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# GENERALIZATION OF A CLASS OF POLYNOMIALS

**Abstract.** An attempt is made to investigate a class of polynomials defined in form of Rodrigues type formula and Mittag-Leffler Function. Some generating relations and finite summation formulae have also been obtained.

## 1. Introduction, notation and results

Chak [1] defined a class of polynomials as:

$$G_{n,k}^{(\alpha)}(x) = x^{-\alpha-kn+n} e^{x} (x^k D)^n [x^\alpha e^{-x}], \tag{1.1}$$

where $D = \dfrac{d}{dx}$, $k$ is constant and $n = 0, 1, 2, \ldots$.

Chatterjea [2] studied a class of polynomials for generalized Laguerre polynomial as:

$$T_m^{(\alpha)}(x,p) = \frac{1}{n!}\, x^{-\alpha-n-1} \exp(px^r)(x^2 D)^n [x^{\alpha+1} \exp(-px^r)]. \tag{1.2}$$

Gould and Hopper [3] introduced generalized Hermite polynomials as:

$$H_n^r(x,a,p) = (-1)^n\, x^{-a} \exp(px^r) D^n [x^a \exp(-px^r)]. \tag{1.3}$$

Singh [11] obtained generalized Truesdell polynomials by using Rodrigues formula, which is defined as:

$$T_n^{(\alpha)}(x,r,p) = x^{-\alpha} \exp(px^r)(xD)^n [x^\alpha \exp(-px^r)]. \tag{1.4}$$

Mittal [5] proved a Rodrigues formula for a class of polynomials $T_{kn}^{(\alpha)}(x)$, which is given by

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!}\, x^{-\alpha} \exp\{p_k(x)\} D^n [x^{\alpha+n} \exp(-p_k(x)\}] \tag{1.5}$$

where, $p_k(x)$ is a polynomial in $x$ of degree $k$ and $x \in (0, \infty)$.

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Mittal [6] also proved the following relation

$$(1.6) \qquad T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!}\, x^{-\alpha-n}\{\exp p_k(x)\}\theta^n[x^\alpha \exp\{-p_k(x)\}]$$

where, $\theta \equiv x(s+xD)$, $D = \dfrac{d}{dx}$ and $s$ is constant, which can be generalized as:

$$\theta^n = x^n(s+xD)(s+1+xD)(s+2+xD)\ldots(s+(n-1)+xD).$$

In this paper, we introduced a new class of polynomial defined as

$$(1.7) \qquad T_{kn}^{(\alpha,\beta+s-1)}(x) = \frac{1}{n!}\, x^{-\beta-n}\, E_\alpha\{p_k(x)\}\theta^n\left[x^\beta E_\alpha\{-p_k(x)\}\right]$$

where $\alpha \geq 0$, $n = 0, 1, 2, \ldots$, $k$ is finite and non-negative integer, $E_\alpha(z)$ is Mittag-Leffler function (Mario N. Berberan e Santos [4]) which is defined as:

$$(1.8) \qquad E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}.$$

**Generalized Laguerre polynomials** (Srivastava and Manocha[14]):

It is denoted by the symbol $L_n^{(\alpha)}(x)$ and defined as

$$(1.9) \qquad L_n^{(\alpha)}(x) = \frac{x^{-\alpha-n-1}}{n!}\,(x^2D)^n[x^{\alpha+1}\,e^{-x}].$$

**Hermite polynomials** (Rainville [10]):

It is denoted by the symbol $H_n(x)$ and defined as

$$(1.10) \qquad H_n(x) = (-1)^n \exp(x^2) D^n[\exp(-x^2)].$$

**Konhauser polynomials of first kind** (Srivastava [13]):

It is denoted by the symbol $Y_n^\alpha(x;k)$ and defined as:

$$(1.11) \qquad Y_n^\alpha(x;k) = \frac{x^{-kn-\alpha-1}}{k^n\, n!}\,(x^{k+1}\,D)^n[x^{\alpha+1}\,e^{-x}].$$

**Konhauser polynomials of second kind** (Srivastava [13]):

It is denoted by the symbol $Z_n^\alpha(x;k)$ and defined as:

$$(1.12) \qquad Z_n^\alpha(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}$$

where $k$ is a positive integer.

$L_n^{\alpha,\beta}(x)$ **polynomials** (Prabhakar and Suman [9]) which is defined as:

$$L_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k\, x^k}{\Gamma(\alpha k+\beta+1)\, k!}, \tag{1.13}$$

$\mathrm{Re}(\beta) > -1$ and $\alpha$ is any complex number with $\mathrm{Re}(\alpha) > 0$.

The following relations are also established by Prabhakar and Suman [9]

$$L_n^{k,\beta}(x^k) = Z_n^{\beta}(x;k), \tag{1.14}$$

$$L_n^{1,\beta}(x) = L_n^{\beta}(x). \tag{1.15}$$

Srivastava and Manocha [14] verified following result by using induction method,

$$(x^2D)^n\{f(x)\} = x^{n+1}\, D^n\{x^{n-1}f(x)\}. \tag{1.16}$$

We are also using the operational formulae (based on Mittal [7], Patil and Thakare [8] work)

$$\theta^n(xuv) = x\sum_{m=0}^{\infty}\binom{n}{m}\theta^{n-m}(v)\theta_1^m(u) \quad \text{where} \quad \theta_1 \equiv x(1+xD), \tag{1.17}$$

$$e^{t\theta}\Big(x^{\alpha}f(x)\Big) = x^{\alpha}(1-xt)^{-(\alpha+s)}\; f[x(1-xt)^{-1}], \tag{1.18}$$

$$e^{t\theta}\Big(x^{\alpha-n}f(x)\Big) = x^{\alpha}(1+t)^{-1+\alpha+s}\; f[x(1+t)]. \tag{1.19}$$

## 2. Generating relations

We obtained some generating relations of (1.7) as:

$$\sum_{n=0}^{\infty}T_{kn}^{(\alpha,\beta+s-1)}(x)t^n = (1-t)^{-(\beta+s)}\; E_{\alpha}[p_k(x)]\, E_{\alpha}[-p_k\{x(1-t)^{-1}\}], \tag{2.1}$$

$$\sum_{n=0}^{\infty}T_{kn}^{(\alpha,\beta-n+s-1)}(x)t^n = (1+t)^{\beta+s-1}\; E_{\alpha}[p_k(x)]\, E_{\alpha}[-p_k\{x(1+t)\}], \tag{2.2}$$

$$\sum_{n=0}^{\infty}\binom{m+n}{n}T_{k(n+m)}^{(\alpha,\beta+s-1)}(x)t^m = (1-t)^{-(\beta+s+n)}\frac{E_{\alpha}[p_k(x)]}{E_{\alpha}[p_k\{x(1-t)^{-1}\}]} \times\; T_{kn}^{(\alpha,\beta+s-1)}\{x(1-t)^{-1}\} \tag{2.3}$$

where $0 < t < 1$.

Proof of (2.1): From (1.7), we consider

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha,\beta+s-1)}(x)\, t^n = x^{-\beta}\, E_\alpha\{p_k(x)\}\, e^{t\theta}[x^\beta\, E_\alpha\{-p_k(x)\}]$$

using (1.18), above equation reduces to,

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha,\beta+s-1)}(x)\, t^n$$
$$= x^{-\beta}\, E_\alpha\{p_k(x)\}\, x^\beta(1-xt)^{-(\beta+s)}\, E_\alpha[-p_k\{x(1-xt)^{-1}\}]$$

and replacing $t$ by $t/x$, which gives (2.1).

Proof of (2.2): From (1.7), we consider

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha,\beta-n+s-1)}(x) t^n = x^{-(\beta-n)-n}\, E_\alpha\{p_k(x)\}\, e^{t\theta}[x^{\beta-n}\, E_\alpha\{-p_k(x)\}]$$

using (1.19) and simplifying the above equation which reduces to:

$$\sum_{n=0}^{\infty} T_{kn}^{(\alpha,\beta-n+s-1)}(x) t^n$$
$$= x^{-\beta}\, E_\alpha\{p_k(x)\}\, x^\beta(1+t)^{-1+(\beta+s)}\, E_\alpha[-p_k\{x(1+t)\}].$$

Proof of (2.3): Again from (1.7) we consider

$$\theta^n[x^\beta\, E_\alpha\{-p_k(x)\}] = n!\; x^{\beta+n}\, \frac{1}{E_\alpha\{p_k(x)\}}\, T_{kn}^{(\alpha,\beta+s-1)}(x)$$

or

$$e^{t\theta}(\theta^n[x^\beta\, E_\alpha\{-p_k(x)\}]) = n!\; e^{t\theta}\left[x^{\beta+n}\, \frac{1}{E_\alpha\{p_k(x)\}}\, T_{kn}^{(\alpha,\beta+s-1)}(x)\right]$$

applying operational formula, above equation can be written as

$$\sum_{m=0}^{\infty} \frac{t^m\, t^{m+n}}{m!}[x^\beta\, E_\alpha\{-p_k(x)\}]$$
$$= n!\; x^{\beta+n}(1-xt)^{-(\beta+s+n)}\, \frac{1}{E_\alpha[p_k\{x(1-xt)^{-1}\}]}\, T_{kn}^{(\alpha,\beta+s-1)}\{x(1-xt)^{-1}\}.$$

Therefore, we get

$$\sum_{m=0}^{\infty} \frac{1}{m!\; n!}\, (m+n)!\; x^{\beta+m+n}\, \frac{1}{E_\alpha\{p_k(x)\}}\, T_{k(m+n)}^{(\alpha,\beta+s-1)}(x)\, t^m$$
$$= x^{\beta+n}(1-xt)^{-(\beta+s+n)}\, \frac{1}{E_\alpha[p_k\{x(1-xt)^{-1}\}]}\, T_{kn}^{(\alpha,\beta+s-1)}\{x(1-xt)^{-1}\}.$$

Above equation can be written as:

$$\sum_{m=0}^{\infty} x^m \binom{m+n}{n} T_{k(m+n)}^{(\alpha,\beta+s-1)}(x) t^m$$
$$= (1-xt)^{-(\beta+s+n)} \frac{E_\alpha\{p_k(x)\}}{E_\alpha[p_k\{x(1-xt)^{-1}\}]} T_{kn}^{(\alpha,\beta+s-1)}\{x(1-xt)^{-1}\}$$

and replacing $t$ by $t/x$, which immediately leads to (2.3).

## 3. Finite summation formulae

We obtained two finite summation formulae for (1.7) as

$$\text{(3.1)} \qquad T_{kn}^{(\alpha,\beta+s-1)}(x) = \sum_{m=0}^{n} \frac{1}{m!} (\beta)_m T_{k(n-m)}^{(\alpha,s-1)}(x), \quad \text{where } \beta > 0,$$

$$\text{(3.2)} \qquad T_{kn}^{(\alpha,\beta+s-1)}(x) = \sum_{m=0}^{n} \frac{1}{m!} (\beta-\gamma)_m T_{k(n-m)}^{(\alpha,\gamma+s-1)}(x), \quad \text{where } \beta-\gamma > 0.$$

Proof of (3.1): We can write (1.7) as,

$$T_{kn}^{(\alpha,\beta+s-1)}(x) = \frac{1}{n!} x^{-\beta-n} E_\alpha\{p_k(x)\}\theta^n[xx^{\beta-1}E_\alpha\{-p_k(x)\}],$$

using (1.17) we get,

$$T_{kn}^{(\alpha,\beta+s-1)}(x) = \frac{1}{n!} x^{-\beta-n} E_\alpha\{p_k(x)\} x \sum_{m=0}^{n} \binom{n}{m} \theta^{n-m}[E_\alpha\{-p_k(x)\}]\theta_1^m(x^{\beta-1})$$

which yields

$$\text{(3.1.1)} \qquad \frac{1}{n!} x^{-\beta-n} E_\alpha\{p_k(x)\} x \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}$$
$$\times x^{n-m}\big[(s+xD)(s+1+xD)(s+2+xD)\dots(s+(n-m-1)+xD)\big]$$
$$\times \big[E_\alpha\{-p_k(x)\}\big] x^m \big[(1+xD)(2+xD)(3+xD)\dots(m+xD)\big] x^{\beta-1}$$
$$= E_\alpha\{p_k(x)\} \sum_{m=0}^{n} \frac{1}{m!\,(n-m)!} \prod_{i=0}^{n-m-1} (s+i+xD)\Big[E_\alpha\{-p_k(x)\}\Big](\beta)_m,$$

where $(\beta)_m$ is a factorial function defined (Rainville [10]) as

$$(\beta)_m = \beta(\beta+1)(\beta+2)\dots(\beta+m-1), \qquad m-1 \ \geq\ 0$$

and

$$(\beta)_0 = 1.$$

If $\beta = 0$ in (1.7) and $n$ is replaced by $n-m$ then (1.7) becomes

$$T_{k(n-m)}^{(\alpha,s-1)}(x) = \frac{1}{(n-m)!}\, x^{-(n-m)}\, E_\alpha\{p_k(x)\}\theta^{n-m}[E_\alpha\{-p_k(x)\}]$$

thus, we have

$$\frac{1}{(n-m)!}\,\theta^{n-m}\Big[E_\alpha\{-p_k(x)\}\Big] = x^{n-m}\,\frac{1}{E_\alpha\{p_k(x)\}}\,T_{k(n-m)}^{(\alpha,s-1)}(x).$$

Above equation can be written as:

$$(3.1.2)\quad \frac{1}{(n-m)!}\prod_{i=0}^{n-m-1}(s+i+xD)[E_\alpha\{-p_k(x)\}] = \frac{1}{E_\alpha\{p_k(x)\}}\,T_{k(n-m)}^{(\alpha,s-1)}(x).$$

Use of (3.1.2) and (3.1.1), yields (3.1).

P r o o f o f (3.2): From (1.7) we consider,

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha,\beta+s-1)}(x)\, t^n = x^{-\beta}\, E_\alpha\{p_k(x)\}e^{t\theta}[x^\beta\, E_\alpha\{-p_k(x)\}],$$

using (1.18) above equation reduces to,

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha,\beta+s-1)}(x)\, t^n = (1-xt)^{-(\beta+s)}\, E_\alpha[p_k(x)]\, E_\alpha[-p_k\{x(1-xt)^{-1}\}].$$

Above result can be written as:

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha,\beta+s-1)}(x)\, t^n$$
$$= (1-xt)^{-(\gamma+s)}\sum_{m=0}^{\infty}(\beta-\gamma)_m\,\frac{(xt)^m}{m!}\, E_\alpha[p_k(x)]\, E_\alpha[-p_k\{x(1-xt)^{-1}\}].$$

Using result (2.1), we get

$$\sum_{n=0}^{\infty} x^n\, T_{kn}^{(\alpha,\beta+s-1)}(x)\, t^n$$
$$= \sum_{m=0}^{\infty}(\beta-\gamma)_m\,\frac{(xt)^m}{m!}\, x^{-\gamma}\, E_\alpha\{p_k(x)\}e^{t\theta}\Big[x^\gamma\, E_\alpha\{-p_k(x)\}\Big]$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\beta-\gamma)_m\,\frac{x^m t^{n+m}}{m!\, n!}\, x^{-\gamma}\, E_\alpha\{p_k(x)\}\theta^n[x^\gamma\, E_\alpha\{-p_k(x)\}]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(\beta-\gamma)_m \frac{x^m t^n}{m!\,(n-m)!}\, x^{-\gamma}\, E_\alpha\{p_k(x)\}\theta^{n-m}[x^\gamma\, E_\alpha\{-p_k(x)\}]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m!}\, x^m\, (\beta-\gamma)_m \frac{x^{-\gamma}}{(n-m)!}\, E_\alpha\{p_k(x)\}\theta^{n-m}[x^\gamma\, E_\alpha\{-p_k(x)\}]t^n.$$

Equating the coefficients of $t^n$, we get

$$x^n\, T_{kn}^{(\alpha,\beta+s-1)}(x) = \sum_{m=0}^{n}\frac{1}{m!}\, x^m\, (\beta-\gamma)_m \frac{x^{-\gamma}}{(n-m)!}\, E_\alpha\{p_k(x)\}\theta^{n-m}[x^\gamma\, E_\alpha\{-p_k(x)\}]$$

therefore, we obtain

$$T_{kn}^{(\alpha,\beta+s-1)}(x) = \sum_{m=0}^{n}\frac{1}{m!}\, (\beta-\gamma)_m \frac{x^{-\gamma-(n-m)}}{(n-m)!}\, E_\alpha\{p_k(x)\}\theta^{n-m}[x^\gamma\, E_\alpha\{-p_k(x)\}]$$

applying (1.7), which follows (3.2).

## 4. Special cases

Here, we obtained some special cases of $T_{kn}^{(\alpha,\beta+s-1)}(x)$ polynomials.

Putting $\alpha = 1$ and replacing $\beta$ by $\alpha$ in (1.7), then (1.7) immediately leads to,

$$T_{kn}^{(1,\alpha+s-1)}(x) = T_{kn}^{(\alpha+s-1)}(x) \tag{4.1}$$

therefore, we can say that (1.6) is a particular case of (1.7).

If $\alpha = 1$, replacing $\beta$ by $\alpha+1$, $p_k(x) = p_1(x) = x$ and $s = 0$ in (1.7), then this equation reduces to

$$T_n^{(1,\alpha)}(x) = L_n^{(\alpha)}(x) = L_n^{1,\alpha}(x) = Z_n^\alpha(x;1) = Y_n^\alpha(x;1). \tag{4.2}$$

If we replace $\beta$ by $\alpha+1$, $\alpha = 1$, $p_k(x) = px^r$ and $s = 0$ in (1.7), gives

$$T_{rn}^{(1,\alpha)}(x) = T_{rn}^{(\alpha)}(x,p). \tag{4.3}$$

Substituting $\alpha = 1$, replacing $\beta$ by $1-n$, $p_k(x) = x^2$, $s = 0$ in (1.7) and using (1.16), which yields

$$T_{2n}^{(1,1-n)}(x) = \frac{(-x)^n}{n!}\, H_n(x). \tag{4.4}$$

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