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ON VOLTERRA-FREDHOLM INTEGRAL EQUATION IN TWO VARIABLES

Abstract. The aim of this paper is to study the existence, uniqueness and other properties of solution of a certain Volterra-Fredholm integral equation in two independent variables. The main tools employed in the analysis are based on the applications of the well known Banach fixed point theorem and the new integral inequality with explicit estimate.

1. Introduction

Consider the system of Volterra-Fredholm integral equation

$$\text{(VF)} \qquad u(x,y) = h(x,y) + \int_0^x \int_0^y F(x,y,s,t,u(s,t))\,dtds + \int_0^\infty \int_0^\infty G(x,y,s,t,u(s,t))\,dtds,$$

for $(x,y) \in [0,\infty) \times [0,\infty)$, where $u, h, F, G \in R^n$, the n-dimensional Euclidean space with appropriate norm $|.|$. We denote by $R_+ = [0,\infty)$ the given subset of R, the set of real numbers and $\Delta = R_+ \times R_+$, $\Delta^2 = \{(x,y,s,t) : 0 \le s \le x < \infty,\ 0 \le t \le y < \infty\}$ and throughout assume that $h \in C(\Delta, R^n); F, G \in C(\Delta^2 \times R^n, R^n)$.

The mathematical literature concerning Volterra and Fredholm integral equations involving functions of one independent variable is particularly rich. A good deal of information on such equations may be found in [1, 4, 5, 10–13]. For the study of integral equations in several variables, we refer the interested readers to [2, 6-9, 18-20]. Volterra-Fredholm integral equations of the form (VF) are of particular interest, since the special versions of

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the same arise in a variety of applications, see [20]. The main objective of this paper is to study the existence, uniqueness and other properties of the solutions of equation (VF) by using Banach fixed point theorem (see [5, p. 37]) coupled with Bielecki type norm [3] and the new integral inequality recently established by the present author (see [17, p. 111]).

2. Existence and uniqueness

Let E be the space of those functions $\phi : \Delta \to R^n$ which are continuous and fulfil the condition

$$|\phi(x,y)| = O(\exp(\lambda(x+y))), \tag{2.1}$$

for $(x,y) \in \Delta$, where $\lambda > 0$ is a constant. In the space E we define the norm (see [3])

$$|\phi|_E = \sup_{(x,y)\in\Delta} [|\phi(x,y)| \exp(-\lambda(x+y))]. \tag{2.2}$$

It is easy to see that E with norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a constant $N \geq 0$ such that $|\phi(x,y)| \leq N \exp(\lambda(x+y))$ for $(x,y) \in \Delta$. Using this fact in (2.2) we observe that

$$|\phi|_E \leq N. \tag{2.3}$$

We need the following new integral inequality established by Pachpatte (see [17, Theorem 2.5.7 part (r_1), p. 111]). For detailed account on such inequalities, see [16, 17].

LEMMA. *Let* $u(x,y), a(x,y), b(x,y), c(x,y), f(x,y), g(x,y) \in C(R_+^2, R_+)$ *and*

$$u(x,y) \leq a(x,y) + b(x,y) \int_0^x \int_0^y f(s,t) u(s,t)\, dt ds + c(x,y) \int_0^\infty \int_0^\infty g(s,t) u(s,t)\, dt ds, \tag{2.4}$$

for $x, y \in R_+$. *If*

$$p = \int_0^\infty \int_0^\infty g(s,t) D(s,t)\, dt ds < 1, \tag{2.5}$$

then

$$u(x,y) \leq B(x,y) + M D(x,y), \tag{2.6}$$

for $x, y \in R_+$, *where*

$$B(x,y) = a(x,y) + b(x,y) A(x,y) \int_0^x \int_0^y f(s,t) a(s,t) \, dtds, \tag{2.7}$$

$$D(x,y) = c(x,y) + b(x,y) A(x,y) \int_0^x \int_0^y f(s,t) c(s,t) \, dtds, \tag{2.8}$$

$$A(x,y) = \exp\left(\int_0^x \int_0^y f(s,t) b(s,t) \, dtds \right), \tag{2.9}$$

and

$$M = \frac{1}{1-p} \int_0^\infty \int_0^\infty g(s,t) B(s,t) \, dtds. \tag{2.10}$$

Our main result in this section is given in the following theorem.

THEOREM 1. *Assume that*

(i) F, G *satisfy the conditions*

$$|F(x,y,s,t,u(s,t)) - F(x,y,s,t,v(s,t))| \leq k(x,y,s,t) |u(s,t) - v(s,t)|, \tag{2.11}$$

$$|G(x,y,s,t,u(s,t)) - G(x,y,s,t,v(s,t))| \leq r(x,y,s,t) |u(s,t) - v(s,t)|, \tag{2.12}$$

where $k, r \in C(\Delta^2, R_+)$,

(ii) *for* λ *as in (2.1), there exist nonnegative constants* α_1, α_2 *such that* $\alpha_1 + \alpha_2 < 1$ *and*

$$\int_0^x \int_0^y k(x,y,s,t) \exp(\lambda(s+t)) \, dtds \leq \alpha_1 \exp(\lambda(x+y)), \tag{2.13}$$

$$\int_0^\infty \int_0^\infty r(x,y,s,t) \exp(\lambda(s+t)) \, dtds \leq \alpha_2 \exp(\lambda(x+y)), \tag{2.14}$$

(iii) *for* λ *as in (2.1), there exists a nonnegative constant* β *such that*

$$|h(x,y)| + \int_0^x \int_0^y |F(x,y,s,t,0)| \, dtds + \int_0^\infty \int_0^\infty |G(x,y,s,t,0)| \, dtds \leq \beta \exp(\lambda(x+y)). \tag{2.15}$$

Then the equation (VF) *has a unique solution* $u(x,y)$ *on* Δ.

Proof. Let $u(x,y) \in E$ and define the operator T by

$$(2.16)\qquad (Tu)(x,y) = h(x,y) + \int_0^x\int_0^y F(x,y,s,t,u(s,t))\,dtds + \int_0^\infty\int_0^\infty G(x,y,s,t,u(s,t))\,dtds.$$

First we show that Tu maps E into itself. Evidently Tu is continuous on Δ and Tu is in R^n. We verify that (2.1) is fulfilled. From (2.16) and using the hypotheses and (2.3) we have

$$\begin{aligned}(2.17)\qquad &|(Tu)(x,y)| \\ &\le |h(x,y)| + \int_0^x\int_0^y |F(x,y,s,t,u(s,t)) - F(x,y,s,t,0)|\,dtds \\ &+ \int_0^\infty\int_0^\infty |G(x,y,s,t,u(s,t)) - G(x,y,s,t,0)|\,dtds \\ &+ \int_0^x\int_0^y |F(x,y,s,t,0)|\,dtds + \int_0^\infty\int_0^\infty |G(x,y,s,t,0)|\,dtds \\ &\le \beta\exp(\lambda(x+y)) \\ &+ \int_0^x\int_0^y k(x,y,s,t)\exp(\lambda(s+t))\,|u(s,t)|\exp(-\lambda(s+t))\,dtds \\ &+ \int_0^\infty\int_0^\infty r(x,y,s,t)\exp(\lambda(s+t))\,|u(s,t)|\exp(-\lambda(s+t))\,dtds \\ &\le \beta\exp(\lambda(x+y)) + |u|_E\int_0^x\int_0^y k(x,y,s,t)\exp(\lambda(s+t))\,dtds \\ &+ |u|_E\int_0^\infty\int_0^\infty r(x,y,s,t)\exp(\lambda(s+t))\,dtds \\ &\le [\beta + N(\alpha_1+\alpha_2)]\exp(\lambda(x+y)).\end{aligned}$$

From (2.17) it follows that $Tu \in E$. This proves that the operator T maps E into itself.

Next we verify that the operator T is a contraction map. Let $u(x,y)$, $v(x,y) \in E$. From (2.16) and using the hypotheses we have

$$
\begin{aligned}
&|(Tu)(x,y) - (Tv)(x,y)| \\
&\quad \le \int_0^x\int_0^y |F(x,y,s,t,u(s,t)) - F(x,y,s,t,v(s,t))|\,dtds \\
&\qquad + \int_0^\infty\int_0^\infty |G(x,y,s,t,u(s,t)) - G(x,y,s,t,v(s,t))|\,dtds \\
&\quad \le \int_0^x\int_0^y k(x,y,s,t)\exp(\lambda(s+t))\,|u(s,t) - v(s,t)|\exp(-\lambda(s+t))\,dtds \\
&\qquad + \int_0^\infty\int_0^\infty r(x,y,s,t)\exp(\lambda(s+t))\,|u(s,t) - v(s,t)|\exp(-\lambda(s+t))\,dtds \\
&\quad \le |u-v|_E \int_0^x\int_0^y k(x,y,s,t)\exp(\lambda(s+t))\,dtds \\
&\qquad + |u-v|_E \int_0^\infty\int_0^\infty r(x,y,s,t)\exp(\lambda(s+t))\,dtds \\
&\quad \le |u-v|_E(\alpha_1+\alpha_2)\exp(\lambda(x+y)).
\end{aligned}
$$

Consequently we have

$$|Tu - Tv|_E \le (\alpha_1+\alpha_2)\,|u-v|_E\,.$$

Since $\alpha_1 + \alpha_2 < 1$, it follows from Banach fixed point theorem (see [5, p.37]) that T has a unique fixed point in E. The fixed point of T is however a solution of equation (VF).

REMARK 1. We note that Theorem 1 given above yields existence and uniqueness of solutions of equation (VF) in E.

Indeed the following theorem holds concerning the uniqueness of solution of equation (VF) in R^n:

THEOREM 2. *Suppose that the functions* F, G *in equation* (VF) *satisfy the conditions* (2.11), (2.12) *with* $k(x,y,s,t) = b(x,y)f(s,t)$, $r(x,y,s,t) = c(x,y)g(s,t)$, *where* $b, f, c, g \in C(\Delta, R_+)$. *Let* p *be as in* (2.5). *Then the equation* (VF) *has at most one solution on* Δ *in* R^n.

Proof. Let $u_1(x,y)$ and $u_2(x,y)$ be two solutions of equation (VF). Then we have

$$(2.18)\quad u_1(x,y) - u_2(x,y) = \int_0^x\int_0^y \{F(x,y,s,t,u_1(s,t)) - F(x,y,s,t,u_2(s,t))\}\,dtds + \int_0^\infty\int_0^\infty \{G(x,y,s,t,u_1(s,t)) - G(x,y,s,t,u_2(s,t))\}\,dtds.$$

From (2.18) and using the hypotheses we have

$$(2.19)\quad |u_1(x,y) - u_2(x,y)| \le b(x,y)\int_0^x\int_0^y f(s,t)\,|u_1(s,t) - u_2(s,t)|\,dtds + c(x,y)\int_0^\infty\int_0^\infty g(s,t)\,|u_1(s,t) - u_2(s,t)|\,dtds.$$

Here it is easy to observe that $B(x,y)$ and M defined in (2.7) and (2.10) reduces to $B(x,y) = 0$ and $M = 0$. Now an application of Lemma (with $a(x,y) = 0$) to (2.19) yields $|u_1(x,y) - u_2(x,y)| \le 0$, and hence $u_1(x,y) = u_2(x,y)$. Thus there is at most one solution to the equation (VF) on Δ in R^n.

3. Estimates on solutions

In this section we obtain estimates on the solutions of equation (VF) under some suitable conditions on the functions involved in equation (VF).

First we shall give the following theorem concerning the estimate on the solution of equation (VF).

THEOREM 3. *Assume that the functions F, G in equation* (VF) *satisfy the conditions*

$$(3.1)\quad |F(x,y,s,t,u(s,t))| \le b(x,y)\,f(s,t)\,|u(s,t)|,$$

$$(3.2)\quad |G(x,y,s,t,u(s,t))| \le c(x,y)\,g(s,t)\,|u(s,t)|,$$

where $b, f, c, g \in C(\Delta, R_+)$. Let $p, D(x,y)$ be as in (2.5), (2.8) *and*

$$(3.3)\quad M_1 = \frac{1}{1-p}\int_0^\infty\int_0^\infty g(s,t)\,B_1(s,t)\,dtds,$$

where $B_1(x,y)$ is defined by the right hand side of (2.7) *by replacing $a(x,y)$ by $|h(x,y)|$. If $u(x,y)$, $(x,y) \in \Delta$ is any solution of equation* (VF) *then*

$$(3.4)\quad |u(x,y)| \le B_1(x,y) + M_1 D(x,y),$$

for $(x,y) \in \Delta$.

Proof. Using the fact that $u(x,y)$, $(x,y) \in \Delta$ is a solution of equation (VF) and the hypotheses we have

$$(3.5) \qquad |u(x,y)| \le |h(x,y)| + \int_0^x \int_0^y |F(x,y,s,t,u(s,t))|\,dtds + \int_0^\infty \int_0^\infty |G(x,y,s,t,u(s,t))|\,dtds$$
$$\le |h(x,y)| + b(x,y) \int_0^x \int_0^y f(s,t)\,|u(s,t)|\,dtds + c(x,y) \int_0^\infty \int_0^\infty g(s,t)\,|u(s,t)|\,dtds.$$

Now an application of Lemma to (3.5) yields (3.4).

Next, we shall obtain the estimation on the solution of equation (VF) assuming that the functions F, G satisfy Lipschitz type conditions.

THEOREM 4. *Suppose that the functions F, G be as in Theorem 2. Let $p, D(x,y)$ be as in (2.5), (2.8) and*

$$(3.6) \qquad h_0(x,y) = \int_0^x \int_0^y |F(x,y,s,t,h(s,t))|\,dtds + \int_0^\infty \int_0^\infty |G(x,y,s,t,h(s,t))|\,dtds,$$

$$(3.7) \qquad M_2 = \frac{1}{1-p} \int_0^\infty \int_0^\infty g(s,t)\,B_2(s,t)\,dtds,$$

where $B_2(x,y)$ is defined by the right hand side of (2.7) *by replacing $a(x,y)$ by $h_0(x,y)$. If $u(x,y)$, $(x,y) \in \Delta$ is any solution of equation* (VF) *then*

$$(3.8) \qquad |u(x,y) - h(x,y)| \le B_2(x,y) + M_2 D(x,y),$$

for $(x,y) \in \Delta$.

Proof. Using the fact that $u(x,y)$, $(x,y) \in \Delta$ is a solution of equation (VF) we observe that

$$(3.9) \qquad u(x,y) - h(x,y) = \int_0^x \int_0^y \{F(x,y,s,t,u(s,t)) - F(x,y,s,t,h(s,t))\}\,dtds + \int_0^\infty \int_0^\infty \{G(x,y,s,t,u(s,t)) - G(x,y,s,t,h(s,t))\}\,dtds$$

$$+\int_0^x\int_0^y F(x,y,s,t,h(s,t))\,dtds+\int_0^\infty\int_0^\infty G(x,y,s,t,h(s,t))\,dtds.$$

From (3.9) and using the hypotheses we have

$$(3.10)\quad |u(x,y)-h(x,y)| \le |h_0(x,y)|+b(x,y)\int_0^x\int_0^y f(s,t)\,|u(s,t)-h(s,t)|\,dtds + c(x,y)\int_0^\infty\int_0^\infty g(s,t)\,|u(s,t)-h(s,t)|\,dtds.$$

Now an application of Lemma to (3.10) yields (3.8).

The following theorem deals with the estimate on the difference between the solutions of equation (VF) and the system of Volterra integral equations

$$(3.11)\qquad v(x,y)=h(x,y)+\int_0^x\int_0^y F(x,y,s,t,v(s,t))\,dtds,$$

for $(x,y)\in\Delta$, where the functions h, F are as given in equation (VF).

THEOREM 5. *Suppose that the functions F, G be as in Theorem* 2 *and* $G(x,y,s,t,0)=0$. *Let $v(x,y)$, $(x,y)\in\Delta$ be a solution of equation* (3.11) *such that $|v(x,y)|\le Q$, where $Q\ge 0$ is a constant. Let*

$$\bar{a}(x,y)=Qc(x,y)\int_0^\infty\int_0^\infty g(s,t)\,dtds,$$

and $p, D(x,y)$ be as in (2.5), (2.8) *and*

$$(3.12)\qquad M_3=\frac{1}{1-p}\int_0^\infty\int_0^\infty g(s,t)\,B_3(s,t)\,dtds,$$

where $B_3(x,y)$ is defined by the right hand side of (2.7) *by replacing $a(x,y)$ by $\bar{a}(x,y)$. If $u(x,y)$, $(x,y)\in\Delta$ is a solution of equation* (VF), *then*

$$(3.13)\qquad |u(x,y)-v(x,y)|\le B_3(x,y)+M_3D(x,y),$$

for $(x,y)\in\Delta$.

Proof. Using the facts that $u(x,y)$ and $v(x,y)$ for $(x,y)\in\Delta$ are the solutions of equations (VF) and (3.11) we observe that

$$(3.14)\quad u(x,y)-v(x,y) = \int_0^x\int_0^y \{F(x,y,s,t,u(s,t))-F(x,y,s,t,v(s,t))\}\,dtds$$

$$+\int_0^\infty\int_0^\infty \{G(x,y,s,t,u(s,t)) - G(x,y,s,t,v(s,t))$$

$$+ G(x,y,s,t,v(s,t)) - G(x,y,s,t,0)\}\, dtds.$$

From (3.14) and using the hypotheses we have

$$(3.15)\qquad |u(x,y) - v(x,y)|$$
$$\le \bar{a}(x,y) + b(x,y)\int_0^x\int_0^y f(s,t)\,|u(s,t) - v(s,t)|\, dtds$$
$$+ c(x,y)\int_0^\infty\int_0^\infty g(s,t)\,|u(s,t) - v(s,t)|\, dtds.$$

Now an application of Lemma to (3.15) yields (3.13).

4. Continuous dependence

In this section we study the continuous dependence of solutions of equation (VF) on functions involved on the right hand side of equation (VF) and also the continuous dependence of solutions of equations of the form (VF) on parameters.

Consider the equation (VF) and the system of Volterra-Fredholm integral equations

$$(4.1)\qquad v(x,y) = \bar{h}(x,y) + \int_0^x\int_0^y \bar{F}(x,y,s,t,v(s,t))\, dtds$$
$$+ \int_0^\infty\int_0^\infty \bar{G}(x,y,s,t,v(s,t))\, dtds,$$

for $(x,y) \in \Delta$, where $\bar{h} \in C(\Delta, R^n)\, ; \bar{F}, \bar{G} \in C\left(\Delta^2 \times R^n, R^n\right)$.

The following theorem shows the continuous dependence of solutions of equation (VF) on the right hand side of equation (VF).

THEOREM 6. *Suppose that the functions* F, G *in equation* (VF) *satisfy the conditions* (2.11), (2.12) *with* $k(x,y,s,t) = b(x,y)f(s,t)$, $r(x,y,s,t) = c(x,y)g(s,t)$, *where* $b, f, c, g \in C(\Delta, R_+)$. *Assume that*

$$(4.2)\qquad |h(x,y) - \bar{h}(x,y)| + \int_0^x\int_0^y |F(x,y,s,t,v(s,t)) - \bar{F}(x,y,s,t,v(s,t))| dtds$$
$$+ \int_0^\infty\int_0^\infty \left|G(x,y,s,t,v(s,t)) - \bar{G}(x,y,s,t,v(s,t))\right| dtds \le \varepsilon,$$

where h, F, G and $\bar{h}, \bar{F}, \bar{G}$ are the functions involved in equations (VF) *and* (4.1), *$v(x,y)$ is a solution of* (4.1) *and $\varepsilon > 0$ is an arbitrarily small constant. Let $p, D(x,y)$ be as in* (2.5), (2.8) *and*

$$(4.3) \qquad M_4 = \frac{1}{1-p} \int_0^\infty \int_0^\infty g(s,t)\, B_4(s,t)\, dtds,$$

where $B_4(x,y)$ is defined by the right hand side of (2.7) *by replacing $a(x,y)$ by ε. Then the solution $u(x,y)$, $(x,y) \in \Delta$ of equation* (VF) *depends continuously on the functions involved on the right hand of equation* (VF).

Proof. Since $u(x,y)$ and $v(x,y)$ for $(x,y) \in \Delta$ are the solutions of equations (VF) and (4.1) we have

$$\begin{aligned}(4.4) \quad u(x,y) - v(x,y) = {} & h(x,y) - \bar{h}(x,y) \\ & + \int_0^x \int_0^y \{F(x,y,s,t,u(s,t)) - F(x,y,s,t,v(s,t)) \\ & + F(x,y,s,t,v(s,t)) - \bar{F}(x,y,s,t,v(s,t))\}\, dtds \\ & + \int_0^\infty \int_0^\infty \{G(x,y,s,t,u(s,t)) - G(x,y,s,t,v(s,t)) \\ & + G(x,y,s,t,v(s,t)) - \bar{G}(x,y,s,t,v(s,t))\} dtds.\end{aligned}$$

From (4.4) and using the hypotheses we have

$$\begin{aligned}(4.5) \quad & |u(x,y) - v(x,y)| \\ & \le \left|h(x,y) - \bar{h}(x,y)\right| \\ & \quad + \int_0^x \int_0^y |F(x,y,s,t,u(s,t)) - F(x,y,s,t,v(s,t))| dtds \\ & \quad + \int_0^x \int_0^y \left|F(x,y,s,t,v(s,t)) - \bar{F}(x,y,s,t,v(s,t))\right| dtds \\ & \quad + \int_0^\infty \int_0^\infty |G(x,y,s,t,u(s,t)) - G(x,y,s,t,v(s,t))| dtds \\ & \quad + \int_0^\infty \int_0^\infty \left|G(x,y,s,t,v(s,t)) - \bar{G}(x,y,s,t,v(s,t))\right| dtds \\ & \le \varepsilon + b(x,y) \int_0^x \int_0^y f(s,t)\, |u(s,t) - v(s,t)| dtds \\ & \quad + c(x,y) \int_0^\infty \int_0^\infty g(s,t)\, |u(s,t) - v(s,t)| dtds.\end{aligned}$$

Now an application of Lemma to (4.5) yields

$$|u(x,y) - v(x,y)| \leq B_4(x,y) + M_4 D(x,y), \tag{4.6}$$

for $(x,y) \in \Delta$. From (4.6) it follows that the solutions of equation (VF) depends continuously on the functions involved on the right hand side of equation (VF).

REMARK 2. From (4.6), it is easy to observe that if $B_4(x,y)$ and $D(x,y)$ are bounded for $(x,y) \in \Delta$ and $\varepsilon \to 0$, then $|u(x,y) - v(x,y)| \to 0$ on Δ.

We next consider the following systems of Volterra-Fredholm integral equations

$$\begin{aligned} z(x,y) = h(x,y) &+ \int_0^x \int_0^y H(x,y,s,t,z(s,t),\mu)\,dtds \\ &+ \int_0^\infty \int_0^\infty L(x,y,s,t,z(s,t),\mu)\,dtds, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} z(x,y) = h(x,y) &+ \int_0^x \int_0^y H(x,y,s,t,z(s,t),\mu_0)\,dtds \\ &+ \int_0^\infty \int_0^\infty L(x,y,s,t,z(s,t),\mu_0)\,dtds, \end{aligned} \tag{4.8}$$

for $(x,y) \in \Delta$, where h, H, L are in R^n, μ, μ_0 are real parameters and $h \in C(\Delta, R^n)$; $H, L \in C(\Delta^2 \times R^n \times R, R^n)$.

The next theorem shows the dependency of solutions of equations (4.7) and (4.8) on parameters.

THEOREM 7. *Assume that the functions H, L satisfy the conditions*

$$\begin{aligned} &|H(x,y,s,t,z(s,t),\mu) - H(x,y,s,t,\bar{z}(s,t),\mu)| \\ &\qquad \leq b(x,y) f(s,t) |z(s,t) - \bar{z}(s,t)|, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &|H(x,y,s,t,z(s,t),\mu) - H(x,y,s,t,z(s,t),\mu_0)| \\ &\qquad \leq r_1(x,y,s,t) |\mu - \mu_0|, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &|L(x,y,s,t,z(s,t),\mu) - L(x,y,s,t,\bar{z}(s,t),\mu)| \\ &\qquad \leq c(x,y) g(s,t) |z(s,t) - \bar{z}(s,t)|, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &|L(x,y,s,t,z(s,t),\mu) - L(x,y,s,t,z(s,t),\mu_0)| \\ &\qquad \leq r_2(x,y,s,t) |\mu - \mu_0|, \end{aligned} \tag{4.12}$$

where $b, f, c, g \in C(\Delta, R_+)$; $r_1, r_2 \in C(\Delta^2, R_+)$. Let

$$a_0(x,y) = |\mu - \mu_0| \left[\int_0^x \int_0^y r_1(x,y,s,t)\,dtds + \int_0^\infty \int_0^\infty r_2(x,y,s,t)\,dtds \right], \tag{4.13}$$

and $p, D(x,y)$ *be as in* (2.5), (2.8) *and*

$$M_5 = \frac{1}{1-p} \int_0^\infty \int_0^\infty g(s,t)\, B_5(s,t)\,dtds, \tag{4.14}$$

where $B_5(x,y)$ *is defined by the right hand side of* (2.7) *by replacing* $a(x,y)$ *by* $a_0(x,y)$. *Let* $z_1(x,y)$ *and* $z_2(x,y)$ *be the solutions of equations* (4.7) *and* (4.8), *respectively. Then*

$$|z_1(x,y) - z_2(x,y)| \le B_5(x,y) + M_5 D(x,y), \tag{4.15}$$

for $(x,y) \in \Delta$.

Proof. Let $z(x,y) = z_1(x,y) - z_2(x,y)$. Since $z_1(x,y)$ and $z_2(x,y)$ are the solutions of equations (4.7) and (4.8) we have

$$\begin{aligned} z(x,y) &= z_1(x,y) - z_2(x,y) \\ &= \int_0^x \int_0^y \{H(x,y,s,t,z_1(s,t),\mu) - H(x,y,s,t,z_2(s,t),\mu) \\ &\quad + H(x,y,s,t,z_2(s,t),\mu) - H(x,y,s,t,z_2(s,t),\mu_0)\}\,dtds \\ &\quad + \int_0^\infty \int_0^\infty \{L(x,y,s,t,z_1(s,t),\mu) - L(x,y,s,t,z_2(s,t),\mu) \\ &\quad + L(x,y,s,t,z_2(s,t),\mu) - L(x,y,s,t,z_2(s,t),\mu_0)\}\,dtds. \end{aligned} \tag{4.16}$$

From (4.16) and using the hypotheses we have

$$\begin{aligned} |z(x,y)| &\le b(x,y) \int_0^x \int_0^y f(s,t)\,|z_1(s,t) - z_2(s,t)|dtds \\ &\quad + \int_0^x \int_0^y r_1(x,y,s,t)\,|\mu - \mu_0|dtds \\ &\quad + c(x,y) \int_0^\infty \int_0^\infty g(s,t)\,|z_1(s,t) - z_2(s,t)|dtds \\ &\quad + \int_0^\infty \int_0^\infty r_2(x,y,s,t)\,|\mu - \mu_0|dtds \\ &= a_0(x,y) + b(x,y) \int_0^x \int_0^y f(s,t)\,|z(s,t)|\,dtds \\ &\quad + c(x,y) \int_0^\infty \int_0^\infty g(s,t)\,|z(s,t)|\,dtds. \end{aligned} \tag{4.17}$$

Now an application of Lemma to (4.17) yields (4.15), which shows the dependency of solutions of equations (4.7) and (4.8) on parameters.

REMARK 3. We note that the ideas of this paper can be extended to study the Volterra-Fredholm integral equations of the form (VF) involving more than two independent variables. The details of the formulation of such results are very close to those of given above with suitable modifications. Here we omit the details.

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