1}{p}} < \infty \right\} \tag{1.1}$$

equipped with the norm

$$\|x\| = \left(\sum_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{\frac{1}{p}}.$$

This space was introduced by Shue [12]. Some geometric properties of the Cesàro sequence space ces_p were studied by many mathematicians. It is known that ces_p is locally uniform rotund and posses property-H [5]. Cui and Hudzik [3] proved that ces_p has the Banach- Saks of type p if $p > 1$, and it was shown in [4] that ces_p has β-property.

2. The sequence space $\mathcal{A}$

The space $ces(p)$ [11] is defined by

$$ces(p) = \{x \in w : \rho(tx) < \infty \quad \text{for some} \quad t > 0\}, \tag{2.1}$$

where $\rho(x) = \sum_n \left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n}$. The space $ces(p)$ is a Banach space with the norm

$$\|x\| = \inf\left\{t > 0 : \rho\left(\frac{x}{t}\right) \le 1\right\}$$

and if $p = (p_n)$ is bounded then we have

$$ces(p) = \left\{x \in w : \sum_n \left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n} < \infty\right\}.$$

Several geometric properties of $ces(p)$ were studied in [11]. Define the sequence $y = (y_n)$, which will be frequently used, as the A-transform of a sequence $x = (x_k)$, i.e.,

$$(Ax)_n = y_n = a_n \sum_{k=0}^{n} x_k \tag{2.2}$$

where, $A = (a_{nk})$ is defined by

$$a_{nk} = \begin{cases} a_n, & (0 \le k \le n) \\ 0, & (k > n) \end{cases}; \quad (n, \quad k \in \mathbb{N}), \tag{2.3}$$

$a_n > 0$ for all $n \in \mathbb{N}$, $a = (a_n)$ is monotone decreasing and A is regular.

Now, we wish to introduce the generalized Nakano sequence space $\mathcal{A}(p)$, as the set of all sequences such that $A-$ transforms of them are in the space $\ell(p)$, that is

$$\mathcal{A}(p) = \{x = (x_k) \in w : (Ax) \in \ell(p)\} \tag{2.4}$$

or in another word

$$\mathcal{A}(p) = \{x \in w : m(tx) < \infty \ \text{ for some } \ t > 0\},$$

where $m(x) = \sum_n (a_n \sum_{i=0}^{n} |x_i|)^{p_n} < \infty$. We consider the space $\mathcal{A}(p)$ equipped with the so-called Luxemburg norm

$$\|x\| = \inf\left\{t > 0 : m\left(\frac{x}{t}\right) \le 1\right\}.$$

If $p = (p_n)$ is bounded, then we have

$$\mathcal{A}(p) = \Big\{x \in w : \sum_n \Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n} < \infty\Big\}.$$

The purpose of this note is to define and to investigate the generalized Nakano sequence space $\mathcal{A}(p)$ and show that the sequence space $\mathcal{A}(p)$ equipped with the Luxemburg norm is rotund and posses H-property when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

Clearly, in the special cases $a_n = (n+1)^{-1}$ and $a_n = 1$, we have $\mathcal{A}(p) = ces(p)$ and $\mathcal{A}(p) = \ell(p)$, respectively. Also, throughout this paper we assume that $p = (p_i)$ is bounded with $p_i > 1$ for all $i \in \mathbb{N}$ and $K = \sup_i p_i$.

Now, we may begin with the following theorem which is essential in the text:

THEOREM 2.1. *The set $\mathcal{A}(p)$ is the BK- spaces with the norm $\|x\|_{\mathcal{A}(p)} = \|Ax\|_{\ell(p)}$.*

Proof. Since (2.2) holds and $\ell(p)$ is the $BK-$space [10] with respect to its norm and the matrix A is normal, Theorem 4.3.2 of Wilansky [13, pp. 61] gives the fact that the space $\mathcal{A}(p)$ is $BK-$ space. □

PROPOSITION 2.2. *The functional m on the space $\mathcal{A}(p)$ is a convex modular.*

Proof. $m(x) = 0 \Leftrightarrow x = 0$ and $m(\alpha x) = m(x)$ for all scalar α with $|\alpha| = 1$ is clear so, we omit it. Let $x, y \in \mathcal{A}(p)$ and $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $u \to |u|^{p_n}$; $n \in \mathbb{N}$, we have:

$$\begin{aligned} m(\alpha x + \beta y) &= \sum_n \Big(a_n \sum_{i=0}^{n} |(\alpha x_i + \beta y_i)|\Big)^{p_n} \\ &\leq \sum_n \Big(\Big(a_n \sum_{i=0}^{n} |\alpha x_i|\Big) + (a_n \sum_{i=0}^{n} |\beta y_i|)\Big)^{p_n} \\ &\leq \alpha \sum_n \Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n} + \beta \sum_n \Big(a_n \sum_{i=0}^{n} |y_i|\Big)^{p_n} \\ &= \alpha m(x) + \beta m(y). \end{aligned}$$

□

PROPOSITION 2.3. *For $x \in \mathcal{A}(p)$ the modular m on $\mathcal{A}(p)$ satisfies the following properties:*

P1. if $0 < r < 1$ then $r^K m(r^{-1}x) \leq m(x)$ and $m(rx) \leq rm(x)$,

P2. if $r > 1$, then $m(x) \leq r^K m(r^{-1}x)$,

P3. if $r \geq 1$, then $m(x) \leq rm(x) \leq m(rx)$.

Proof. It is obvious that *P3* is satisfied by the convexity of m. It remains to prove *P1* and *P2*. For $0 < r < 1$, we have

$$\begin{aligned} m(x) &= \sum_n \Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n} = \sum_n \Big(r a_n \sum_{i=0}^{n} |r^{-1} x_i|\Big)^{p_n} \\ &= \sum_n r^{p_n} \Big(a_n \sum_{i=0}^{n} |r^{-1} x_i|\Big)^{p_n} \geq \sum_n r^{K} \Big(a_n \sum_{i=0}^{n} |r^{-1} x_i|\Big)^{p_n} \\ &= r^K \sum_n \Big(a_n \sum_{i=0}^{n} |r^{-1} x_i|\Big)^{p_n} = r^K m(r^{-1}x), \end{aligned}$$

and it implies by the convexity of m that $m(rx) \leq rm(x)$, hence *P1* is satisfied. Note that *P2* follows directly from *P1*. Namely, if $r > 1$ then $0 < r^{-1} < 1$. Therefore, by *P1*, $(r^{-1})^k m(rx) \leq m(x)$, we get $m(x) \leq r^K m(r^{-1}x)$ and so *P2* is obtained. □

Now, we give relationships between the Luxemburg norm and the modular m on the space $\mathcal{A}(p)$.

PROPOSITION 2.4. *For any $x \in \mathcal{A}(p)$, if $p = (p_n)$ is bounded, we have*

P4. *if* $\|x\| < 1$ *then* $m(x) \leq \|x\|$
P5. *if* $\|x\| > 1$ *then* $m(x) \geq \|x\|$
P6. $\|x\| = 1$ *if and only if* $m(x) = 1$
P7. $\|x\| < 1$ *if and only if* $m(x) < 1$
P8. $\|x\| > 1$ *if and only if* $m(x) > 1$
P9. *if* $0 < r < 1$ *and* $\|x\| > r$ *then* $m(x) > r^K$
P10. *if* $r \geq 1$ *and* $\|x\| < r$ *then* $m(x) < r^K$.

Proof. *P4.* Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \|x\|$. Then we have $\|x\| + \epsilon < 1$. By definition of $\|.\|$ there exists $\mu > 0$ such that $\|x\| + \epsilon > \mu$ and $m(\mu^{-1}x)$. From Proposition 2.3 (*P1.* and *P3.*), we have

$$m(x) \leq m\left((\|x\| + \epsilon)\mu^{-1}x\right) \leq (\|x\| + \epsilon) m\left(\mu^{-1}x\right) \leq \|x\| + \epsilon$$

which implies that $m(x) \leq \|x\|$. So *P4* is satisfied.

P5. Let $\epsilon > 0$ be such that $0 < \epsilon < (\|x\| - 1)\|x\|^{-1}$ then $1 < (1-\epsilon)\|x\| < \|x\|$. By definition of $\|.\|$ and by part *P1* of Proposition 2.3 we have

$$1 < m(x[(1-\epsilon)\|x\|]^{-1}) \leq [(1-\epsilon)\|x\|]^{-1} m(x).$$

So $(1-\epsilon)\|x\| < m(x)$ for all $\epsilon \in (0, (\|x\| - 1)\|x\|^{-1})$. This implies that $\|x\| \leq m(x)$, hence *P5* is obtained.

P6. We have that $m(x) = 1$ implies that $\|x\| = 1$. Now assume that $\|x\| = 1$. By the definition of $\|x\|$ we have that for any $\epsilon > 0$ there exists $\mu > 0$ such that $1 + \epsilon > \mu > \|x\|$ and $m(x\mu^{-1}) \leq 1$. By part *P2* of Proposition 2.3, we have

$$m(x) \leq \mu^K m(x\mu^{-1}) \leq \mu^K < (1+\epsilon)^K.$$

So, $(m(x))^{K^{-1}} < 1+\epsilon$ for all $\epsilon > 0$, which implies $m(x) \leq 1$. If $m(x) < 1$, then we can choose $r \in (0,1)$ such that $m(x) < r^K < 1$. By part *P1* of Proposition 2.3, we have $m(r^{-1}x) \leq (r^K)^{-1}m(x) < 1$ hence $\|x\| \leq r < 1$ which is a contradiction. Therefore $m(x) = 1$.

P7. Follows directly from *P4.* and *P6.*

P8. Follows from *P6* and *P7.*

P9. Suppose $0 < r < 1$ and $\|x\| > r$. Then $\|xr^{-1}\| > 1$. By *P5* we have $m(xr^{-1}) > 1$. Hence by part *P1* of Proposition 2.3, we obtain that $m(x) \geq r^K m(r^{-1}x) > r^K$.

P10. Suppose that $r \geq 1$ and $\|x\| < r$. Then $\|xr^{-1}\| < 1$. By *P7* we have $\|xr^{-1}\| < 1$. If $r = 1$, it is obvious that $m(x) < 1 = r^K$. If $r > 1$, then by part *P2* of Proposition 2.3; we obtain that $m(x) \leq r^K m(r^{-1}x) < r^K$. □

PROPOSITION 2.5. *Let (x_n) be a sequence in $\mathcal{A}(p)$, where $p = (p_k)$ is bounded. Then;*

P11. If $\|x_n\| \to 1$ as $n \to \infty$, then $m(x_n) \to 1$ as $n \to \infty$.

P12. If $m(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.

Proof. *P11.* Suppose that $\|x\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1-\epsilon < \|x_n\| < 1+\epsilon$ for all $n \in \mathbb{N}$. By part *P9.* and *P10.* of Proposition 2.4 we have $(1-\epsilon)^K < m(x_n) < (1+\epsilon)^K$ for all $n \geq N$ which implies $m(x_n) \to 1$ as $n \to \infty$.

P12. Suppose that $\|x_n\| \nrightarrow 0$ as $n \to \infty$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By part *P9.* of Proposition 2.4 we have $m(x_{n_k}) > \epsilon^K$ for all $k \in \mathbb{N}$. This implies $m(x_{n_k}) \nrightarrow 0$ as $n \to \infty$. □

Now we shall show that $\mathcal{A}(p)$ has the H-property but first we give a lemma:

LEMMA 2.6. *Let $x \in \mathcal{A}(p)$ and $(x^n) \subseteq \mathcal{A}(p)$. If $\lim_n m(x^n) = m(x)$ and $\lim_n x_i^n = x_i$ for all $i \in \mathbb{N}$ then $\lim_n x^n = x$ in $\mathcal{A}(p)$, that is $\|x^n - x\| \to 0$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $m(x) = \sum_n (a_n \sum_{i=0}^n |x_i|)^{p_n} < \infty$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \left(a_n \sum_{i=0}^{n} |x_i|\right)^{p_n} < \epsilon(2^{K+1}3)^{-1}. \tag{2.5}$$

Since

$$m(x^n) - \sum_{n=0}^{n_0}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n} \to m(x) - \sum_{n=0}^{n_0}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n}$$

as $(n \to \infty)$ and $x_i^n \to x_i$ as $n \to \infty$ as for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that

$$m(x^n) - \sum_{n=0}^{n_0}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n} < m(x) - \sum_{n=0}^{n_0}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n} + (2^{K+1}3)^{-1} \tag{2.6}$$

for all $n \geq n_0$, and

$$\sum_{n=0}^{n_0}\Big(a_n \sum_{i=0}^{n} |x_i^n - x_i|\Big)^{p_n} < 3^{-1}\epsilon \tag{2.7}$$

for all $n \geq n_0$. It follows from (2.5), (2.6) and (2.7) that for $n \geq n_0$

$$\begin{aligned}
m(x^n - x) &= \sum_{n}\Big(a_n \sum_{i=0}^{n} |x_i^n - x_i|\Big)^{p_n} \\
&= \sum_{n=0}^{n_0}\Big(a_n \sum_{i=0}^{n} |x_i^n - x_i|\Big)^{p_n} + \sum_{n=n_0+1}^{\infty}\Big(a_n \sum_{i=0}^{n} |x_i^n - x_i|\Big)^{p_n} \\
&< 3^{-1}\epsilon + 2^K\Big[\sum_{n=n_0+1}^{\infty}\Big(a_n \sum_{i=0}^{n} |x_i^n|\Big)^{p_n} + \sum_{n=n_0+1}^{\infty}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n}\Big] \\
&= 3^{-1}\epsilon + 2^K\Big[m(x^n) - \sum_{n=0}^{n_0}\Big(a_n \sum_{i=0}^{n} |x_i^n|\Big)^{p_n} + \sum_{n=n_0+1}^{\infty}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n}\Big] \\
&< 3^{-1}\epsilon + 2^K\Big[m(x^n) - \sum_{n=0}^{n_0}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n} + (2^K 3)^{-1}\epsilon \\
&\quad + \sum_{n=n_0+1}^{\infty}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n}\Big] \\
&= 3^{-1}\epsilon + 2^K\Big[\sum_{n=n_0+1}^{\infty}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n} + (2^K 3)^{-1}\epsilon \\
&\quad + \sum_{n=n_0+1}^{\infty}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n}\Big] \\
&= 3^{-1}\epsilon + 2^K\Big[(2^K 3)^{-1}\epsilon + 2\sum_{n=n_0+1}^{\infty}\Big(a_n \sum_{i=0}^{n} |x_i|\Big)^{p_n}\Big] \\
&\quad < 3^{-1}\epsilon + 3^{-1}\epsilon + 3^{-1}\epsilon = \epsilon.
\end{aligned}$$

This show that $m(x^n - x) \to 0$ as $n \to \infty$. Hence by part *P8* of Proposition 2.5, we have that $\|x^n - x\| \to 0$ as $n \to \infty$. □

THEOREM 2.7. *The $\mathcal{A}(p)$ has the H-property.*

Proof. Let $x \in S(\mathcal{A}(p))$ and $(x^n) \subseteq \mathcal{A}(p)$ be such that $\|x^n\| \to 1$ and $x^n \to x$ *weakly* as $n \to \infty$. From Proposition 2.2, we have $m(x) = 1$ so it follows from Proposition 2.3 that $m(x^n) \to m(x)$ as $n \to \infty$. Since the mapping $p_i : \mathcal{A}(p) \longrightarrow \mathbb{R}$, defined by $p_i(y) = y_i$ is a continuous linear functional on $\mathcal{A}(p)$ it follows that $x_i^n \to x_i$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.6, we get $x^n \to x$ as $n \to \infty$. □

THEOREM 2.8. *The space $\mathcal{A}(p)$ is rotund whenever $p = (p_n)$ is bounded.*

Proof. Let $x \in S(\mathcal{A}(p))$ and $y, z \in B(\mathcal{A}(p))$ with $x = 2^{-1}(y+z)$. By Proposition 2.2 and convexity of m, we have

$$1 = m(x) \leq 2^{-1}(m(y) + m(z)) \leq 2^{-1}(1+1),$$

so that $m(x) = 2^{-1}(m(y) + m(z)) = 1$. This implies that

$$\Big(a_k \sum_{i=0}^{k} |2^{-1}(y_i + z_i)|\Big)^{p_k} = 2^{-1}\Big(a_k \sum_{i=0}^{k} |y_i|\Big)^{p_k} + 2^{-1}\Big(a_k \sum_{i=0}^{n} |z_i|\Big)^{p_k} \tag{2.8}$$

for all $k \in \mathbb{N}$. We shall show that $y_i = z_i$ for all $i \in \mathbb{N}$. From (2.8), we have

$$|x_1|^{p_1} = 2^{-1}[|y_1| + |z_1|]^{p_1}. \tag{2.9}$$

Since the mapping $u \to |u|^{p_1}$ is strictly convex, it implies by (2.8) that $y_1 = z_1$. Now assume that $y_i = z_i$ for all $i = 1, 2, \ldots, n-1$. Then $y_i = z_i = x_i$ for all $i = 1, 2, \ldots, n-1$. From (2.8) we have

$$\Big(a_n \sum_{i=0}^{n} |2^{-1}(y_i + z_i)|\Big)^{p_n} = \Big(2^{-1}[a_n \sum_{i=0}^{n} |y_i| + a_n \sum_{i=0}^{n} |z_i|]\Big)^{p_n} \tag{2.10}$$

$$= 2^{-1}\Big(a_n \sum_{i=0}^{n} |y_i|\Big)^{p_n} + 2^{-1}\Big(a_n \sum_{i=0}^{n} |z_i|\Big)^{p_n}. \tag{2.11}$$

By the convexity of the mapping $u \to |u|^{p_1}$ it implies that $a_n \sum_{i=0}^{n} |y_i| = a_n \sum_{i=0}^{n} |z_i|$. Since $y_i = z_i$ for all $i = 1, 2, \ldots, n-1$ we get that

$$|y_n| = |z_n|. \tag{2.12}$$

If $y_n = 0$, then we have $y_n = z_n = 0$. Suppose that $y_n \neq 0$. Then $z_n \neq 0$. If $y_n z_n < 0$ it follows from (2.12) that $y_n + z_n = 0$. This implies by (2.10) and (2.12)

$$\Big(a_n \sum_{i=0}^{n-1} |x_i|\Big)^{p_n} = \Big(a_n\Big(\sum_{i=0}^{n-1} |x_i| + |y_i|\Big)\Big)^{p_n},$$

which is a contradiction. Thus, we have $y_n z_n > 0$. This implies that, by (2.9) $y_n = z_n$. Thus, by induction, we have $y_n = z_n$ for all $n \in \mathbb{N}$, so $y = z$. □

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INTERNATIONAL TURKISH SCHOOL
RIYADH, SAUDI ARABIA
E-mail: msengonul@yahoo.com

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N. Hussain, A. Latif, S. Al-Mezel

NONCOMMUTING MAPS AND INVARIANT APPROXIMATIONS

Abstract. We obtain common fixed point results for generalized I-nonexpansive compatible as well as weakly compatible maps. As applications, various best approximation results for this class of maps are derived in the setup of certain metrizable topological vector spaces.

1. Introduction and preliminaries

Let X be a linear space. A p-norm on X is a real-valued function $\|\cdot\|_p$ on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$,
(ii) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$,
(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$,

for all x, $y \in X$ and all scalars α. The pair $(X, \|, \|_p)$ is called a p-normed space. It is a metric linear space with a translation invariant metric d_p defined by $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of the usual normed space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \leq 1$(see [15]). The spaces l_p and L_p, $0 < p \leq 1$ are p-normed spaces. A p-normed space is not to necessarily a locally convex space. Recall that dual space X^* separates points of X (or equivalently X^* is total [18]) if for each nonzero $x \in X$, there exists $f \in X^*$ such that $f(x) \neq 0$. In this case the weak topology on X is well-defined and is Hausdorff. Notice that if X is not locally convex space, then X^* need not separate the points of X. For example, if $X = L_p[0, 1]$, $0 < p < 1$, the space of to the power p integrable functions, or $X = S[0, 1]$, the space of measurable functions, then

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$X^* = \{0\}$(see [15, 18, 20]). However, there are some non-locally convex spaces X (such as the p-normed spaces l_p, $0 < p < 1$) whose dual X^* separates the points of X.

Let X be a metric linear space and M a nonempty subset of X. The set $P_M(u) = \{x \in M : d(x,u) = dist(u,M)\}$ is called the set of best approximants to $u \in X$ from M, where $dist(u,M) = inf\ \{d(y,u) : y \in M\}$. We shall use N to denote the set of positive integers, $cl(S)$ to denote the closure of a set S. The diameter of M is denoted and defined by $\delta(M) = \sup\ \{\|x-y\| : x,y \in M\}$. A mapping $I : X \to X$ has diminishing orbital diameters (d.o.d.) [13] if for each $x \in X$, $\delta(O(x)) < \infty$ and whenever $\delta(O(x)) > 0$, there exists $n = n_x \in N$ such that $\delta(O(x)) > \delta(O(I^n(x)))$, where $O(x) = \{I^k(x) : k \in N \cup \{0\}\}$ is the orbit of I at x and $O(I^n(x)) = \{I^k(x) : k \in N \cup \{0\}$ and $k \geq n\}$ is the orbit of I at $I^n(x)$ for $n \in N \cup \{0\}$. Let I be a self-map of a topological space X. The orbit $O(x)$ of I at x is proper if and only if $O(x) = \{x\}$ or there exists $n = n_x \in N$ such that $cl(O(I^n(x)))$ is a proper subset of $cl(O(x))$. If $O(x)$ is proper for each $x \in M \subset X$, we shall say that I has proper orbits on M. Observe that in metric space (X,d) if I has d.o.d. on X, then I has proper orbits [10, 11]. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called an I-contraction if, there exists $0 \leq k < 1$ such that $d(Tx,Ty) \leq kd(Ix,Iy)$ for any $x,y \in M$. If $k = 1$, then T is called I-nonexpansive. A mapping $T : M \to M$ is called (1) completely continuous if $\{x_n\}$ converges weakly to x implies that $\{Tx_n\}$ converges strongly to Tx; (2) demiclosed at 0 if for every sequence $\{x_n\} \in M$ such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to 0, we have $Tx = 0$. The mappings I and T are said to satisfy the condition (A^0) if for any sequence $\{x_n\}$ in M, $D \in C(M)$ such that $dist(x_n, D) \to 0$ and $d(Ix_n, Tx_n) \to 0$ as $n \to \infty$, there exists $y \in D$ with $Iy = Ty$, where $C(M)$ denotes the class of nonempty closed subsets of M. The set of fixed points of T (resp. I) is denoted by $F(T)$ (resp. $F(I)$). A point $x \in M$ is a common fixed (coincidence) point of I and T if $x = Ix = Tx(Ix = Tx)$. The set of coincidence points of I and T is denoted by $C(I,T)$. The pair $\{I,T\}$ is called (3) commuting if $TIx = ITx$ for all $x \in M$; (4) R-weakly commuting if for all $x \in M$ there exists $R > 0$ such that $d(ITx, TIx) \leq \mathrm{R}d(Ix,Tx)$. If $R = 1$, then the maps are called weakly commuting; (5) compatible [9] if $\lim_n d(TIx_n, ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M; (6) weakly compatible if they commute at their coincidence points, i.e.,if $ITx = TIx$ whenever $Ix = Tx$. If I and T are weakly compatible and do have a coincidence point, I and T are called [3, 10] nontrivially weakly compatible. The subset M of a linear space is called q-starshaped with $q \in M$ if the segment $[q,x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to

x, is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called;

(7) R-subcommuting on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(ITx, TIx) \leq \frac{R}{k} d((1-k)q + kTx, Ix)$ for each $k \in (0,1]$. If $R = 1$, then the maps are called 1-subcommuting [7]; (8) R-subweakly commuting on M (see [8, 23]) if for all $x \in M$, there exists a real number $R > 0$ such that $d(ITx, TIx) \leq R dist(Ix, [q, Tx])$. Clearly, R-weakly commuting, and compatible maps are weakly compatible but not conversely in general. R-subcommuting and R-subweakly commuting maps are compatible but the converse does not hold in general [11].

In 1995, Jungck and Sessa [12] extended the results of Meinardus [17], Singh [25], Habiniak [4] and Sahab, Khan and Sessa [21] to the pair of commuting maps defined on weakly compact subset of a Banach space. Latif [16], further extended these results to the setting of p-normed spaces. More recently, Shahzad [23, 24]], Hussain and Jungck [11], Hussain et al. [8], Jungck and Hussain [11] and O'Regan and Hussain [19] further extended the above mentioned results to R-subweakly commuting and weakly compatible maps. The aim of this paper is to establish a general common fixed point theorem for compatible and weakly compatible generalized I-nonexpansive maps in the setting of locally bounded topological vector spaces and locally convex topological vector spaces. As application, we derive some results on the existence of best approximations. Our results unify and extend the results of Dotson [1, 2], Habiniak [4], Hussain and Berinde [5], Hussain and Khan [7], Hussain, O'Regan and Agarwal [8], Jungck and Sessa [12], Khan et al. [13], Khan and Khan [14], Latif [16], O'Regan and Hussain [19], Sahab et al. [21], Sahney et al. [22], Shahzad [23, 24], and Singh [25, 26].

Here, we state some useful results.

THEOREM 1.1 [3]. *Let X be a Hausdorff topological space, and I, T be continuous and nontrivially weakly compatible self-maps of X. Then there exists a point z in X such that $Iz = Tz = z$, provided T satisfies following condition*

(*C*) $\quad A \cap F(T) \neq \emptyset$ *for any nonempty T-invariant closed set $A \subset X$.*

The next theorem gives conditions under which condition (C) is satisfied.

THEOREM 1.2 ([10], Theorem 3.1). *Let X be a Hausdorff topological space and T be a continuous self-map of X. If T has relatively compact proper orbits then T satisfies condition (C).*

2. Common fixed point and approximation results

The following recent result will be needed in the sequel.

THEOREM 2.1 [11]. *Let M be a subset of a metric space (X,d), and I and T be self-maps of M. Assume that $clT(M) \subset I(M)$, $clT(M)$ is complete and I, T satisfy for all $x,y \in M$ and $0 \leq h < 1$ the condition*

$$(2.1) \quad d(Tx,Ty) \leq h\max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)\}.$$

Then I and T have a unique coincidence point in M.

Throughout this section, we shall assume that X^* separates points of a p-normed space X whenever weak topology is under consideration.

THEOREM 2.2. *Let I and T be self-maps on a q-starshaped subset M of a p-normed space X. Assume that T satisfies condition (C), $clT(M) \subset I(M)$, $q \in F(I)$ and I is affine. Suppose that I and T are continuous, and satisfy*

$$(2.2) \quad \|Tx - Ty\|_p \leq max \left\{ \begin{matrix} \|Ix - Iy\|_p, dist(Ix,[Tx,q]), dist(Iy,[Ty,q]), \\ dist(Ix,[Ty,q]), dist(Iy,[Tx,q]) \end{matrix} \right\}$$

for all $x,y \in M$. Then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds;

- (i) *$clT(M)$ is compact and I and T are compatible,*
- (ii) *M is complete and bounded, T is a compact map and I and T are compatible,*
- (iii) *M is complete and bounded, I and T satisfy condition (A^0) and I and T are weakly compatible,*
- (iv) *X is complete, M is weakly compact, $I - T$ is demiclosed at 0 and I and T are weakly compatible,*
- (v) *X is complete, M is weakly compact, I and T are completely continuous and I and T are weakly compatible.*

Proof. Define $T_n : M \to M$ by $T_n x = (1-k_n)q + k_n Tx$ for some q and all $x \in M$ and a fixed sequence of real numbers $k_n \in (0,1)$ converging to 1. Then, for each n, $clT_n(M) \subset I(M)$ as M is q-starshaped, $clT(M) \subset I(M)$, I is affine and $Iq = q$. By (2.2),

$$\begin{aligned} \|T_n x - T_n y\|_p &= (k_n)^p \|Tx - Ty\|_p \\ &\leq (k_n)^p max\{\|Ix - Iy\|_p, dist(Ix,[Tx,q]), dist(Iy,[Ty,q]), \\ &\quad dist(Ix,[Ty,q]), dist(Iy,[Tx,q])\} \\ &\leq (k_n)^p max\{\|Ix - Iy\|_p, \|Ix - T_n x\|_p, \|Iy - T_n y\|_p, \\ &\quad \|Ix - T_n y\|_p, \|Iy - T_n x\|_p\}, \end{aligned}$$

for each $x,y \in M$.

(i) Since $clT(M)$ is compact, $clT_n(M)$ is also compact and hence complete. By Theorem 2.1, for each $n \geq 1$, there exists $x_n \in M$ such that $Ix_n = T_n x_n$. The compactness of $cl(T(M)$ implies that there exists a subsequence $\{Tx_m\}$of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Since $k_m \to 1$, $Ix_m = (1 - k_m)q + k_m Tx_m$ converges to y. Since T and I are continuous, then $TIx_m \to Ty$ and $ITx_m \to Iy$ as $m \to \infty$. By the compatibility of I and T, we obtain $0 = \lim_{m\to\infty} \|ITx_m - TIx_m\|_p = \|Iy - Ty\|_p$. Thus $Iy = Ty$. Hence the pair $\{I, T\}$ is nontrivially compatible. Theorem 1.1 guarantees that $M \cap F(I) \cap F(T) \neq \emptyset$.

(ii) As in (i), there is a unique $x_n \in M$ such that $T_n x_n = Ix_n$. As T is compact and $\{x_n\}$ being in M is bounded so $\{Tx_n\}$ has a subsequence $\{Tx_m\}$ such that $\{Tx_m\} \to z$ as $m \to \infty$. Then the definition of $T_m x_m$ implies $Ix_m \to z$. So by the continuity of T and I, $TIx_m \to Tz$ and $ITx_m \to Iz$ as $m \to \infty$. By the compatibility of I and T, we obtain $Iz = Tz$. Hence the pair $\{I, T\}$ is nontrivially compatible. Theorem 1.1 guarantees that $M \cap F(I) \cap F(T) \neq \emptyset$.

(iii) As in (i) there exists $x_n \in M$ such that $Ix_n = T_n x_n$. But M is bounded, so $\|Ix_n - Tx_n\|_p = \|((1 - k_n)q + k_n Tx_n) - Tx_n\|_p \leq (1 - k_n)^p(\|q\|_p + \|Tx_n\|_p) \to 0$ as $n \to \infty$. By condition (A^0), $Ix_0 = Tx_0$ for some $x_0 \in M$. Hence the pair $\{I, T\}$ is nontrivially weakly compatible. Theorem 1.1 guarantees that $M \cap F(I) \cap F(T) \neq \emptyset$.

(iv) Since M is weakly compact and hence complete, then $cl(T_n(M))$ is complete. By Theorem 2.1, for each $n \geq 1$, there exists $x_n \in M$ such that $Ix_n = T_n x_n$. The weak compactness of M implies that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y$ weakly as $m \to \infty$. Since $\{x_m\}$ is bounded, $k_m \to 1$, so $\|(Ix_m - Tx_m\|_p = \|((1 - k_m)q + k_m Tx_m) - Tx_m\|_p \leq (1 - k_m)^p(\|q\|_p + \|Tx_m\|_p)$ converges to 0. Since $(I - T)$ is demiclosed at 0 so $(I - T)y = 0$ and hence $Iy = Ty$. Thus the pair $\{I, T\}$ is nontrivially weakly compatible and the conclusion follows from Theorem 1.1.

(v) As in (iv), we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in M converging weakly to $y \in M$ as $m \to \infty$. Since I and T are completely continuous, then $Ix_m \to Iy$ and $Tx_m \to Ty$ as $m \to \infty$. Since $k_m \to 1$, then $Ix_m = T_m x_m = k_m Tx_m + (1 - k_m)q \to Ty$ as $m \to \infty$. Using the uniqueness of the limit, we have $Iy = Ty$. Thus the pair $\{I, T\}$ is nontrivially weakly compatible and the conclusion follows from Theorem 1.1.

COROLLARY 2.3. *Let M be a q-starshaped subset of a p-normed space X, and I and T continuous self-maps of M. Suppose that I is affine with $q \in F(I)$, $clT(M) \subset I(M)$ and $clT(M)$ is compact. If T has d.o.d., the pair $\{I, T\}$ is compatible and satisfy (2.2) for all $x, y \in M$, then $M \cap F(T) \cap F(I) \neq \emptyset$.*

Proof. Since T has d.o.d, T has proper orbits [10]. As $clT(M))$ is compact, T has relatively compact orbits. Therefore by Theorem 1.2, T satisfies condition (C). The result now follows by Theorem 2.2(i).

REMARK 2.4. Theorem 2.2 and Corollary 2.3 extend and improve Theorems 1 and 2 of Dotson [1], Theorem 4 of Habiniak [4], Theorem 2.3 and Corollary 2.4 of Jungck and Hussain [11], Theorem 6 of Jungck and Sessa [12], Theorem 2.4 of O'Regan and Hussain [19], Theorem 2.2 of Shahzad [24], and corresponding results in [14, 16, 21, 23, 25].

The following result extends Theorem 3 of [21], Theorem 8 of [4], and the main results in [14, 16, 17, 25].

THEOREM 2.5. *Let M be subset of a p-normed space X and let $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Assume that T satisfies condition (C), $I(P_M(u)) = P_M(u)$ and the pair $\{I, T\}$ is continuous and compatible on $P_M(u)$ and satisfy for all $x \in P_M(u) \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ max\{\|Ix - Iy\|_p, dist(Ix, [q, Tx]), dist(Iy, [q, Ty]), & \\ dist(Ix, [q, Ty]), dist(Iy, [q, Tx])\} & \text{if } y \in P_M(u). \end{cases} \tag{2.3}$$

If $P_M(u)$ is closed, q-starshaped with $q \in F(I)$, I is affine and $clT(P_M(u))$ is compact then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in P_M(u)$. Then $\|x - u\|_p = dist(u, M)$. Note that for any $k \in (0, 1)$,

$$\|ku + (1 - k)x - u\|_p = (1 - k)^p \|x - u\|_p < dist(u, M).$$

It follows that the line segment $\{ku + (1 - k)x : 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Also since $Ix \in P_M(u)$, $u \in F(T) \cap F(I)$ and T and I satisfy (2.3), we have

$$\|Tx - u\|_p = \|Tx - Tu\|_p \leq \|Ix - Iu\|_p = \|Ix - u\|_p = dist(u, M).$$

Thus $Tx \in P_M(u)$. Theorem 2.2(i) further guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Let $D = P_M(u) \cap C_M^I(u)$, where $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$.

The following result provides a non-locally convex space analogue of Theorem 3.3 [7] for more general class of maps.

THEOREM 2.6. *Let M be subset of a p-normed space X and $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that T satisfies condition (C), D is closed q-starshaped with $q \in F(I)$, I is affine, $clT(D)$ is compact, $I(D) = D$ and the pair $\{I, T\}$ is compatible and continuous on D and, for all $x \in D \cup \{u\}$, satisfies the following inequality,*

$$(2.4) \qquad \|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ max\{\|Ix - Iy\|_p, dist(Ix, [q, Tx]), dist(Iy, [q, Ty]), \\ dist(Ix, [q, Ty]), dist(Iy, [q, Tx])\} & \text{if } y \in D. \end{cases}$$

If I is nonexpansive on $P_M(u) \cup \{u\}$, then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in D$ then proceeding as in the proof of Theorem 2.5, we obtain $Tx \in P_M(u)$. Moreover, since I is nonexpansive on $P_M(u) \cup \{u\}$ and T satisfies (2.4), we obtain

$$\|ITx - u\|_p \leq \|Tx - Tu\|_p \leq \|Ix - Iu\|_p = dist(u, M).$$

Thus $ITx \in P_M(u)$ and so $Tx \in C_M^I(u)$. Hence $Tx \in D$. Consequently, $clT(D) \subset D = I(D)$. Now Theorem 2.2(i) guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

REMARK 2.7. (a) It is worth to mention that approximation results similar to Theorem 2.5 and Theorem 2.6 can be obtained, using Theorem 2.2(ii)-(v) which extend and improve the corresponding results in [12, 14, 16, 17, 21, 24, 25].
(b) As an application of Theorem 2.2(i), we can prove Theorem 2.7 of [11] in the setup of p-normed space X.
(c) The results of this section hold true for the the nonlocally convex spaces, for example, the sequences spaces l_p, $0 < p < 1$ and Hardy spaces H^p, $0 < p < 1$ whose topological duals are total. When topological dual is not total the situation becomes more complicated. The topological dual of $X = L_p[0, 1]$, $0 < p < 1$, and $X = S[0, 1]$, vanish and Shauder's conjecture is still open even for these spaces (see for details [18, 20] and references therein).

3. Further results

(1) All results of the paper (Theorem 2.2-Remark 2.7) remain valid in the setup of a metrizable locally convex topological vector space (X, d), where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x, y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$ (recall that d_p is translation invariant

and satisfies $d_p(\alpha x, \alpha y) \leq (\alpha)^p d_p(x,y)$ for any scalar $\alpha \geq 0$). Consequently, Theorem 2.2-Theorem 3.3 due to Hussain and Khan [7] and corresponding results in [5, 22, 26] are improved and extended.

We define $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$ and denote by $\Im_0$ the class of closed convex subsets of X containing 0. For $M \in \Im_0$, we define $M_u = \{x \in M : d(0,x) \leq 2d(0,u)\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$.

Following result extends Theorem 8 in [4], Theorem 3.3 in [5], Theorems 2.9-2.10 in [11], Theorem 2.6 in [19], Theorem 2.3-2.4 in [23], Theorem 2.9 in [24] and many others.

THEOREM 3.1. *Let X be a metrizable locally convex space (X,d) where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x,y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$, and I and T be self-mappings of X with $u \in F(I) \cap F(T)$ and $M \in \Im_0$ such that $T(M_u) \subset I(M) \subset M$. Suppose that I is affine, $d(Ix,u) \leq d(x,u)$, $d(Tx,u) \leq d(Ix,u)$ for all $x \in M$, the pair $\{I,T\}$ is continuous on M and one of the following two conditions is satisfied:*

(a) *$clI(M)$ is compact,*
(b) *$clT(M)$ is compact.*

Then

(i) *$P_M(u)$ is nonempty, closed and convex,*
(ii) *$T(P_M(u)) \subset I(P_M(u)) \subset P_M(u)$ provided that $d(Ix,u) \leq d(x,u)$ for all $x \in C_M^I(u)$,*
(iii) *$P_M(u) \cap F(I) \cap F(T) \neq \emptyset$ provided that $d(Ix,u) \leq d(x,u)$ for all $x \in C_M^I(u)$, I and T satisfy condition (C), $I(P_M(u))$ is closed, the pair $\{I,T\}$ is compatible on $P_M(u)$ and satisfies for all $q \in F(I)$,*

$$d(Tx,Ty) \leq \max\{d(Ix,Iy), dist(Ix,[q,Tx]), dist(Iy,[q,Ty]), dist(Ix,[q,Ty]), dist(Iy,[q,Tx])\},$$

for all $x, y \in P_M(u)$.

Proof.

(i) Let $r = dist(u,M)$. Then there is a minimizing sequence $\{y_n\}$ in M such that $lim_n d(u,y_n) = r$. As $clI(M)$ is compact so $\{Iy_n\}$ has a convergent subsequence $\{Iy_m\}$ with $\lim_m Iy_m = x_0$ (say) in M. Now by using $d(Ix,u) \leq d(x,u)$ we get

$$r \leq d(x_0,u) = \lim_m d(Iy_m,u) \leq \lim_m d(y_m,u) = \lim_n d(y_n,u) = r.$$

Hence $x_0 \in P_M(u)$. Thus $P_M(u)$ is nonempty closed and convex. Similarly, when $clT(M)$ is compact we get same conclusion by using inequalities $d(Ix,u) \leq d(x,u)$ and $d(Tx,u) \leq d(Ix,u)$ for all $x \in M$.

(ii) Let $z \in P_M(u)$. Then $d(Tz,u) \leq d(Iz,u) = dist(u,M)$. This implies that $Tz \in P_M(u)$ and so $T(P_M(u)) \subset P_M(u)$. Also we have $I(P_M(u)) \subset P_M(u)$. Let $y \in T(P_M(u)$. Since $T(M_u) \subset I(M)$ and $P_M(u) \subset M_u$, then there exist $z \in P_M(u)$ and $x \in M$ such that $y = Tz = Ix$. Thus, we have $d(Ix,u) = d(Tz,u) \leq d(Iz,u) \leq d(z,u) = dist(u,M)$. Hence $x \in C_M^I(u) = P_M(u)$ and so (ii) holds.

(iii) (a) By (i) $P_M(u)$ is closed and by (ii) $P_M(u)$ is I-invariant, so by condition (C) of I, $P_M(u) \cap F(I) \neq \emptyset$. It follows that there exists $q \in P_M(u)$ such that $q \in F(I)$. By (ii), the compactness of $clI(M_u)$ implies that $clT(P_M(u))$ is compact. The conclusion now follows from Theorem 2.2(i)(which holds for metrizable locally convex space) applied to $P_M(u)$.

(iii) (b) By (i) $P_M(u)$ is closed and by (ii) $P_M(u)$ is I-invariant, so by condition (C) of I, $P_M(u) \cap F(I) \neq \emptyset$, it follows that there exists $q \in P_M(u)$ such that $q \in F(I)$. Theorem 2.2(i) further guarantees that $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.

(2) Let M be subset of a $p-$normed space X and $F = \{f_x\}_{x \in M}$ a family of functions from $[0,1]$ into M such that $f_x(1) = x$ for each $x \in M$. The family F is said to be contractive [2, 13] if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $x, y \in M$ and all $t \in (0,1)$, we have $\|f_x(t) - f_y(t)\|_p \leq [\phi(t)]^p \|x - y\|_p$. The family F is said to be jointly (weakly) continuous if $t \to t_0$ in [0,1] and $x \to x_o$ ($x \to x_0$ weakly) in M, then $f_x(t) \to f_{x_0}(t_0)$ ($f_x(t) \to f_{x_0}(t_o)$ weakly) in M. We observe that if $M \in X$ is q-starshaped and $f_x(t) = (1-t)q + tx$, ($x \in M; t \in (0,1)$), then $F = \{f_x\}_{x \in M}$ is a contractive jointly continuous and jointly weakly continuous family with $\phi(t) = t$. Thus the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets ((see [2, 8]). Following the arguments as above and those in [8, 13], we can obtain all of the results of the paper (Theorem 2.2-Remark 2.7) provided I is assumed to be surjective, and affinity of I is replaced by $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in M$, $\alpha \in [0,1]$, and the q-starshapedness of the set M is replaced by the property of contractivity and joint continuity or weak joint continuity. Consequently, recent results due to Hussain et al. [8], and Khan et al [13] are extended to the class of weakly compatible pair $\{I, T\}$ where T satisfies property (C).

(3) A subset M of a linear space X is said to have property (N) with respect to T [5, 8] if,

  (i) $T : M \to M$,

  (ii) $(1 - k_n)q + k_nTx \in M$, for some $q \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in M$.

A mapping I is said to be affine on a set M with property (N) if $I((1-k_n)q+k_nTx)=(1-k_n)Iq+k_nITx$ for each $x\in M$ and $n\in N$. All of the results of the paper (Theorem 2.3-Remark 2.7) remain valid, provided I is assumed to be surjective and the q-starshapedness of the set M is replaced by the property (N), in the setup of p-normed spaces and metrizable locally convex topological vector space(tvs) (X,d) where d is translation invariant and $d(\alpha x,\alpha y)\le \alpha d(x,y)$, for each α with $0<\alpha<1$ and $x,y\in X$. Consequently, recent results due to Hussain and Berinde [5], and Hussain, O'Regan and Agarwal [8] are extended to the class of weakly compatible maps, where T satisfies property (C).

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DEPARTMENT OF MATHEMATICS
KING ABDUL AZIZ UNIVERSITY
P. O. BOX 80203
JEDDAH 21589, SAUDI ARABIA

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