DEMONSTRATIO MATHEMATICA
Vol. XL No 4 2007

Feng Wei, Zhankui Xiao

GENERALIZED JORDAN DERIVATIONS ON SEMIPRIME RINGS*

Abstract. It is shown that, given a 2-torsion-free semiprime ring with unit e, every generalized Jordan derivation on $\mathcal{R}$ is a generalized derivation. Let n be a fixed positive integer, $\mathcal{R}$ be a noncommutative $(n+1)!$-torsion-free prime ring with the center $\mathcal{C}_{\mathcal{R}}$. It is proved that, if $\mu : \mathcal{R} \to \mathcal{R}$ is a generalized Jordan derivation of $\mathcal{R}$ such that $\mu(x)x^n + x^n\mu(x) \in \mathcal{C}_{\mathcal{R}}$ for all $x \in \mathcal{R}$, then $\mu = 0$.

1. Introduction

The notion of generalized derivations appeared in operator algebras at first [8]. In the theory of operator algebras, they are considered as an important class of so-called *elementary operators.* Analysts have investigated these maps in the context of algebras on certain normed spaces [4]. Later, it was introduced to pure algebra field by Hvala in [10]. Since then many people began to study generalized derivations of rings in various ways, we refer the readers to [1]–[4], [10], [14], [15]. Let $\mathcal{R}$ be an associative ring. An additive map $\mu : \mathcal{R} \longrightarrow \mathcal{R}$ is called a *generalized derivation* of $\mathcal{R}$ if there exists a derivation d of $\mathcal{R}$ such that

$$\mu(xy) = \mu(x)y + xd(y)$$

for all $x, y \in \mathcal{R}$. d is called an *associated derivation* of the generalized derivation μ. Obviously, the following map

$$\mu : \mathcal{R} \longrightarrow \mathcal{R}, \quad x \longmapsto ax - xb$$

is a generalized derivation of $\mathcal{R}$, where a and b are fixed elements in $\mathcal{R}$. Indeed, for all $x, y \in \mathcal{R}$,

$$\mu(xy) = axy - xyb = (ax - xb)y + x(by - yb) = \mu(x)y + xd(y),$$

Key words and phrases: generalized Jordan derivations.
1991 *Mathematics Subject Classification*: 16W25.
*This work is supported by a research fellowship the China Scholarship Council and the Basic Research Foundation of Beijing Institute of Technology.

where d is an inner derivation of $\mathcal{R}$ induced by the element b. Such generalized derivations are called *generalized inner derivations*. It is easy to check that if the associated derivation d of a generalized derivation μ is inner, then μ is also inner. Moreover, all derivations of $\mathcal{R}$ and all right or left multiplier maps of $\mathcal{R}$ are also generalized derivations of $\mathcal{R}$. The notion of generalized Jordan derivation was introduced by Nakajima in [15]. An additive map $\mu : \mathcal{R} \longrightarrow \mathcal{R}$ is called a *generalized Jordan derivation* of $\mathcal{R}$ if there exists a Jordan derivation d of $\mathcal{R}$ such that

$$\mu(x^2) = \mu(x)x + xd(x)$$

for all $x \in \mathcal{R}$. The map d is called an *associated Jordan derivation* of the generalized Jordan derivation μ. When $\mathcal{R}$ is a 2-torsion-free ring, this definition is equivalent to saying that there exists a Jordan derivation d of $\mathcal{R}$ such that

$$\mu(xy + yx) = \mu(x)y + xd(y) + \mu(y)x + yd(x)$$

for all $x \in \mathcal{R}$. Obviously, all Jordan derivations of $\mathcal{R}$ and all generalized derivations of $\mathcal{R}$ are generalized Jordan derivations. It is well-known that all derivations of rings are Jordan derivations of rings. The converse is in general not true. Likewise, generalized derivations of rings are generalized Jordan derivations of rings. This converse is also in general false. Argac and Albas [2] gave a counterexample with respect to this. It is natural to ask whether every generalized Jordan derivation on a (semi-)prime ring is a generalized derivation. Recently, Jing and Lu proved that every generalized Jordan derivation on a 2-torsion-free prime ring is a generalized derivation [11].

In [15] and [16] Nakajima gathered together some elementary observations concerning categorical properties of generalized Jordan (Lie) derivations. Some results known for Jordan derivations and generalized derivations are extended to generalized Jordan derivations [15] and [16]. Jing and Lu studied generalized Jordan derivations and generalized Jordan triple derivations on prime rings and standard operator algebras [12].

The main objective of this paper is to study the generalized Jordan derivations on rings. We prove in Theorem 2.6 that every generalized Jordan derivation on a 2-torsion-free semiprime ring with unit e is a generalized derivation. Let n be a fixed positive integer, $\mathcal{R}$ be a noncommutative $(n+1)!$-torsion-free prime ring with the center $\mathcal{C}_{\mathcal{R}}$. It is proved in Theorem 2.8, that if μ is a generalized Jordan derivation of $\mathcal{R}$, such that $\mu(x)x^n + x^n\mu(x) \in \mathcal{C}_{\mathcal{A}}$ for all $x \in \mathcal{R}$, then $\mu = 0$. Let n be a fixed positive integer, $\mathcal{R}$ be a noncommutative $(n+1)!$-torsion-free semiprime ring with unit e and the center $\mathcal{C}_{\mathcal{R}}$. We prove in Theorem 2.11, that if μ is a generalized Jordan derivation of $\mathcal{R}$, such that $\mu(x)x^n + x^n\mu(x) \in \mathcal{C}_{\mathcal{A}}$ for all $x \in \mathcal{R}$, then μ maps $\mathcal{R}$ into $\mathcal{C}_{\mathcal{R}}$.

2. Generalized Jordan derivations on rings

Throughout this paper $\mathcal{R}$ always denotes an *associative ring* with the center $C_{\mathcal{R}}$ and $\mathcal{A}$ always denotes a *unital Banach algebra* which is a complex normed algebra and its underlying vector space is a Banach space. μ always denotes a generalized Jordan derivation with the associated Jordan derivation d on $\mathcal{R}$ or $\mathcal{A}$. A ring $\mathcal{R}$ is said to be *n-torsion-free* if $nx = 0$ implies $x = 0$ for all $x \in \mathcal{R}$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. Moreover, we assume that all maps on the Banach algebra $\mathcal{A}$ are linear maps in this paper.

Let us recall the following lemma.

LEMMA 2.1 ([4, Proposition 4.1.2]). *If $\mathcal{R}$ is a ring with unit e and $\mu : \mathcal{R} \longrightarrow \mathcal{R}$ is an additive map, then the following statements are equivalent:*

(i) *μ is a generalized derivation of $\mathcal{R}$.*
(ii) *There exists a derivation $d : A \longrightarrow A$ such that $\mu(y) = \mu(e)y + d(y)$ for all $y \in \mathcal{R}$.*
(iii) *The equality $\mu(xyz) = \mu(xy)z - x\mu(y)z + x\mu(yz)$ holds for all $x, y, z \in \mathcal{R}$.*

Let $\mathcal{R}$ be a ring with unit e. Then the definition of generalized Jordan derivation on $\mathcal{R}$ is equivalent to $\mu(x^2) = \mu(x)x + x\mu(x) - x\mu(e)x$ for all $x \in \mathcal{R}$ for all $x \in \mathcal{R}$. From now we will use the definition form of

$$\mu(x^2) = \mu(x)x + x\mu(x) - x\mu(e)x,$$

for all $x \in \mathcal{R}$. Using similar methods of [11] we can get the following results, and their proofs are omitted here.

LEMMA 2.2. *If $\mathcal{R}$ is a 2-torsion-free ring with unit e and μ is a generalized Jordan derivation of $\mathcal{R}$, then the following statements hold:*

(i) *$\mu(xy + yx) = \mu(x)y + x\mu(y) + \mu(y)x + y\mu(x) - x\mu(e)y - y\mu(e)x$ for all $x, y \in \mathcal{R}$.*
(ii) *$\mu(xyx) = \mu(x)yx + x\mu(y)x + xy\mu(x) - x\mu(e)yx - xy\mu(e)x$ for all $x, y \in \mathcal{R}$.*
(iii) *$\mu(xyz + zyx) = \mu(x)yz + x\mu(y)z + xy\mu(z) + \mu(z)yx + z\mu(y)x + zy\mu(x) - x\mu(e)yz - xy\mu(e)z - z\mu(e)yx - zy\mu(e)x$ for all $x, y, z \in \mathcal{R}$.*

LEMMA 2.3. *If $\mathcal{R}$ is a 2-torsion-free semiprime ring with unit e and μ is a generalized Jordan derivation of $\mathcal{R}$, then $(\mu(xy) - \mu(x)y - x\mu(y) + x\mu(e)y)z[x, y] = 0$ for all $x, y, z \in \mathcal{R}$.*

LEMMA 2.4. *If $\mathcal{R}$ is a 2-torsion-free semiprime ring with unit e and μ is a generalized Jordan derivation of $\mathcal{R}$, then $\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b \in C_{\mathcal{R}}$ for all $a, b \in \mathcal{R}$.*

LEMMA 2.5. *If $\mathcal{R}$ is a 2-torsion-free semiprime ring with unit e and μ is a generalized Jordan derivation of $\mathcal{R}$, then $(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)[r,s] = 0$ for all $a, b, r, s \in \mathcal{R}$.*

We are in a position to state the first main result of this paper.

THEOREM 2.6. *If $\mathcal{R}$ is a 2-torsion-free semiprime ring with unit e, then every generalized Jordan derivation on $\mathcal{R}$ is a generalized derivation.*

Proof. By Lemma 2.5, we have that $(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)[r,s] = 0$ for all $a, b, r, s \in \mathcal{R}$. Thus

$$\begin{aligned}
(2.1)\quad & 2(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)^2 \\
&= (\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)(\mu(ab) \\
&\quad - \mu(a)b - a\mu(b) + a\mu(e)b - (\mu(ba) - \mu(b)a - b\mu(a) + b\mu(e)a)) \\
&= (\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b \\
&\quad - \mu(ba) + \mu(b)a + b\mu(a) - b\mu(e)a) \\
&= (\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)(\mu([a,b]) + [\mu(b),a] + [b,\mu(a)] \\
&\quad + [a\mu(e),b] + [a,b\mu(e)] + [b,a]\mu(e)) \\
&= (\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)\mu([a,b]).
\end{aligned}$$

By Lemma 2.4, we get

$$\begin{aligned}
&(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)[a,b] + [a,b](\mu(ab) - \mu(a)b - ad(b) + a\mu(e)b) \\
&\qquad = 2(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)[a,b] = 0.
\end{aligned}$$

By (i) of Lemma 2.2, it follows that

$$\begin{aligned}
(2.2)\quad 0 &= \mu((\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)[a,b] \\
&\quad + [a,b](\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)) \\
&= \mu(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)[a,b] \\
&\quad + (\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)\mu([a,b]) \\
&\quad + \mu([a,b])(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b) \\
&\quad + [a,b]\mu(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b) \\
&\quad - (\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)\mu(e)[a,b] \\
&\quad - [a,b]\mu(e)(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b) \\
&= \mu(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)[a,b] + [a,b]\mu(\mu(ab) - \mu(a)b \\
&\quad - a\mu(b) + a\mu(e)b) + 4(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)^2.
\end{aligned}$$

The left multiplication of (2.2) by $\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b$ leads to

$$4(\mu(ab) - \mu(a)b - a\mu(b) + a\mu(e)b)^3 = 0.$$

Since $\mathcal{R}$ is 2-torsion-free, $(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)^3=0$. So $(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)^2x(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)^2=(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)^4x=0$ for all $x\in\mathcal{R}$. Since $\mathcal{R}$ is a semiprime ring, $(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)^2=0$. Furthermore, $(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)x(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)=(\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b)^2x=0$ for all $x\in\mathcal{R}$. By the semiprimeness of $\mathcal{R}$ again, it follows that $\mu(ab)-\mu(a)b-a\mu(b)+a\mu(e)b=0$. The proof is complete.

In [18] Sinclair has proved that every continuous Jordan derivation on a semisimple Banach algebra is a derivation. Simultaneously, Sinclair also posed a question: Is a Jordan derivation on a semisimple Banach algebra continuous? Brešar [7] gave an affirmative answer to Sinclair's question. In [12] Johnson and Sinclair proved that every derivation on a semisimple Banach algebra is continuous. Using this result and Lemma 2.1 it is easy to check that every generalized derivation on a semisimple Banach algebra is continuous. Thus we immediately get

COROLLARY 2.7. *Every generalized Jordan derivation on a semisimple Banach algebra $\mathcal{A}$ is continuous.*

In [1] Albas and Argac proved that if $\mathcal{R}$ is a noncommutative prime ring with char$\mathcal{R}\neq 2$ and μ is a generalized derivation of $\mathcal{R}$ such that $\mu(x)x+x\mu(x)\in\mathcal{C}_{\mathcal{R}}$ for all $x\in\mathcal{R}$, then $\mu=0$. We next consider a more general situation concerning generalized Jordan derivations of a prime ring and prove the following theorem.

THEOREM 2.8. *Let n be a fixed positive integer, $\mathcal{R}$ be a noncommutative $(n+1)!$-torsion-free prime ring and μ be a generalized Jordan derivation of $\mathcal{R}$. If*

$$\mu(x)x^n+x^n\mu(x)\in\mathcal{C}_{\mathcal{R}},$$

for all $x\in\mathcal{R}$, then $\mu=0$.

For the proof of Theorem 2.8, we need some basic results. From now on $\mathcal{R}$ always denotes a (semi-)prime ring and $\mathcal{U}$ always denotes the *left Utumi quotient ring* of $\mathcal{R}$. $\mathcal{U}$ can be characterized as a ring satisfying the following properties:

(1) $\mathcal{R}$ is a subring of $\mathcal{U}$.

(2) For each $q\in\mathcal{U}$, there exists a dense left ideal $\mathcal{I}_q$ of $\mathcal{R}$ such that $\mathcal{I}_q q\subseteq\mathcal{R}$.

(3) If $q\in\mathcal{U}$ and $\mathcal{I}q=0$ for some dense left ideal $\mathcal{I}$ of $\mathcal{R}$, then $q=0$.

(4) If $\phi:\mathcal{I}\to\mathcal{R}$ is a left $\mathcal{R}$-module map from a dense left ideal $\mathcal{I}$ of $\mathcal{R}$ into $\mathcal{R}$, then there exists an element $q\in\mathcal{U}$ such that $\phi(i)=iq$ for all $i\in\mathcal{I}$. Up to isomorphisms, $\mathcal{U}$ is uniquely determined by the above four properties. If $\mathcal{R}$ is a (semi-)prime ring, then $\mathcal{U}$ is also a (semi-)prime ring. The center

of $\mathcal{U}$ is called the *extended centroid* of $\mathcal{R}$ and is denoted by $\mathcal{C}$. It is well known that $\mathcal{C}$ is a Von Neumann regular ring. It turns out that $\mathcal{C}$ is a field if and only if $\mathcal{R}$ is a prime ring. The set of all idempotents of $\mathcal{C}$ is denoted by $\mathcal{E}$. The elements of $\mathcal{E}$ are called *central idempotents.* For the basic facts and results of left Utumi quotient ring $\mathcal{U}$ we refer the reader to [6].

Another related object we have to mention is the generalized differential identities on semiprime rings. A generalized differential polynomial over $\mathcal{U}$ means a generalized polynomial with coefficients in $\mathcal{U}$ and with noncommutative variables involving generalized derivations. A generalized differential identity for some subset of $\mathcal{U}$ is a generalized differential polynomial satisfied by the given subset. Obviously, the definition of a generalized differential polynomial(or identity) is a common generalization of the definition of a differential polynomial(or identity).

LEMMA 2.9 ([14, Theorem 4]). *Let $\mathcal{R}$ be a semiprime ring. Then every generalized derivation μ on a dense left ideal of $\mathcal{R}$ can be uniquely extended to be a generalized derivation of $\mathcal{U}$ and has the form $\mu(x) = \mu(e)x + d(x)$ for all $x \in \mathcal{U}$, where e is the identity element in $\mathcal{U}$ and d is a derivation on $\mathcal{U}$.*

Proof of Theorem 2.8. By [11, Theorem 2.5], μ is a generalized derivation with the associated derivation d on $\mathcal{R}$. According to the assumption, we have

$$[\mu(x)x^n + x^n\mu(x), z] = 0, \tag{2.3}$$

for all $x, z \in \mathcal{R}$. Substituting $x + \lambda y$ for x in (2.3), we get

$$\lambda P_1(x,y,z) + \lambda^2 P_2(x,y,z) + \cdots + \lambda^n P_n(x,y,z) = 0,$$

where $\lambda \in \mathbf{Z}$, $x, y, z \in \mathcal{R}$, $P_i(x,y,z)$ denotes the sum of terms involving i factors of y in the expansion of $[\mu(x+\lambda y)(x+\lambda y)^n+(x+\lambda y)^n\mu(x+\lambda y), z] = 0$. By [9, Lemma 1], we obtain

$$\begin{aligned} P_1(x,y,z) = [&\mu(x)x^{n-1}y + \mu(x)x^{n-2}yx + \mu(x)x^{n-3}yx^2 + \cdots + \mu(x)yx^{n-1} \\ &+ \mu(y)x^n + x^{n-1}y\mu(x) + x^{n-2}yx\mu(x) + x^{n-3}yx^2\mu(x) + \cdots \\ &+ yx^{n-1}\mu(x) + x^n\mu(y), z] = 0, \end{aligned}$$

for all $x, y, z \in \mathcal{R}$. This shows that

$$\begin{aligned} &\mu(x)x^{n-1}y + \mu(x)x^{n-2}yx + \mu(x)x^{n-3}yx^2 + \cdots + \mu(x)yx^{n-1} + \mu(y)x^n \\ &+ x^{n-1}y\mu(x) + x^{n-2}yx\mu(x) + x^{n-3}yx^2\mu(x) + \cdots + yx^{n-1}\mu(x) + x^n\mu(y) \in \mathcal{C}_{\mathcal{R}}, \end{aligned} \tag{2.4}$$

for all $x, y \in \mathcal{R}$.

If $\mathcal{C}_{\mathcal{R}} \neq 0$, then there exists a nonzero element $c \in \mathcal{C}_{\mathcal{R}}$. Taking $x = c$ in (2.3), we get

$$2\mu(c)c^n \in \mathcal{C}_{\mathcal{R}}$$

for all $x \in \mathcal{R}$. The fact that $c \in \mathcal{C}_{\mathcal{R}}$ is nonzero element and $\mathcal{R}$ is prime implies that $2\mu(c) \in \mathcal{C}_{\mathcal{R}}$. Note that $\mathcal{R}$ is $(n+1)!$-torsion-free. Therefore

$$\mu(c) \in \mathcal{C}_{\mathcal{R}}.$$

Taking $y = c$ in (2.4) gives

$$n(\mu(x)x^{n-1} + x^{n-1}\mu(x))c + \mu(c)x^n + x^n\mu(c) \in \mathcal{C}_{\mathcal{R}}$$

for all $x \in \mathcal{R}$. That is

$$n(\mu(x)x^{n-1} + x^{n-1}\mu(x))c + 2\mu(c)x^n \in \mathcal{C}_{\mathcal{R}} \tag{2.5}$$

for all $x \in \mathcal{R}$. Substituting xy for x in (2.3) produces

$$\mu(xy)(xy)^n + (xy)^n\mu(xy) \in \mathcal{C}_{\mathcal{R}},$$

for all $x, y \in \mathcal{R}$. That is

$$((\mu(x)y + xd(y))(xy)^n + (xy)^n(\mu(x)y + xd(y)) \in \mathcal{C}_{\mathcal{R}}, \tag{2.6}$$

for all $x, y \in \mathcal{R}$, where d is an associated derivation of μ. Taking $x = c$ in (2.6) gives

$$((\mu(c)y + cd(y))c^ny^n + c^ny^n((\mu(c)y + cd(y)) \in \mathcal{C}_{\mathcal{R}},$$

for all $y \in \mathcal{R}$. Since $c \in \mathcal{C}_{\mathcal{R}}$ is a nonzero element and $\mathcal{R}$ is a prime ring,

$$2\mu(c)y^{n+1} + cd(y)y^n + cy^nd(y) \in \mathcal{C}_{\mathcal{R}},$$

for all $y \in \mathcal{R}$. This implies that

$$[2\mu(c)y^{n+1} + cd(y)y^n + cy^nd(y), z] = 0, \tag{2.7}$$

for all $y, z \in \mathcal{R}$. Replacing y by $x + \lambda y$ in (2.7), we get

$$[2\mu(c)(x+\lambda y)^{n+1} + cd(x+\lambda y)(x+\lambda y)^n + c(x+\lambda y)^nd(x+\lambda y), z] = 0$$

for all $x, y, z \in \mathcal{R}$. We have

$$\lambda P_1(x,y,z) + \lambda^2P_2(x,y,z) + \cdots + \lambda^nP_n(x,y,z) = 0,$$

where $\lambda \in \mathbf{Z}$, $x, y, z \in \mathcal{R}$, $P_i(x,y,z)$ denotes the sum of terms involving i factors of y in the expansion of $[2\mu(c)(x+\lambda y)^{n+1} + cd(x+\lambda y)(x+\lambda y)^n + c(x+\lambda y)^nd(x+\lambda y), z] = 0$. By [9, Lemma 1] again, we get

$$\begin{aligned} P_1(x,y,z) = {} & [2\mu(c)(x^ny + x^{n-1}yx + x^{n-2}yx^2 + x^{n-3}yx^3 + \cdots + xyx^{n-1} \\ & + yx^n) + cd(x)(x^{n-1}y + x^{n-2}yx + x^{n-3}yx^2 + \cdots + yx^{n-1}) + cd(y)x^n \\ & + cx^nd(y) + c(x^{n-1}y + x^{n-2}yx + x^{n-3}yx^2 + \cdots + yx^{n-1})d(x), z] = 0 \end{aligned} \tag{2.8}$$

for all $x, y, z \in \mathcal{R}$. Taking $x = c$ in (2.8) leads to

$$2(n+1)\mu(c)c^ny + 2nd(c)c^ny + 2c^{n+1}d(y) \in \mathcal{C}_{\mathcal{R}}$$

for all $y \in \mathcal{R}$. This shows

$$(n+1)\mu(c)y + nd(c)y + cd(y) \in \mathcal{C}_{\mathcal{R}}, \tag{2.9}$$

for all $y \in \mathcal{R}$. By (2.9) we get

$$[(n+1)\mu(c)y + nd(c)y + cd(y), y] = 0,$$

for all $y \in \mathcal{R}$. The fact $\mu(c), d(c) \in \mathcal{C}_{\mathcal{R}}$ leads to

$$[d(y), y] = 0,$$

for all $y \in \mathcal{R}$. By [17, Lemma 3] we have

$$d = 0. \tag{2.10}$$

Combining (2.10) with (2.9), we have

$$(n+1)\mu(c)y \in \mathcal{C}_{\mathcal{R}},$$

for all $y \in \mathcal{R}$. Since $\mathcal{R}$ is a noncommutative $(n+1)!$-torsion-free prime ring, $\mu(c) = 0$. By (2.5) we know that

$$n(\mu(x)x^{n-1} + x^{n-1}\mu(x))c \in \mathcal{C}_{\mathcal{R}},$$

for all $x \in \mathcal{R}$. This implies that

$$\mu(x)x^{n-1} + x^{n-1}\mu(x) \in \mathcal{C}_{\mathcal{R}},$$

for all $x \in \mathcal{R}$. Continuing this process, we obtain

$$\mu(x)x + x\mu(x) \in \mathcal{C}_{\mathcal{R}},$$

for all $x \in \mathcal{R}$. By [1, Corollary 3.9], we get $\mu = 0$.

If $\mathcal{C}_{\mathcal{R}} = 0$, then

$$\mu(x)x^n + x^n\mu(x) = 0,$$

for all $x \in \mathcal{R}$. It is well known that $\mathcal{R}$ and $\mathcal{U}$ satisfy the same differential identities [13, Theorem 2] and hence also satisfy the same generalized differential identities, by Lemma 2.9. Therefore

$$\mu(x)x^n + x^n\mu(x) = 0, \tag{2.11}$$

for all $x \in \mathcal{U}$. Note that $\mathcal{U}$ has the identity element e. Choosing $x = e$ in (2.11) we obtain $2\mu(e) = 0$. Note that $\mathcal{U}$ is still a $(n+1)!$-torsion-free prime ring. It follows that $\mu(e) = 0$. By Lemma 2.9, μ can be uniquely extended to be a generalized derivation of $\mathcal{U}$ and has the form $\mu(x) = \mu(e)x + d(x)$ for all $x \in \mathcal{U}$. Hence $\mu(x) = d(x)$ for all $x \in \mathcal{U}$. Thus (2.11) becomes

$$d(x)x^n + x^n d(x) = 0$$

for all $x \in \mathcal{U}$. It follows from [19, Theorem 1]that $d = 0$ and hence $\mu = 0$. The proof of the theorem is complete.

Now, we will use the orthogonal completeness method [5] to extend Theorem 2.8 to the case of semiprime rings.

LEMMA 2.10 ([14, Theorem 2]). *If $\mathcal{R}$ is a semiprime ring with unit e, then every derivation d on a dense left ideal of $\mathcal{R}$ can be uniquely extended to be a derivation of $\mathcal{U}$.*

THEOREM 2.11. *Let n be a fixed positive integer, $\mathcal{R}$ be a noncommutative $(n+1)!$-torsion-free semiprime ring with unit e and μ be a generalized Jordan derivation of $\mathcal{R}$. If $\mu(x)x^n + x^n\mu(x) \in \mathcal{C}_{\mathcal{R}}$ for all $x \in \mathcal{R}$, then μ maps $\mathcal{R}$ into $\mathcal{C}_{\mathcal{R}}$.*

Proof. By Theorem 2.6, μ is a generalized derivation with the associated derivation d on $\mathcal{R}$. Let $\mathcal{B}$ be the complete Boolean algebra of $\mathcal{E}$. We choose a maximal ideal $\mathcal{M}$ of $\mathcal{B}$. According to [5], $\mathcal{M}\mathcal{U}$ is a prime ideal of $\mathcal{U}$, which is invariant under any derivation of $\mathcal{U}$. By Lemma 2.10, we know that the associated derivation d of μ can be uniquely extended to be derivations of $\mathcal{U}$. Let $\overline{d}$ be the canonical derivation of $\overline{\mathcal{U}} = \mathcal{U}/\mathcal{M}\mathcal{U}$ induced by d. By Lemma 2.9, we can set $\overline{\mu}(\overline{x}) = \overline{\mu(e)}\overline{x} + \overline{d}(\overline{x})$ for all $\overline{x} \in \overline{\mathcal{U}}$. It is easy to check that $\overline{\mu}$ is a generalized derivation of the prime ring $\overline{\mathcal{U}}$. The assumption implies that

$$[\mu(x)x^n + x^n\mu(x), z] = 0,$$

for all $x \in \mathcal{R}$. It is well known that $\mathcal{R}$ and $\mathcal{U}$ satisfy the same differential identities [13, Theorem 2] and hence also satisfy the same generalized differential identities by Lemma 2.9. Thus

$$[\mu(x)x^n + x^n\mu(x), z] = 0,$$

for all $x, z \in \mathcal{U}$. Furthermore,

$$[\overline{\mu}(\overline{x})\overline{x}^n + \overline{x}^n\overline{\mu}(\overline{x}), \overline{z}] = 0,$$

for all $\overline{x}, \overline{z} \in \overline{\mathcal{U}}$. By Theorem 2.8, either $\overline{\mu}(\overline{x}) = 0$ or $[\overline{\mathcal{U}}, \overline{\mathcal{U}}] = 0$. In any case

$$\mu(\mathcal{U})[\mathcal{U},\mathcal{U}] \in \mathcal{M}\mathcal{U},$$

for all $\mathcal{M}$. Note that $\bigcap\{\mathcal{M}\mathcal{U} | \mathcal{M}$ *is any maximal ideal of* $\mathcal{B}\} = 0$. So $\mu(\mathcal{U})[\mathcal{U},\mathcal{U}] = 0$. In particular, we have $\mu(\mathcal{R})[\mathcal{R},\mathcal{R}] = 0$. This implies

$$0 = \mu(\mathcal{R})[\mathcal{R}^2,\mathcal{R}] = \mu(\mathcal{R})\mathcal{R}[\mathcal{R},\mathcal{R}] + \mu(\mathcal{R})[\mathcal{R},\mathcal{R}]\mathcal{R} = \mu(\mathcal{R})\mathcal{R}[\mathcal{R},\mathcal{R}].$$

Therefore $[\mathcal{R}, \mu(\mathcal{R})]\mathcal{R}[\mathcal{R}, \mu(\mathcal{R})] = 0$. By the semiprimeness of $\mathcal{R}$, we get $[\mathcal{R}, \mu(\mathcal{R})] = 0$, that is $\mu(\mathcal{R}) \in \mathcal{C}_{\mathcal{R}}$. This completes the proof of the theorem.

By the theory of orthogonal completion for semiprime rings [6], we also have

COROLLARY 2.12. *Let n be a fixed positive integer, $\mathcal{R}$ be a noncommutative $(n+1)!$-torsion-free semiprime ring with a unit e and μ be a generalized Jordan derivation of $\mathcal{R}$. If $\mu(x)x^n + x^n\mu(x) \in \mathcal{C}_{\mathcal{R}}$, for all $x \in \mathcal{R}$, then there exists a central idempotent element e of $\mathcal{U}$ such that on the direct decomposition $e\mathcal{U} \oplus (1-e)\mathcal{U}$, μ vanishes identically on $e\mathcal{U}$ and the ring $(1-e)\mathcal{U}$ is commutative.*

Acknowledgements. The author would like to express his sincere thanks to the referee for the valuable suggestions which help to clarify the whole paper.

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DEPARTMENT OF APPLIED MATHEMATICS
BEIJING INSTITUTE OF TECHNOLOGY
BEIJING 100081, P. R. CHINA
E-mail: daoshuo@bit.edu.cn, zhkxiao@bit.edu.cn

Received December 4, 2006; revised version January 25, 2007.