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CERTAIN SUBCLASSES OF MULTIVALENT PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

Abstract. The object of the present paper is to investigate coefficient estimates and closure theorems for functions belonging to the class $R_{\gamma,\lambda}^{p,n}[A,B,\alpha]$ of p-valent prestarlike functions with negative coefficients. We also consider integral operators associalted with functions belonging to the class $R_{\gamma,\lambda}^{p,n}[A,B,\alpha]$. Furthermore, distortion theorems involving a generalized fractional integral (derivative) opertaor for functions in this class are proved.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2 \dots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p)$ is called p-valent starlike of order $\gamma (0 \le \gamma < p)$ if $f(z)$ satisfies the conditions

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in U) \tag{1.2}$$

and

$$\int_{\gamma}^{2\pi} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2\pi p \quad (z \in U). \tag{1.3}$$

We denote by $S^*(p,\gamma)$ the class of all p-valent starlike functions of order γ. The class $S^*(p,\gamma)$was introduced by Patil and Thakare [14]. The function

$$s_\gamma^p(z) = \frac{z^p}{(1-z)^{2(p-\gamma)}} \quad (0 \le \gamma < p; p \in N) \tag{1.4}$$

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is the familiar extremal function for the class $S^*(p,\gamma)$. Setting

$$(1.5)\qquad G^p(\gamma,k)=\frac{\prod_{m=2}^{k}[2(p-\gamma)+m-2]}{(k-1)!}\qquad (k\in N\backslash\{1\};\ 0\le\gamma<p),$$

$s^p_\gamma(z)$ can be written in the form

$$(1.6)\qquad s^p_\gamma(z)=z^p+\sum_{k=1}^{\infty}G^p(\gamma,k+1)z^{p+k}.$$

Clearly , $s^p_\gamma\ (z)\in S^*(p,\gamma)$ and $G^p(\gamma,k+1)$ is a decreasing function in γ $(0\le\gamma\le\frac{2p-1}{2};\ p\in N)$ and satisfies

$$\lim_{k\to\infty}G^p(\gamma,k+1)=\begin{cases}\infty & (\gamma<\frac{2p-1}{2})\\ 1 & (\gamma=\frac{2p-1}{2})\\ 0 & (\gamma>\frac{2p-1}{2}).\end{cases}$$

For A and B fixed, $-1\le A<B\le 1$, $0<B\le 1$, $0\le\alpha<p$ and $p\in N$, we say that a function $f(z)\in A(p)$ is in the class $S^*_p(A,B,\alpha)$ if and only if

$$(1.7)\qquad \left|\frac{\frac{zf'(z)}{f(z)}-p}{B\frac{zf'(z)}{f(z)}-[pB+(A-B)(p-\alpha)]}\right|<1\qquad (z\in U).$$

The class $S^*_p(A,B,\alpha)$ was introduced by Aouf [3].

Let $(f*g)(z)$ denote the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$,that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$(1.8)\qquad g(z)=z^p+\sum_{k=1}^{\infty}b_{p+k}z^{p+k},$$

then

$$(1.9)\qquad (f*g)(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^{p+k}.$$

A function $f(z)\in A(p)$ is said to be p-valent prestarlike function of order γ $(0\le\gamma<p)$ if

$$(1.10)\qquad (f*s^p_\gamma)(z)\in S^*(p,\gamma),$$

where $s^p_\gamma(z)$ is defined by (1.4).We denote by R^p_γ the class of all p-valent prestarlike functions of order γ (see [9] and [18]).

For a function $f(z)\in A(p)$, we define the following differential operator:

$$(1.11)\qquad D^0_{\lambda,p}f(z)=f(z),$$

$$D^1_{\lambda,p}f(z) = (1-\lambda)f(z)+\frac{\lambda}{p}zf'(z) = D_{\lambda,p}(D^0_{\lambda,p}f(z)) \quad (\lambda \geq 0;\ p \in N), \tag{1.12}$$

$$D^2_{\lambda,p}f(z) = D_{\lambda,p}(D^1_{\lambda,p}f(z)), \tag{1.13}$$

and

$$D^n_{\lambda,p}f(z) = D_{\lambda,p}(D^{n-1}_{\lambda,p}f(z)) \quad (p, n \in N;\ \lambda \geq 0). \tag{1.14}$$

It can be easily seen that

$$D^n_{\lambda,p}f(z) = z^p + \sum_{k=1}^{\infty}(\frac{p+\lambda k}{p})^n a_{p+k}z^{p+k}. \tag{1.15}$$

We note that:

(i) $D^n_{1,1}f(z) = D^n f(z)$ (Salagean [16]);
(ii) $D^n_{\lambda,1}f(z) = D^n_\lambda f(z)$ (AL-Oboudi [2]);
(iii) When $\lambda = 1$, the operator $D^n_{1,p}f(z) = D^n_p f(z)$ was introduced by Shenan et al. [18].

Let $R^{p,n}_{\gamma,\lambda}(A, B, \alpha)$ be the subclass of $A(p)$ consisting of functions $f(z)$ such that

$$\left|\frac{\dfrac{zh'(z)}{h(z)} - p}{B\dfrac{zh'(z)}{h(z)} - [pB + (A-B)(p-\alpha)]}\right| < 1 \quad (z \in U) \tag{1.16}$$

$(A$ and B fixed; $\ -1 \leq A < B \leq 1;\ 0 < B \leq 1;\ 0 \leq \alpha < p, p \in N),$

where

$$h(z) = (D^n_{\lambda,p}f * s^p_\gamma)(z) \quad (n \in N_0 = N \cup \{0\};\ \lambda \geq 0;\ 0 \leq \gamma < p;\ p \in N).$$

We note that:

(i) $R^{1,0}_{\gamma,0}(-1, 1, \alpha) = R_\gamma(\alpha)(0 \leq \gamma, \alpha < 1)$, is the class of $\gamma-$ prestarlike functions of order α, which was introduced by Sheil-Small et al. [17];

(ii) $R^{1,0}_{\gamma,0}(-\beta, \beta, \alpha) = R_\gamma(\alpha, \beta)(0 \leq \gamma, \alpha < 1)$, is the class of $\gamma-$ prestarlike functions of order α and type β, which was studied by Ahuja and Silverman [1];

(iii) $R^{1,0}_{\gamma,0}(A, B, \alpha) = R_\gamma(A, B, \alpha)$ (Aouf et al. [6]).

Denoting by $T(p)$ the subclass of $A(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k}z^{p+k} \quad (a_{p+k} \geq 0;\ p \in N). \tag{1.17}$$

We denote by $T_p^*(A,B,\alpha)$ and $R_{\gamma,\lambda}^{p,n}[A,B,\alpha]$ the classes obtained by taking intersections, respectively, of the classes $S_p^*(A,B,\alpha)$ and $R_{\gamma,\lambda}^{p,n}[A,B,\alpha]$ with the class $T(p)$. The class $T_p^*(A,B,\alpha)$ was studied by Aouf [3].

Also we note that, by specializing the parameters λ, p, n, A and B, we obtain the following subclasses studied by various authors:

(i) $R_{\gamma,0}^{1,0}[-1,1,\alpha] = R_\gamma[\alpha]$ $(0 \le \gamma, \alpha < 1)$ (Silverman an and Silvia [19], Uralegaddi and Sarangi [25], Aouf and Salgean [7], Srivastava and Aouf [20] and Raina and Sirvastava [15]);

(ii) $R_{\gamma,0}^{1,0}(-\beta,\beta,\alpha) = R_\gamma[\alpha,\beta]$ $(0 \le \gamma, \alpha < 1;\ 0 < \beta \le 1)$ (Ahuja and Silverman [1] and Owa and Ahuja [11]);

(iii) $R_{\gamma,1}^{1,n}[-1,1,\alpha] = R_\gamma[\gamma,\alpha,n)]$ $(0 \le \gamma,\ \alpha < 1,\ n \in N_0)$ (Aouf and Salagean [8] and Aouf et al. [4]);

(iv) $R_{\gamma,0}^{1,0}(A,B,\alpha) = R_\gamma[A,B,\alpha]$ (Aouf et al. [6]);

(v) $R_{p\alpha',0}^{p,0}[-1,1,p\alpha'] = R^p[\alpha']$ $(0 \le \alpha' < 1; p \in N)$ (Kumar and Reddy [9]);

(vi) $R_{\gamma,1}^{p,0}[-\beta,\beta,\alpha] = R_\gamma^p[\alpha,\beta]$

$$= \left\{ f(z) \in T(p) : \left| \frac{\dfrac{zh'(z}{h(z)} - p}{\dfrac{zh'(z)}{h(z)} + p - 2\alpha} \right| < \beta \quad (z \in U), \right.$$

where $h(z) = (f * s_\gamma^p)(z);\ 0 \le \alpha < p;\ 0 < \beta \le 1;\ p \in N\}$.

We further, observe that the special choices of n, λ, A and B our class $R_{\gamma,\lambda}^{p,n}[A,B,\alpha]$ give rise the following new subclasses of $T(p)$:

(i) $R_{\gamma,0}^{p,0}[-1,1,\alpha] = R_\gamma^p[\alpha]$, is the class of p-valent γ-prestarlike functions of order α $(0 \le \gamma, \alpha < p)$;

(ii) $R_{\gamma,0}^{p,0}[-\beta,\beta,\alpha] = R_\gamma^p[\alpha,\beta]$ $(0 \le \gamma, \alpha < p;\ 0 < \beta \le 1;\ p \in N)$; is the class of p-valent γ-prestarlike functions of order α and type β;

(iii) $R_{\gamma,0}^{p,0}[A,B,\alpha] = R_\gamma^p[A,B,\alpha]$

$$= \left\{ f(z) \in T(p) : \left| \frac{\dfrac{zh'(z)}{h(z)} - p}{B\dfrac{zh'(z)}{h(z)} - [pB + (A-B)(p-\alpha)]} \right| < 1 \quad (z \in U), \right.$$

where $h(z) = (f * s_\gamma^p)(z)$, $-1 \le A < B \le 1;\ 0 < B \le 1;\ 0 \le \alpha < p;\ p \in N\}$;

(iv) $R^{p,n}_{\gamma,1}[A,B,\alpha] = R^{p,n}_{\gamma}[A,B,\alpha]$

$$= \left\{ f(z) \in T(p) : \left| \frac{\dfrac{zh'(z)}{h(z)} - p}{B\dfrac{zh'(z)}{h(z)} - [pB + (A-B)(p-\alpha)]} \right| < 1 \quad (z \in U), \right.$$

where $h(z) = (D^n_p f * s^p_\gamma)(z)$, $-1 \leq A < B \leq 1$; $0 < B \leq 1$; $0 \leq \alpha < p$; $p \in N\}$;

(v) $R^{p,n}_{\gamma,1}[-\beta,\beta,\alpha] = R^{p,n}_{\gamma}[\alpha,\beta]$

$$= \left\{ f(z) \in T(p) : \left| \frac{\dfrac{zh'(z)}{h(z)} - p}{\dfrac{zh'(z)}{h(z)} + p - 2\alpha} \right| < \beta \quad (z \in U), \right.$$

where $h(z) = (D^n_p f * s^p_\gamma)(z)$, $-1 \leq A < B \leq 1$; $0 < B \leq 1$; $0 \leq \alpha < p$; $p \in N\}$.

In the present paper we propose to investigate several important properties and characterisics of the class $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$. Furthermore, distortion theorems involving generalized fractional derivative operator for functions in the class $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$ are given.

2. Coefficients estimates

THEOREM 1. *Let the function $f(z)$ be defined by (1.17). Then $f(z) \in R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$ if and only if*

$$\sum_{k=1}^{\infty}[(1+B)k + (B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma, k+1)a_{p+k} \leq (B-A)(p-\alpha). \tag{2.1}$$

Proof. It is known that [3] a necessary and sufficient condition for $f(z) \in T(p)$ to be in the class $T^*_p(A,B,\alpha)$ is that

$$\sum_{k=1}^{\infty}[(1+B)k + (B-A)(p-\alpha)]a_{p+k} \leq (B-A)(p-\alpha).$$

Since

$$(D^n_{\lambda,p} f * s^p_\gamma)(z) = z^p - \sum_{k=1}^{\infty}\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)a_{p+k}z^{p+k} \quad (a_{p+k} \geq 0),$$

where $s^p_\gamma(z)$ is given by (1.4), the result (2.1) follows. This completes the proof of Theorem 1.

COROLLARY 1. *Let the function* $f(z)$ *defined by* (1.17) *be in the class* $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$. *Then*

$$a_{p+k} \leq \frac{(B-A)(p-\alpha)}{[(1+B)k+(B-A)(p-\alpha)](\frac{p+\lambda k}{p})^n G^p(\gamma,k+1)} \quad (p,k \in N). \tag{2.2}$$

Equality in (2.2) holds true for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)(p-\alpha)}{[(1+B)k+(B-A)(p-\alpha)](\frac{p+\lambda k}{p})^n G^p(\gamma,k+1)} z^{p+k} \quad (p,k \in N). \tag{2.3}$$

3. Closure theorems

THEOREM 2. *The class* $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$ *is closed under convex linear combination.*

Proof. Let each of the functions $f_1(z)$ and $f_2(z)$ given by

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} \quad (a_{p+k,j} \geq 0; p \in N; j = 1,2) \tag{3.1}$$

be in the class $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$. Then it is sufficient to show that the function $h(z)$ defined by

$$h(z) = tf_1(z) + (1-t)f_2(z) \quad (0 \leq t \leq 1) \tag{3.2}$$

is also in the class $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$. Since, for $0 \leq t \leq 1$,

$$h(z) = z^p - \sum_{k=1}^{\infty} \{ta_{p+k,1} + (1-t)a_{p+k,2}\} z^{p+k}, \tag{3.3}$$

with the aid of Theorem 1, we have

$$\begin{aligned} \sum_{k=1}^{\infty} &[(1+B)k+(B-A)(p-\alpha)] \\ &\times \left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)\,[ta_{p+k,1} + (1-t)a_{p+k,2}] \\ &\leq (B-A)(p-\alpha) \quad (0 \leq t \leq 1) \end{aligned} \tag{3.4}$$

which implies that $h(z) \in R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$.

As a consequence of Theorem 2, there exist the extreme points of the class $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$.

THEOREM 3. *Let*

$$f_p(z) = z^p \tag{3.5}$$

and

$$(3.6)\quad f_{p+k}(z) = z^p - \frac{(B-A)(p-\alpha)}{[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)} z^{p+k} \qquad (p,k \in N).$$

Then $f(z) \in R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \mu_{p+k} f_{p+k}(z), \tag{3.7}$$

where $\mu_{p+k} \geq 0$ *and* $\sum\limits_{k=0}^{\infty} \mu_{p+k} = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=0}^{\infty} \mu_{p+k} f_{p+k}(z) \tag{3.8}$$

$$= z^p - \sum_{k=1}^{\infty} \frac{(B-A)(p-\alpha)}{[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)} \mu_{p+k} z^{p+k}.$$

Then it follows that

$$(3.9)\quad \sum_{k=1}^{\infty} \frac{[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)}{(B-A)(p-\alpha)} \times$$

$$\times \frac{(B-A)(p-\alpha)}{[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)} \mu_{p+k}$$

$$= \sum_{k=1}^{\infty} \mu_{p+k} = 1 - \mu_p \leq 1.$$

Therefore, by Theorem 1, $f(z) \in R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$.

Conversely, assume that the function $f(z)$ defined by (1.17) belongs to the class $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$. Then

$$(3.10)\quad a_{p+k} \leq \frac{(B-A)(p-\alpha)}{[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)} \qquad (p,k \in N).$$

Setting

$$\mu_{p+k} = \frac{[(1+B)k + (B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma, k+1)}{(B-A)(p-\alpha)} a_{p+k} \qquad (p, k \in N), \tag{3.11}$$

and

$$\mu_p = 1 - \sum_{k=1}^{\infty} \mu_{p+k}. \tag{3.12}$$

Hence, we can see that $f(z)$ can be expressed in the form (3.7). This completes the proof of Theorem 3.

COROLLARY 2. *The extreme points of the class $R^{p,n}_{\gamma,\lambda}[A, B, \alpha]$ are the functions $f_p(z) = z^p$ and*

$$f_{p+n}(z) = z^p - \frac{(B-A)(p-\alpha)}{[(1+B)k + (B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma, k+1)} z^{p+k} \qquad (p, k \in N).$$

4. Integral operators

THEOREM 4. *Let the function $f(z)$ defined by (1.17) be in the class $R^{p,n}_{\gamma,\lambda}[A, B, \alpha]$ and let c be a real number such that $c > -p$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(z)\, dt \tag{4.1}$$

also belongs to the class $R^{p,n}_{\gamma,\lambda}[A, B, \alpha]$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k}, \tag{4.2}$$

where

$$b_{p+k} = \left(\frac{c+p}{c+p+k}\right) a_{p+k}.$$

Therefore

$$\sum_{k=1}^{\infty} [(1+B)k + (B-A)(p-\alpha)] \left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma, k+1)\, b_{p+k}$$

$$= \sum_{k=1}^{\infty} [(1+B)k + (B-A)(p-\alpha)] \left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma, k+1) \left(\frac{c+p}{c+p+k}\right) a_{p+k}$$

$$\leq \sum_{k=1}^{\infty} [(1+B)k + (B-A)(p-\alpha)] \left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma, k+1)\, a_{p+k}$$
$$\leq (B-A)(p-\alpha),$$

since $f(z) \in R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$. Hence by Theorem 1, $F(z) \in R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$.

THEOREM 5. *Let the function $F(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (a_{p+k} \geq 0)$ be in the class $R^{p,n}_{\gamma,\lambda}[A,B,\alpha]$ and let c be a real number such that $c > -p$. Then the function $f(z)$ defined by (4.1) is p-valent in $|z| < R_p^*$, where*

(4.3)
$$R_p^* = \inf_k \left\{ \frac{p(c+p)[(1+B)k + (B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma, k+1)}{(p+k)(c+p+k)(B-A)(p-\alpha)} \right\}^{\frac{1}{k}}$$

$(k \geq 1)$. The result is sharp.

Proof. From (4.1), we have

$$f(z) = \frac{z^{1-c}\left[z^c F(z)\right]'}{(c+p)} (c > -p)$$
$$= z^p - \sum_{k=1}^{\infty} \left(\frac{c+p+k}{c+p}\right) a_{p+k} z^{p+k}.$$

To prove the result it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p \quad \text{for} \quad |z| < R_p^*,$$

where R_p^* is defined by (4.3). Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|\sum_{k=1}^{\infty} (p+k) \frac{c+p+k}{c+p} a_{p+k} z^k\right|$$
$$\leq \sum_{k=1}^{\infty} (p+k) \left(\frac{c+p+k}{c+p}\right) a_{p+k} |z|^k.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p$ if

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) \left(\frac{c+p+k}{c+p}\right) a_{p+k} |z|^k \leq 1. \tag{4.4}$$

But Theorem 1 confirms that

$$\sum_{k=1}^{\infty} \frac{[(1+\beta)k + (B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right) G^p(\gamma, k+1)}{(B-A)(p-\alpha)} a_{p+k} \leq 1. \tag{4.5}$$

Thus (4.4) will be satisfied if

$$\frac{(p+k)(c+p+k)}{p(c+p)}|z|^k \leq \frac{[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)}{(B-A)(p-\alpha)}$$

or if

$$(4.6)\quad |z| \leq \left\{\frac{p(c+p)[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)}{(p+k)(c+p+k)(B-A)(p-\alpha)}\right\}^{\frac{1}{k}}$$

($k \geq 1$). The required result follows now from (4.6). The result is sharp for the function

(4.7)

$$f(z) = z^p - \frac{(c+p+k)(B-A)(p-\alpha)}{(c+p)[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)} z^{p+k}$$

$(p, k \in N)$.

5. Distorion properties associated with generalized fractional calculus

In terms of the Gauss hypergeometric function:

$$(5.1)\qquad {}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}$$

$$(z \in U;\ \alpha,\beta,\gamma \in C;\ \gamma \neq 0,-1,-2,\ldots),$$

where (and in what follows) $(a)_n$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0) \\ a(a+1)\ldots(a+n-1) & (n \in N), \end{cases}$$

the generalized fractional calculus operators$I_{0,z}^{\beta,\delta,\eta}$ and $J_{0,z}^{\beta,\delta,\eta}$ are defined below (cf., e.g., [12] and [24]).

DEFINITION 1 (Generalized Fractional Integral Operator). For real numbers $\beta > 0, \delta$ andη, the generalized fractional integral of order β is defined, for a function $f(z)$, by

$$(5.2)\qquad I_{0,z}^{\beta,\delta,\eta} f(z) = \frac{z^{-\beta-\delta}}{\Gamma(\beta)} \int_0^z (z-\zeta)^{\beta-1}_2 F_1\left(\beta+\delta,-\eta;\beta;1-\frac{\zeta}{z}\right) f(\zeta)d\zeta$$

$$(\beta > 0; \in > \max\{0, \delta-\eta\}-1),$$

where $f(z)$ is analytic function in a simply - connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\beta-1}$ is removed by requiring log $(z-\zeta)$ to be real when $z-\zeta>0$, provided further that

(5.3) $$f(z)=O(|z|^{\in})(z\to 0).$$

DEFINITION 2 (Generalized fractional derivative operator). Under the hypotheses of Definition 1, the generalized fractional derivative of order β is defined, for a function $f(z)$, by

(5.4) $$J_{0,2}^{\beta,\delta,\eta}f(z)=$$

$$\begin{cases}\dfrac{1}{\Gamma(1-\beta)}\dfrac{d}{dz}\left\{z^{\beta-\delta}\int_0^z(z-\zeta)^{-\beta}{}_2F_1(\delta-\beta,1-\eta;1-\beta;1-\frac{\zeta}{z})f(\zeta)d\zeta\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(0\leq\beta<1), \\ \dfrac{d^n}{dz^n}J_{0,z}^{\beta-n,\delta,\eta}f(z)\quad(n\leq\beta<n+1;n\in N)\end{cases}$$

$$(\in>\max\{0,\delta-\eta\}-1),$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\beta}$ is removed, as in Definition 1, and $\in$ is given, as in Definition 1, by the order estimate (5.3).

It follows from Defintion 1 and Definition 2 that

(5.5) $$I_{0,z}^{\beta,-\beta,\eta}f(z)=D_z^{-\beta}f(z)\quad(\beta>0)$$

and

(5.6) $$J_{0,z}^{\beta,\beta,\eta}f(z)=D_z^{\beta}f(z)\quad(0\leq\beta<1),$$

where $D_z^{\beta}(\beta\in R)$ is the fractional calculus operator considered by Owa [10] and (subsequently) by Owa and Srivastava [13] and in many other works (cf., e.g., [5], [21], [22] and [23]). Furthermore, in terms of Gamma function, Definitions 1 and 2 readily yield

LEMMA 1 ([24]). *The generalized fractional integral and the generalized fractional derivative of a power function are given by*

(5.7) $$I_{0,z}^{\beta,\delta,\eta}z^{\rho}=\frac{\Gamma(\rho+1)\Gamma(\rho-\delta+\eta+1)}{\Gamma(\rho-\delta+1)\Gamma(\rho+\beta+\eta+1)}z^{\rho-\delta}$$
$$(\beta>0;\rho>\max\{0,\delta-\eta\}-1)$$

and

(5.8) $$J_{0,z}^{\beta,\delta,\eta}z^{\rho}=\frac{\Gamma(\rho+1)\Gamma(\rho-\delta+\eta+1)}{\Gamma(\rho-\delta+1)\Gamma(\rho-\beta+\eta+1)}z^{\rho-\delta}$$
$$(0\leq\beta<1;\rho>\max\{0,\delta-\eta\}-1).$$

Our main results on the growth and distortion properties of a function $f(z)\in R_{\gamma,\lambda}^{p,n}[A,B,\alpha]$, associated with the (generalized) fractional calculus operators $I_{0,z}^{\beta,\delta,\eta}$ and $J_{0,z}^{\beta,\delta,\eta}$ are contained in the following:

THEOREM 6. *Let the function* $f(z)$ *defined by* (1.17) *is in the class* $R^{p,n}_{\gamma,\lambda}[A, B, \alpha]$ (*with* $0 \le \gamma \le \frac{2p-1}{2}$; $p \in N$). *Then*

$$\left|I^{\beta,\delta,\eta}_{0,z} f(z)\right| \ge \frac{\Gamma(p+1)\Gamma(p-\delta+\eta+1)}{\Gamma(p-\delta+1)\Gamma(p+\beta+\eta+1)}|z|^{p-\delta}\times \tag{5.9}$$

$$\times\left\{1 - \frac{(B-A)(p-\alpha)(p+1)(p-\delta+\eta+1)}{2[(1+B)+(B-A))(p-\alpha)](p-\gamma)\left(\frac{p+\lambda}{p}\right)^n(p-\delta+1)(p+\beta+\eta+1)}|z|\right\}$$

and

$$\left|I^{\beta,\delta,\eta}_{0,z} f(z)\right| \le \frac{\Gamma(p+1)\Gamma(p-\delta+\eta+1)}{\Gamma(p-\delta+1)\Gamma(p+\beta+\eta+1)}|z|^{p-\delta}\times \tag{5.10}$$

$$\times\left\{1 + \frac{(B-A)(p-\alpha)(p+1)(p-\delta+\eta+1)}{2[(1+B)+(B-A))(p-\alpha)](p-\gamma)\left(\frac{p+\lambda}{p}\right)^n(p-\delta+1)(p+\beta+\eta+1)}|z|\right\}$$

$$(z \in U_0;\ \beta > 0; p > \max\{\delta-\eta, \delta, -\beta-\eta\} - 1;\ p+2 \ge \delta(1+\frac{\eta}{\beta});\ p \in N)$$

where

$$U_0 = \begin{cases} U & (\delta \le p) \\ U\backslash\{0\} & (\delta > p). \end{cases} \tag{5.11}$$

Equalities in (5.9) *and* (5.10) *are attained by the function*

$$f(z) = z^p - \frac{(B-A)(p-\alpha)}{2[(1+B)+(B-A))(p-\alpha)](p-\gamma)\left(\frac{p+\lambda}{p}\right)^n} z^{p+1}. \tag{5.12}$$

Proof. First of all, we note that

$$G^p(\gamma, k+2) \ge G^p(\gamma, k+1) \qquad (0 \le \gamma \le \frac{2p-1}{2};\ p, k \in N)$$

by means of (1.5). Consequently, by using Theorem 1, we have

$$\begin{aligned} &[(1+B)+(B-A))(p-\alpha)]\left(\frac{p+\lambda}{p}\right)^n G^p(\gamma,2)\sum_{k=1}^{\infty} a_{p+k} \\ &\le \sum_{k=1}^{\infty}[(1+B)k+(B-A)(p-\alpha)]\left(\frac{p+\lambda k}{p}\right)^n G^p(\gamma,k+1)\, a_{p+k} \\ &\le (B-A)(p-\alpha), \end{aligned}$$

which implies that

$$\sum_{k=1}^{\infty} a_{p+k} \le \frac{(B-A)(p-\alpha)}{2[(1+B)+(B-A))(p-\alpha)](p-\gamma)\left(\frac{p+\lambda}{p}\right)^n}. \tag{5.13}$$

Next, making use of the assertion (5.7) of Lemma 1, we find from (1.17) that

$$F(z) = \frac{\Gamma(p-\delta+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\delta+\eta+1)} z^{\delta} I_{0,z}^{\beta,\delta,\eta} f(z) \tag{5.14}$$
$$= z^p - \sum_{k=1}^{\infty} \Psi(k) a_{p+k} z^{p+k},$$

where, for convenience,

$$\Psi(k) = \frac{(p+1)_k (p-\delta+\eta+1)_k}{(p-\delta+1)_k (p+\beta+\eta+1)} \quad (k \in N).$$

Since

$$0 < \Psi(k) \le \Psi(1) = \frac{(p+1)(p-\delta+\eta+1)}{(p-\delta+1)(p+\beta+\eta+1)} \tag{5.15}$$

$$\left(p > \max\{\delta-\eta, \delta, -\beta-\eta\} - 1;\ p+2 \ge \delta\left(1+\frac{\eta}{\beta}\right);\ p \in N\right)$$

the assertions (5.9) and (5.10) would follow from (5.13), (5.14) and (5.15), respectively.

Finally, it is easily seen that the results (5.9) and (5.10) are sharp for the function $f(z)$ given by (5.12). This evidenlty completes the proof of Theorem 6.

By applying the assertion (5.8) of Lemma 1, instead of (5.7) , we can similarly prove the following theorem:

THEOREM 7. *Let the function $f(z)$ defined by (1.17) be in the class $R_{\gamma,\lambda}^{p,n}[A,B,\alpha]$ (with $0 \le \gamma \le \frac{2p-1}{2}$; $p \in N$). Then*

$$\left|J_{0,z}^{\beta,\delta,\eta} f(z)\right| \ge \frac{\Gamma(p+1)\Gamma(p-\delta+\eta+1)}{\Gamma(p-\delta+1)\Gamma(p-\beta+\eta+1)} |z|^{p-\delta} \times \tag{5.16}$$
$$\times \left\{1 - \frac{(B-A)(p-\alpha)(p+1)(p-\delta+\eta+1)}{2[(1+B)+(B-A))(p-\alpha)](p-\gamma)\left(\frac{p+\lambda}{p}\right)^n (p-\delta+1)(p-\beta+\eta+1)} |z|\right\}$$

and

$$\left|I_{0,z}^{\beta,\delta,\eta} f(z)\right| \le \frac{\Gamma(p+1)\Gamma(p-\delta+\eta+1)}{\Gamma(p-\delta+1)\Gamma(p-\beta+\eta+1)} |z|^{p-\delta} \times \tag{5.17}$$
$$\times \left\{1 + \frac{(B-A)(p-\alpha)(p+1)(p-\delta+\eta+1)}{2[(1+B)+(B-A))(p-\alpha)](p-\gamma)\left(\frac{p+\lambda}{\rho}\right)^n (p-\delta+1)(p-\beta+\eta+1)} |z|\right\}$$

$$\left(z \in U_0;\ 0 \le \beta < 1; p > \max\{\delta-\eta, \delta, \delta, \beta-\eta\}-1;\ p+2 \le \delta\left(1-\frac{\eta}{\beta}\right);\ p \in N\right),$$

where U_0 *is defined as before by* (5.11). *Equalities in* (5.16) *and* (5.17) *are attained by the function* $f(z)$ *given by* (5.12).

6. Applications

In view of the relationships (5.5) and (5.6), by setting $\delta = -\beta$ and $\delta = -\beta$in our assertions (5.9), (5.10), (5.16) and (5.17), respectively, we obtain:

COROLLARY 3. *.Let the function* $f(z)$ *defined by* (1.17) *be in the class* $R^{p,n}_{\gamma,\lambda}[A, B, \alpha]$ (*with* $0 \leq \gamma \leq \frac{2\rho-1}{2}; \rho \in N$). *Then*

$$\frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}|z|^{p+\beta}\times \tag{6.1}$$
$$\times\left\{1 - \frac{(B-A)(p-\alpha)(p+1)}{2[(1+B)+(B-A)(p-\alpha)](p-\gamma)(\frac{p+\lambda}{p})^n(p+\beta+1)}|z|\right\}$$
$$\leq \left|D_z^{-\beta}f(z)\right| \leq$$
$$\frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}|z|^{p+\beta}\times$$
$$\times\left\{\frac{(B-A)(p-\alpha)(p+1)}{2[(1+B)+(B-A)(p-\alpha)](p-\gamma)(\frac{p+\lambda}{p})^n(p+\beta+1)}|z|\right\}$$
$$(z \in U;\ \beta > 0)$$

and

$$\frac{\Gamma(p+1)}{\Gamma(p-\beta+1)}|z|^{p-\beta}\times \tag{6.2}$$
$$\times\left\{1 - \frac{(B-A)(p-\alpha)(p+1)}{2[(1+B)+(B-A)(p-\alpha)](p-\gamma)(\frac{p+\lambda}{p})^n(p-\beta+1)}|z|\right\}$$
$$\leq \left|D_z^{\beta}f(z)\right| \leq$$
$$\frac{\Gamma(p+1)}{\Gamma(p-\beta+1)}|z|^{p-\beta}\times$$
$$\times\left\{1 + \frac{(B-A)(p-\alpha)(p+1)}{2[(1+B)+(B-A)(p-\alpha)](p-\gamma)(\frac{p+\lambda}{p})^n(p-\beta+1)}|z|\right\}$$
$$(z \in U;\ 0 \leq \beta < 1).$$

Each of these results is sharp for the function $f(z)$ given by(5.12).

The assertions (6.1) and (6.2) of Corollary 3 can indeed be applied in order to deduce the following interesting results (Corollary 4 and Corollary 5, respectively) for functions in the class $R^{p,n}_{\gamma,\lambda}[A, B, \alpha]$.

COROLLARY 4. *Under the hypotheses of Corollary* 3, $D_z^{-\beta} f(z)(\beta > 0)$ *is included in a disc with its center at the origin and radius* r_1 *given by*

$$r_1 = \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)} \times \tag{6.3}$$

$$\times \left\{ 1 + \frac{(B-A)(p-\alpha)(p+1)}{2[(1+B)+(B-A)(p-\alpha)](p-\gamma)(\frac{p+\lambda}{p})^n (p+\beta+1)} \right\} \quad (\beta > 0).$$

COROLLARY 5. *Under the hypotheses of Corollary* 3, $D_z^{\beta} f(z)(0 \leq \beta < 1)$ *is included in a disc with its center at the origin and radius* r_2 *given by*

$$r_2 = \frac{\Gamma(p+1)}{\Gamma(p-\beta+1)} \times \tag{6.4}$$

$$\times \left\{ 1 + \frac{(B-A)(p-\alpha)(p+1)}{2[(1+B)+(B-A)(p-\alpha)](p-\gamma)(\frac{p+\lambda}{p})^n (p-\beta+1)} \right\} \quad (0 \leq \beta < 1).$$

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