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OSCILLATION CRITERIA FOR HIGH-ORDER SUBLINEAR NEUTRAL DELAY FORCED DIFFERENCE EQUATIONS WITH OSCILLATING COEFFICIENTS

Abstract. In this paper we are concerned with the oscillation of solutions of higher-order sublinear neutral type difference equation with an oscillating coefficient of the form

$$\Delta^n[y(k)+p(k)y(\tau(k))]+q(k)y^\alpha(\sigma(k))=r(k) \quad N \ni n \geq 2,$$

where $p(k)$ is an oscillatory function which is interesting. We obtain some comparison criteria for oscillatory behaviour. The results are new when $n=2$ and $n=3$.

1. Introduction

We consider the higher-order sublinear difference equation of the form

$$(1) \qquad \Delta^n[y(k)+p(k)y(\tau(k))]+q(k)y^\alpha(\sigma(k))=r(k) \quad n,k \in N, \ n \geq 2,$$

where $\alpha \in (0,1)$ is a ratio of positive odd integers, and the following conditions are always held:

i) $p(k)$ is an oscillating function with $\lim_{k\to\infty} p(k)=0$,
ii) $q(k) \geq 0$ for $k \geq k_0$,
iii) $r(k)$ is an oscillating function with $\Delta^n s(k)=r(k)$ and $\lim_{k\to\infty} s(k)=0$,
iv) $\tau(k)<k$ with $\tau(k)\to+\infty$ as $k\to\infty$ and $\sigma(k)<k$ with $\sigma(k)\to+\infty$ as $k\to\infty$.

By a solution of Eq.(1), we mean any function $y(k): Z \to R$ which is defined for all $k \geq \min_{i\geq 0}\{\tau(i),\sigma(i)\}$ and satisfies Eq. (1) for sufficiently large k. We consider only such solutions which are nontrivial for all large k. As it is customary, a solution $\{y(k)\}$ is said to be oscillatory if the terms $y(k)$ of the sequence are not eventually positive or not eventually negative. Otherwise, the solution is called non-oscillatory. A difference

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equation is called oscillatory if all of its solutions oscillate. Otherwise, it is non-oscillatory. In this paper, we restrict our attention to real valued solutions $y(k)$.

Neutral difference equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. Recently, much researches have been done on the oscillatory and asymptotic behaviour of solutions of higher order delay and neutral delay type difference equations. But there are very scarsely results in the case of coefficient p_k is an oscillating function.

The purpose of this paper is to study oscillatory behaviour of solutions of the Eq. (1). For the general theory of difference equations, one can refer to [1–5]. Many references for the applications of the difference equations can be found in [4–5].

For the sake of convenience, the function $z(k)$ is defined as

$$z(k) = y(k) + p(k)y(\tau(k)) - s(k) \tag{2}$$

and $N(a) = \{a, a+1, \dots\}$.

2. Some auxiliary lemmas

LEMMA 1 ([1]). *Let $y(k)$ be defined for $k \geq k_0 \in N$, and $y(k) > 0$ with $\Delta^n y(k)$ of constant sign for $k \geq k_0$, $n \in N(1)$ and not identically zero. Then there exists an integer $m, 0 \leq m \leq n$ with $(n+m)$ even for $\Delta^n y(k) \geq 0$ or $(n+m)$ odd for $\Delta^n y(k) \leq 0$ such that*

i) *$m \leq n-1$ implies $(-1)^{m+i}\Delta^i y(k) > 0$ for all $k \geq k_0, m \leq i \leq n-1$*
ii) *$m \geq 1$ implies $\Delta^i y(k) > 0$ for all large $k \geq k_0, 1 \leq i \leq m-1$.*

LEMMA 2 ([1]). *Let $y(k)$ be defined for $k \geq k_0$, and $y(k) > 0$ with $\Delta^n y(k) \leq 0$ for $k \geq k_0$ and not identically zero. Then there exist a large $k_1 \geq k_0$ such that*

$$y(k) \geq \frac{1}{(n-1)!}(k-k_1)^{n-1}\Delta^{n-1}y(2^{n-m-1}k), \qquad k \geq k_1,$$

where m is defined as in Lemma 1. Further, if $y(k)$ is increasing, then

$$y(k) \geq \frac{1}{(n-1)!}\left(\frac{k}{2^{n-1}}\right)^{n-1}\Delta^{n-1}y(k), \quad k \geq 2^{n-1}k_1.$$

3. Main results

THEOREM 1. *Assume that n is even and inequality*

$$\Delta z(k) + \frac{1}{[2(2^{n-1})^{n-1}(n-1)!]^{\alpha}}q(k)\sigma^{\alpha(n-1)}(k)z^{\alpha}(\sigma(k)) \leq 0 \tag{C_1}$$

does not have any positive bounded solution for all sufficiently large k. Then every bounded solution of equation (1) is either oscillatory or tends to zero as $k \to \infty$.

Proof. Assume that Eq. (1) has a bounded nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive. That is, $y(k) > 0, y(\tau(k)) > 0$ and $y(\sigma(k)) > 0$ for all $k \geq k_1 \geq k_0$. Further, suppose that $y(k)$ does not tend to zero as $k \to \infty$. By (1), (2) we have

(3) $$\Delta^n z(k) = -q(k)y^\alpha(\sigma(k)), \quad (0 < \alpha < 1, \quad k \geq k_1).$$

That is $\Delta^n z(k) < 0$. It follows that $\Delta^a z(k)$ $(a = 0, 1, 2, \dots, n-1)$ is strictly monotone and eventually of constant sign. Since $y(k)$ is bounded, by virtue of (i), (iii) and (2) there is a $k_2 \geq k_1$ such that $z(k) > 0$ for all $k \geq k_2$ and $z(k)$ is bounded. Because n is even, by Lemma 1, since $m = 1$(otherwise, $z(k)$ is not bounded) there exists $k_3 \geq k_2$ such that for $k \geq k_3$

(4) $$(-1)^{i+1}\Delta^i z(k) > 0 \quad (i = 0, 1, 2, \dots, n-1).$$

In particular, since $\Delta z(k) > 0$ for $k \geq k_3$, $z(k)$ is increasing. Since $y(k)$ is bounded, $\lim_{k\to\infty} p(k)y(\tau(k)) = 0$ by (i). Then, since $\lim_{k\to\infty} s(k) = 0$ by (iii), there exists a $k_4 \geq k_3$ by (2)

$$y(k) = z(k) - p(k)y(\tau(k) + s(k) \geq \frac{1}{2}z(k) > 0$$

for all $k \geq k_4$. We may find a $k_5 \geq k_4$ such that for $k \geq k_5$ we have

$$y(\sigma(k)) \geq \frac{1}{2}z(\sigma(k)) > 0$$

and

(5) $$y^\alpha(\sigma(k)) \geq [\frac{1}{2}z(\sigma(k))]^\alpha > 0, \quad 0 < \alpha < 1.$$

From (3) and (5) we obtain the result of

(6) $$\Delta^n z(k) + q(k)[\frac{1}{2}z(\sigma(k))]^\alpha \leq 0, \quad 0 < \alpha < 1$$

for all large $k \geq k_5$. By Lemma 2, this inequality can be written as

(7) $$\Delta^n z(k) + \frac{1}{[2(2^{n-1})^{n-1}(n-1)!]^\alpha} q(k)\sigma^{\alpha(n-1)}(k)(\Delta^{n-1}z(\sigma(k)))^\alpha \leq 0, \quad k \geq k_5.$$

Let us take $u(k)$ as $\Delta^{n-1}z(k)$ i.e. $u(k) = \Delta^{n-1}z(k)$ in (7). Thus $u(k)$ satisfies for k, which is large enough,

$$\Delta u(k) + \frac{1}{[2(2^{n-1})^{n-1}(n-1)!]^\alpha} q(k)\sigma^{\alpha(n-1)}(k)u^\alpha(\sigma(k)) \leq 0$$

which does not have any eventually positive solutions by (C_1). This contradicts the fact that $\Delta^{n-1}z(k) > 0$ by (4).

In the case, where $y(k)$ is an eventually negative solution, then $-y(k)$ will be an eventually positive solution. The proof of Theorem 1 is completed. ∎

THEOREM 2. *Assume that n is odd and inequality*

$$(C_2) \qquad \Delta z(k) + \frac{(k-k_3)^{\alpha(n-1)}}{[2(n-1)!]^{\alpha}} q(k) z^{\alpha}(\sigma(k)) \le 0$$

does not have any positive bounded solution for all sufficiently large k. Then every bounded solution of equation (1) is either oscillatory or tends to zero as $k \to \infty$.

Proof. Assume that Eq. (1) has a bounded nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive. That is, $y(k) > 0, y(\tau(k)) > 0$ and $y(\sigma(k)) > 0$ for all $k \ge k_1 \ge k_0$. Further, suppose that $y(k)$ does not tend to zero as $k \to \infty$. As in the proof of Theorem 1, we can find that $z(k)$ is bounded. Because n is odd, by Lemma 1 since $m = 0$ (otherwise $z(k)$ is not bounded) there exists $k_1 \ge k_0$ such that

$$(8) \qquad (-1)^i \Delta^i z(k) > 0, \quad (i = 0, 1, 2, ..., n-1) \text{ for all } k \ge k_1.$$

In particular, since $\Delta z(k) < 0$ for $k \ge k_1$ and $z(k)$ is decreasing. Since $y(k)$ is bounded, $\lim_{k\to\infty} p(k)y(\tau(k)) = 0$ by (i). Then there exists a $k_2 \ge k_1$ by (2) and (iii)

$$y(k) = z(k) - p(k)y(\tau(k)) + s(k) \ge \frac{1}{2} z(k) > 0$$

for all $k \ge k_2$ and $z(k)$ is bounded. We may find a $k_3 \ge k_2$ such that

$$y(\sigma(k)) \ge \frac{1}{2} z(\sigma(k)) > 0$$

and we have

$$(9) \qquad y^{\alpha}(\sigma(k)) \ge [\frac{1}{2} z(\sigma(k))]^{\alpha} > 0, \quad 0 < \alpha < 1.$$

for all $k \ge k_3$. From (3) and (9) we can the result of

$$\Delta^n z(k) + \frac{1}{2^{\alpha}} q(k) z^{\alpha}(\sigma(k)) \le 0$$

for all large $k \ge k_3$. Since $z(k)$ is decreasing, we can write this last inequality in the form

$$(10) \qquad \Delta^n z(k) + \frac{1}{2^{\alpha}} q(k) z^{\alpha}(\sigma(k)) \le 0.$$

By Lemma 2, inequality (10) can be written as

$$\Delta^n z(k) + \frac{(k-k_3)^{\alpha(n-1)}}{[2(n-1)!]^{\alpha}} q(k) \Delta^{n-1} z^{\alpha}(\sigma(k)) \le 0, \; k \ge k_3.$$

Let us take $u(k)$ as $\Delta^{n-1}z(k)$ i.e. $u(k) = \Delta^{n-1}z(k)$. Thus $u(k)$ satisfies for all k, which is large enough,

$$\Delta u(k) + \frac{(k-k_3)^{\alpha(n-1)}}{[2(n-1)!]^{\alpha}} q(k)u^{\alpha}(\sigma(k)) \leq 0, \quad k \geq k_3$$

which does not have any eventually positive solutions by (C_2). This contradicts the fact that $\Delta^{n-1}z(k) > 0$ by (8).

In the case, where $y(k)$ is an eventually negative solution, then $-y(k)$ will be an eventually positive solution. The proof of Theorem 2 is completed. ∎

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