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WHEN IS A BCC-ALGEBRA EQUIVALENT TO AN MV-ALGEBRA?

Abstract. The aim of this paper is to characterize BCC-algebras which are term equivalent to MV-algebras. It turns out that they are just the bounded commutative BCC-algebras. Further, we characterize congruence kernels as deductive systems. The explicit description of a principal deductive system enables us to prove that every subdirectly irreducible bounded commutative BCC-algebra is a chain (with respect to the induced order).

1. Introduction

By a **BCC-algebra** we mean an algebra $\mathcal{A} = (A; \to, 1)$ of type $(2,0)$ satisfying the following axioms

(BCC1) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$;

(BCC2) $x \to x = 1$;

(BCC3) $x \to 1 = 1$;

(BCC4) $1 \to x = x$;

(BCC5) $(x \to y = 1$ and $y \to x = 1)$ implies $x = y$.

These algebras were introduced by Y. Komori [9] in connection with the problem whether the class of all BCK-algebras forms a variety. The problem was solved in the negative.

Let us note that for a BCC-algebra $\mathcal{A}$ the relation defined by

$$(*) \qquad x \leq y \quad \text{if and only if} \quad x \to y = 1$$

is an order on A with greatest element 1, see e.g. [9]. Due to this fact, the

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identity (BCC1) can be read as

$$x \to y \le (z \to x) \to (z \to y).$$

This equivalent formulation will be used in our paper.

We can prove the following

LEMMA 1. *Let $\mathcal{A} = (A; \to, 1)$ be a BCC-algebra. Then*

(i) *$x \le y$ implies $z \to x \le z \to y$;*
(ii) *$x \le y$ implies $y \to z \le x \to z$;*
(iii) *$y \le x \to y$.*

Proof. Suppose $x \le y$. Then $x \to y = 1$ and, by (BCC1),

$$1 = (x \to y) \to ((z \to x) \to (z \to y)) = 1 \to ((z \to x) \to (z \to y)) = \\ = (z \to x) \to (z \to y)$$

thus $z \to x \le z \to y$ proving (i). Similarly,

$$1 = (y \to z) \to ((x \to y) \to (x \to z)) = (y \to z) \to (1 \to (x \to z)) = \\ = (y \to z) \to (x \to z)$$

whence $y \to z \le x \to z$ proving (ii).

Applying (ii) and the fact that $x \le 1$ for each $x \in A$ we conclude $y = 1 \to y \le x \to y$. ∎

The concept of a **BCK-algebra** was introduced by K. Iséki and Y. Imai [7] as an algebra $\mathcal{A} = (A; \to, 1)$ of type $(2, 0)$ satisfying the following axioms

(BCK1) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;

(BCK2) $x \to ((x \to y) \to y) = 1$;

(BCK3) $x \to x = 1$;

(BCK4) $x \to 1 = 1$;

(BCK5) $(x \to y = 1$ and $y \to x = 1)$ implies $x = y$.

Moreover, a BCK-algebra $\mathcal{A}$ is called **commutative** if it satisfies the so-called **commutative law** (see [1] for this notation)

(C) $(x \to y) \to y = (y \to x) \to x$.

It is well-known (see [8], [10]) that if $\mathcal{A}$ is a commutative BCK-algebra then it is a $\vee$-semilattice where $x \vee y = (x \to y) \to y$.

Analogously as for BCC-algebras, the relation $\le$ defined by $(*)$ is an order on the support of a BCK-algebra $\mathcal{A}$ and 1 is the greatest element.

The following three lemmas are known but their proofs were published in a different way in several hardly attainable papers. Thus we present our proofs for the reader's convenience.

LEMMA 2. *Let $\mathcal{A} = (A; \to, 1)$ be a BCK-algebra. Then*

(i) $1 \to x = x$;
(ii) $x \leq y$ *implies* $y \to z \leq x \to z$;
(iii) $x \leq y$ *implies* $z \to x \leq z \to y$;
(iv) $y \leq x \to y$.

Proof. (i) Using of (BCK3) and (BCK2), we get $1 \to x = (x \to x) \to x \geq x$. However, $(1 \to x) \to x = ((x \to x) \to x) \to x \geq x \to x = 1$ thus $1 = (1 \to x) \to x$ whence $1 \to x \leq x$. Together we have $1 \to x = x$.

(ii) Suppose $x \leq y$. Then $x \to y = 1$ and, by (i) and (BCK1),

$$1 = (x \to y) \to ((y \to z) \to (x \to z)) = 1 \to ((y \to z) \to (x \to z)) = \\ = (y \to z) \to (x \to z)$$

giving $y \to z \leq x \to z$.

(iii) If $x \leq y$ then by (i) and (BCK1) we derive

$$1 = (z \to x) \to ((x \to y) \to (z \to y)) = (z \to x) \to (1 \to (z \to y)) = \\ = (z \to x) \to (z \to y)$$

proving $z \to x \leq z \to y$.

(iv) Since $x \leq 1$ by (BCK4), we apply (i) and (ii): $y = 1 \to y \leq x \to y$. ■

LEMMA 3. *Every BCK-algebra satisfies the so-called* ***exchange identity***

(EI) $x \to (y \to z) = y \to (x \to z)$.

Proof. Substituting y by $y \to z$ in (BCK1), we get

$$x \to (y \to z) \leq ((y \to z) \to z) \to (x \to z).$$

By (BCK2) we have $y \leq (y \to z) \to z$, thus, by Lemma 2 (ii),

$$((y \to z) \to z) \to (x \to z) \leq y \to (x \to z).$$

Together it yields $x \to (y \to z) \leq y \to (x \to z)$. Swapping x, y, we obtain the converse inequality. ■

LEMMA 4. *Every BCK-algebra is a BCC-algebra. A BCC-algebra is a BCK-algebra if and only if it satisfies the exchange identity* (EI).

Proof. To prove the first assertion, we need to verify only (BCC1). By Lemma 3, a BCK-algebra satisfies (EI) thus, using this and (BCK1), we compute

$$(x \to y) \to ((z \to x) \to (z \to y)) = (z \to x) \to ((x \to y) \to (z \to y)) = 1.$$

Conversely, let a BCC-algebra $\mathcal{A}$ satisfy (EI). We need to verify only (BCK1) and (BCK2). By (BCC1) and (EI) we have

$$1 = (y \to z) \to ((x \to y) \to (x \to z)) = (x \to y) \to ((y \to z) \to (x \to z)),$$

which is (BCK1), and

$$1 = (x \to y) \to (x \to y) = x \to ((x \to y) \to y),$$

which is (BCK2). ∎

We can apply the previous lemmas to state

THEOREM 1. *Every BCC-algebra satisfying* (BCK2) *is a BCK-algebra.*

Proof. By (BCK2) we have $y \leq (y \to z) \to z$. Applying (BCC1) we compute

$$y \leq (y \to z) \to z \leq (x \to (y \to z)) \to (x \to z)$$

thus, by Lemma 1 (ii) and (BCK2),

$$y \to (x \to z) \geq ((x \to (y \to z)) \to (x \to z)) \to (x \to z) \geq x \to (y \to z).$$

Interchanging the roles of x and y, we obtain the converse inequality proving the exchange identity (EI). By Lemma 4, we have shown that the given BCC-algebra is in fact a BCK-algebra. ∎

2. Bounded BCC-algebras

We say that a BCC-algebra $\mathcal{A}$ is **bounded** if it has a least element 0, i.e. if $0 \leq x$ for each $x \in A$. Clearly, this property can be characterized by the identity

(Z) $0 \to x = 1$

and such an algebra will be denoted by $\mathcal{A} = (A; \to, 1, 0)$ to indicate the existence of a new nullary operation explicitly.

A bounded BCC-algebra $\mathcal{A}$ satisfies the **double negation law** if the identity

(DN) $(x \to 0) \to 0 = x$

holds in $\mathcal{A}$.

For the sake of brevity, we will denote $x \to 0$ by $\neg x$ and call it the **negation** of x. Hence, (DN) can be read as

$$\neg\neg x = x.$$

The concept of an MV-algebra was introduced by C.C. Chang [4] as an axiomatization of the Łukasiewicz many-valued logic. We present the definition taken from the monograph [5]:

By an **MV-algebra** we mean an algebra $\mathcal{M} = (M; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(MV2) $x \oplus y = y \oplus x$;

(MV3) $x \oplus 0 = x$;

(MV4) $\neg\neg x = x$;

(MV5) $x \oplus \neg 0 = \neg 0$ ($\neg 0$ is denoted by 1);

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

The following result was proved by D. Mundici [10]:

PROPOSITION. *Let $\mathcal{M} = (M; \oplus, \neg, 0)$ be an MV-algebra. Define $x \to y = \neg x \oplus y$ and $1 = \neg 0$. Then $\mathcal{A}(M) = (M; \to, 1, 0)$ is a bounded commutative BCK-algebra.*

Let $\mathcal{A} = (A; \to, 1, 0)$ be a bounded commutative BCK-algebra. Define $x \oplus y = (x \to 0) \to y$ and $\neg x = x \to 0$. Then $\mathcal{M}(A) = (A; \oplus, \neg, 0)$ is an MV-algebra.

In the sequel, we are going to modify the Proposition for BCC-algebras. At first we prove

LEMMA 5. *Every bounded BCK-algebra satisfying the double negation law satisfies the contraposition law*

(CL) $$x \to y = \neg y \to \neg x.$$

Proof. $\neg y \to \neg x = (y \to 0) \to (x \to 0) = x \to ((y \to 0) \to 0) = x \to \neg\neg y = x \to y$. ∎

REMARK. Of course, the contraposition law entails the double negation law since

$$x = 1 \to x = \neg x \to \neg 1 = \neg x \to 0 = \neg\neg x.$$

It is well-known that commutative BCK-algebras form a variety which can be axiomatized by the following identities

(i) $(x \to y) \to y = (y \to x) \to x$;
(ii) $x \to (y \to z) = y \to (x \to z)$;
(iii) $x \to x = 1$;
(iv) $1 \to x = x$.

Just as in case of BCK-algebras, we say that a BCC-algebra $(A; \to, 1)$ is **commutative** if it satisfies the identity (C).

It easily follows that these algebras are precisely the commutative BCK-algebras:

THEOREM 2. *Every commutative BCC-algebra $(A; \to, 1)$ is a commutative BCK-algebra.*

Proof. By Lemma 1 (iii) and (C) we have

$$x \leq (y \to x) \to x = (x \to y) \to y.$$

Thus $(A; \to, 1)$ satisfies (BCK2), which by Theorem 1 entails that it is a BCK-algebra. ■

COROLLARY. *Let $(A; \to, 1, 0)$ be a bounded commutative BCC-algebra. Define*

$$x \oplus y = (x \to 0) \to y \qquad \text{and} \qquad \neg x = x \to 0.$$

Then $(A; \oplus, \neg, 0)$ is an MV-algebra.

Proof. This is an immediate consequence of the previous Theorem 2 and Mundici's proposition. ■

3. Congruence kernels

Let $\mathcal{A} = (A; \to, 1)$ be a BCC-algebra and $\Theta \in Con\mathcal{A}$. The congruence class $[1]_\Theta$ is called the **kernel of** Θ.

Congruence kernels of BCC-algebras are in a close relationship with congruences and hence it is important to have their characterization.

Let $\mathcal{A} = (A; \to, 1)$ be a BCC-algebra. A subset $D \subseteq A$ is called a **deductive system** of $\mathcal{A}$ if

(a) $1 \in D$;
(b) if $a \in D$ and $a \to b \in D$ then also $b \in D$.

Of course, the condition (b) is in fact the deduction rule **Modus Ponens** which justifies the name "deductive system".

However, the class of BCC-algebras is not closed under homomorphic images thus there is not a one-to-one correspondence between congruences and their kernels. This correspondence exists only for the so-called relative congruences, i.e. such $\Theta \in Con\mathcal{A}$ that $\mathcal{A}/\Theta$ is a BCC-algebra again.

Further, note that every deductive system of a BCC-algebra $\mathcal{A}$ is an order filter (i.e. an upset) with respect to the induced order.

THEOREM 3. *Let $\mathcal{A} = (A; \to, 1, 0)$ be a bounded BCC-algebra satisfying the contraposition law* (CL). *Then $D \subseteq A$ is a congruence kernel if and only if D is a deductive system of $\mathcal{A}$.*

Proof. Obviously, if $D = [1]_\Theta$ for some $\Theta \in Con\mathcal{A}$, $a \in D$ and $a \to b \in D$ then $\langle a, 1\rangle \in \Theta, \langle a \to b, 1\rangle \in \Theta$ thus also $\langle a \to b, b\rangle = \langle a \to b, 1 \to b\rangle \in \Theta$ and, due to transitivity of Θ, $\langle b, 1\rangle \in \Theta$ proving $b \in D$. Thus D is a deductive system.

To prove the converse, we only need to show that the relation defined by

$$\langle x, y\rangle \in \Theta_D \text{ if and only if } x \to y, y \to x \in D$$

is a congruence on $\mathcal{A}$ whenever D is a deductive system. It is clear that then $[1]_{\Theta_D} = D$.

Suppose that D is a deductive system of $\mathcal{A}$. By definition, Θ_D is a reflexive and symmetric binary relation on A. Assume $\langle x, y\rangle \in \Theta_D$ and $\langle y, z\rangle \in \Theta_D$. Then $x \to y, y \to x, y \to z, z \to y \in D$ and, by (BCC1),

$$(y \to x) \to ((z \to y) \to (z \to x)) = 1 \in D.$$

Since $y \to x \in D$ and $z \to y \in D$, we conclude $z \to x \in D$. Analogously, it can be shown $x \to z \in D$, thus $\langle x, z\rangle \in \Theta_D$ proving that Θ_D is an equivalence of A. It remains to show that Θ_D has the Substitution Property with respect to $\to$.

Since $\mathcal{A}$ satisfies the contraposition law, we have

(P) $\langle x, y\rangle \in \Theta_D$ iff $\langle \neg x, \neg y\rangle \in \Theta_D$.

Suppose now $\langle x, y\rangle \in \Theta_D$, i.e. $x \to y, y \to x \in D$. By (BCC1) we have $(x \to y) \to ((z \to x) \to (z \to y)) = 1 \in D$ thus, due to Modus Ponens, also

$$(z \to x) \to (z \to y) \in D.$$

Analogously, we can show $(z \to y) \to (z \to x) \in D$, thus $\langle z \to x, z \to y\rangle \in \Theta_D$. Using (P), we obtain $\langle \neg z \to \neg x, \neg z \to \neg y\rangle \in \Theta_D$, thus also $\langle x \to z, y \to z\rangle \in \Theta_D$. Due to transitivity of Θ_D, we have shown the Substitution Property of Θ_D thus Θ_D is a congruence on $\mathcal{A}$. ∎

Of course, the set of all congruence kernels of a BCC-algebra $\mathcal{A}$ forms a complete lattice with respect to set inclusion. Hence, for a given subset $X \subseteq A$ there exists the least congruence kernel containing X, which will be denoted by $F(X)$. If $X = \{a\}$ is a singleton, $F(X)$ will be denoted briefly by $F(a)$.

In what follows we are going to characterize $F(a)$ explicitly:

LEMMA 6. *Let $\mathcal{A} = (A; \to, 1, 0)$ be a bounded BCC-algebra satisfying* (CL) *and $a \in A$. Define $F_0^a = \{x \in A; a \le x\}$ and $F_i^a = \{x \in A; \alpha \to x = \beta$ for some $\alpha, \beta \in F_{i-1}^a\}$ for $i = 1, 2, \ldots$. Then $F(a) = \bigcup\{F_i^a; i = 0, 1, \ldots\}$.*

Proof. Certainly, $1 \in F_0^a$. If $1 \in F_i^a$ then also $1 \in F_{i+1}^a$ since $1 \to 1 = 1$. Thus $1 \in F_i^a$ for all $i = 0, 1, 2, \ldots$. In particular, $1 \in F = \bigcup\{F_i^a; i = 0, 1, 2, \ldots\}$.

Furthermore, $F_i^a \subseteq F_{i+1}^a$. Indeed, if $x \in F_i^a$ then $1 \to x = x$, thus $x \in F_{i+1}^a$. Now we assume $x, x \to y \in F$. By the previous observation, there is an integer j such that $x, x \to y \in F_j^a$. By definition, this means $y \in F_{j+1}^a$ and hence $y \in F$. We have shown that F is a deductive system.

It is obvious that $F(a) \subseteq F$ since F is a deductive system containg a. We show the converse inclusion by induction. Trivially, $F_0^a \subseteq F(a)$. Assume

$F_i^a \subseteq F(a)$ and let $x \in F_{i+1}^a$. Then there are $\alpha, \beta \in F_i^a \subseteq F(a)$ such that $\alpha \to x = \beta$. This yields $x \in F(a)$ and hence $F(a) = F$. ∎

Moreover, if $\mathcal{A}$ is a bounded commutative BCC-algebra, we can prove the following

LEMMA 7. *Let $\mathcal{A} = (A; \to, 1, 0)$ be a bounded commutative BCC-algebra and $a, b \in A$, $a \parallel b$ and $a \vee b = 1$. Then for each $x \in F(a)$ and $y \in F(b)$ we have $x \vee y = 1$.*

Proof. (i) At first we prove that $x \vee y = 1$ for each $x \in F(a)$ and $y \in F_0^b$. This is clearly equivalent with $x \vee b = 1$. Since $x \in F(a)$, there is an index i such that $x \in F_i^a$. Evidently, $x \vee b = 1$ for $i = 0$. Suppose now $z \vee b = 1$ for all $z \in F_i^a$ and let $x \in F_{i+1}^a$. Then there are $\alpha, \beta \in F_i^a$ with $\alpha \to x = \beta$. Since $\alpha \vee b = \beta \vee b = 1$ by the induction hypothesis, we obtain

$$x \to b \leq (\alpha \to x) \to (\alpha \to b) = \beta \to ((\alpha \vee b) \to b) = \beta \to (1 \to b) =$$
$$= \beta \to b = (\beta \vee b) \to b = 1 \to b = b.$$

Since $b \leq x \to b$, we have $x \to b = b$, thus $x \vee b = (x \to b) \to b = b \to b = 1$.

(ii) Suppose now generally $x \in F(a)$, $y \in F(b)$. Then $y \in F_j^b$ for some index j. If $j = 0$ then $x \vee y = 1$ by (i). Assume $x \vee z = 1$ for all $z \in F_i^b$ and take $y \in F_{i+1}^b$. Then $\alpha \to y = \beta$ for $\alpha, \beta \in F_i^b$ thus, by the induction hypothesis, $x \vee \alpha = x \vee \beta = 1$. Hence

$$y \to x \leq (\alpha \to y) \to (\alpha \to x) = \beta \to (\alpha \to x) = \beta \to ((\alpha \vee x) \to x) =$$
$$= \beta \to (1 \to x) = \beta \to x = (\beta \vee x) \to x = 1 \to x = x.$$

Since $x \leq y \to x$, we conclude $y \to x = x$ thus

$$y \vee x = (y \to x) \to x = x \to x = 1. \ ∎$$

We are able to prove our main result.

THEOREM 4. *Let $\mathcal{A} = (A; \to, 1, 0)$ be a bounded commutative BCC-algebra. If $\mathcal{A}$ is subdirectly irreducible then $\mathcal{A}$ is a chain with respect to the induced order.*

Proof. Suppose that $\mathcal{A}$ is not a chain, i.e. there exist $a', b' \in A$ such that $a' \parallel b'$. Since $\mathcal{A}$ is commutative, it is a $\vee$-semilattice where $x \vee y = (x \to y) \to y$. The commutative law (C) implies

$$\neg\neg x = (x \to 0) \to 0 = (0 \to x) \to x = 1 \to x = x$$

thus $\mathcal{A}$ satisfies also the double negation law and, due to Lemma 2 (ii), the mapping $x \mapsto \neg x$ is an antitone involution. Hence, $\mathcal{A}$ is in fact a lattice where $x \wedge y = \neg(\neg x \vee \neg y)$.

The same is in fact true for every section $[p,1]$. The mapping assigning to $x \in [p,1]$ the element $x \to p$ is an antitone involution on $[p,1]$ and hence

$$x \wedge y = ((x \to p) \vee (y \to p)) \to p$$

for any $x, y \in [p,1]$.

Therefore, $a \to (a \wedge b) = a \to (((a \to 0) \vee (b \to 0)) \to 0) =$

$$= ((a \to 0) \vee (b \to 0)) \to (a \to 0) =$$

$$= (((b \to 0) \to (a \to 0)) \to (a \to 0)) \to (a \to 0) =$$

$$= (b \to 0) \to (a \to 0) = a \to b \quad \text{and hence}$$

$$(a \to b) \vee (b \to a) = (a \to (a \wedge b)) \vee (b \to (a \wedge b)) = (a \wedge b) \to (a \wedge b) = 1.$$

Thus for $a = a' \to b' \neq 1$ and $b = b' \to a' \neq 1$ we obtain

$$a \vee b = (a' \to b') \vee (b' \to a') = (a' \to (a' \wedge b')) \vee (b' \to (a' \wedge b')) =$$

$$= (a' \wedge b') \to (a' \wedge b') = 1.$$

By Lemma 7 we have $x \vee y = 1$ for each $x \in F(a)$ and $y \in F(b)$. This yields $F(a) \cap F(b) = \{1\}$. Since $F(a) \neq \{1\} \neq F(b)$ and $F(a)$ or $F(b)$ uniquely determines the congruence $\Theta(a,1)$ or $\Theta(b,1)$ on $\mathcal{A}$, respectively, we have $\Theta(a,1) \neq \omega \neq \Theta(b,1)$ but $\Theta(a,1) \cap \Theta(b,1) = \omega$ thus $\mathcal{A}$ is not subdirectly irreducible. ∎

This result together with Theorem 2 and the Proposition yields that if an MV-algebra is subdirectly irreducible then it is a chain with respect to the induced order. This result is known, see e.g. [5], however, our new proof is much more simple.

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A COMMON GENERALIZATION OF ORTHOLATTICES AND BOOLEAN QUASIRINGS

Abstract. In [2] a common generalization of Boolean algebras and Boolean rings was introduced. In a similar way we introduce a common generalization of ortholattices and Boolean quasirings.

In [2] a common generalization of Boolean algebras and Boolean rings was considered under the name N-algebra. In [1] the natural one-to-one correspondence between Boolean algebras and Boolean rings was generalized from Boolean algebras to ortholattices. The ring-like structures corresponding to ortholattices this way were called Boolean quasirings. Hence it is natural to ask for a common generalization of ortholattices and Boolean quasirings.

1. Ortholattices and Boolean quasirings

We start with the definition of an ortholattice:

DEFINITION 1.1. An *ortholattice* is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and

$$(x')' = x, (x \vee y)' = x' \wedge y', (x \wedge y)' = x' \vee y', x \vee x' = 1 \text{ and } x \wedge x' = 0$$

for all $x, y \in L$.

Next we define Boolean quasirings.

DEFINITION 1.2 ([1]). A *Boolean quasiring* is an algebra $(R, +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying

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$$x+y=y+x,$$
$$x+x=0,$$
$$x+0=x,$$
$$(xy)z=x(yz),$$
$$xy=yx,$$
$$xx=x,$$
$$x0=0,$$
$$x1=x,$$
$$(xy+1)(x+1)+1=x,$$
$$((x+1)(y+1)+1)(xy+1)=x+y \text{ and}$$
$$x'=x+1.$$

REMARK 1.3. The definition given here is a slight modification of the original one given in [1] since the operation $' : x \mapsto x+1$ was added to the family of fundamental operations.

Now we can state the the correspondence between the two algebras introduced above:

THEOREM 1.4 ([1]). *Let $\mathcal{L}=(L,\vee,\wedge,',0,1)$ be an ortholattice. Define*

$$x+y:=(x\vee y)\wedge(x\wedge y)' \text{ and } xy:=x\wedge y$$

for all $x,y\in L$. Then $\mathbf{R}(\mathcal{L}):=(L,+,\cdot,',0,1)$ is a Boolean quasiring. Conversely, let $\mathcal{R}=(R,+,\cdot,',0,1)$ be a Boolean quasiring. Define

$$x\vee y:=(x+1)(y+1)+1 \text{ and } x\wedge y:=xy$$

for all $x,y\in R$. Then $\mathbf{L}(\mathcal{R}):=(R,\vee,\wedge,',0,1)$ is an ortholattice. Moreover, $\mathbf{L}(\mathbf{R}(\mathcal{L}))=\mathcal{L}$ and $\mathbf{R}(\mathbf{L}(\mathcal{R}))=\mathcal{R}$ for every ortholattice $\mathcal{L}$ and every Boolean quasiring $\mathcal{R}$.

2. QN-algebras

In this section we present a common generalization of ortholattices and Boolean quasirings. Since the common generalization of Boolean algebras and Boolean rings introduced in [2] was called an *N-algebra* we call our algebras *Quasi-N-algebras* or *QN-algebras.*

DEFINITION 2.1. A *QN-algebra* is an algebra $(R,+,\cdot,',0,1)$ of type $(2,2,1,0,0)$ satisfying

$$(xy)z=x(yz),$$
$$xy=yx,$$
$$x0=0,$$

$$
\begin{aligned}
&x1 = x,\\
&(x')' = x,\\
&xx' = 0,\\
&(x'y')'x = x,\\
&((xy)'x')' = x \text{ and}\\
&x + y = ((1+1)'xy)'(x'y')'.
\end{aligned}
$$

We first prove that every QN-algebra induces an ortholattice.

LEMMA 2.2. *Let $(R, +, \cdot, ', 0, 1)$ be a QN-algebra. Define $x \vee y := (x'y')'$ for all $x, y \in R$. Then $(R, \vee, \cdot, ', 0, 1)$ is an ortholattice.*

Proof. Because of

$$
\begin{aligned}
&(xy)z = x(yz),\\
&xy = yx,\\
&(x \vee y) \vee z = (((x'y')')'z')' = ((x'y')z')' = (x'(y'z'))' = (x'((y'z')')')'\\
&\qquad\qquad = x \vee (y \vee z),\\
&x \vee y = (x'y')' = (y'x')' = y \vee x,\\
&(x \vee y)x = (x'y')'x = x,\\
&(xy) \vee x = ((xy)'x')' = x,\\
&x0 = 0,\\
&x1 = x,\\
&(x')' = x,\\
&(x \vee y)' = ((x'y')')' = x'y' \text{ and}\\
&(xy)' = ((x')'(y')')' = x' \vee y'
\end{aligned}
$$

for all $x, y, z \in R$, $(R, \vee, \cdot, 0, 1)$ is a bounded lattice and $'$ an antitone involution. Hence $0' = 1$ and $1' = 0$ which finally for all $x \in R$ implies

$$x \vee x' = (x'(x')')' = (x'x)' = (xx')' = 0' = 1 \text{ and } xx' = 0. \blacksquare$$

THEOREM 2.3. *The ortholattices are exactly the* QN-*algebras* $(R, +, \cdot, ', 0, 1)$ *satisfying* $1 + 1 = 1$.

Proof. Let $(L, \vee, \wedge, ', 0, 1)$ be an ortholattice and define $(L, +, \cdot, ', 0, 1) := (L, \vee, \wedge, ', 0, 1)$. Then $1 + 1 = 1 \vee 1 = 1$ and all axioms of a QN-algebra are satisfied since

$$
\begin{aligned}
((1+1)'xy)'(x'y')' &= ((1 \vee 1)' \wedge x \wedge y)' \wedge (x' \wedge y')' =\\
&= (1' \wedge x \wedge y)' \wedge (x \vee y) =\\
&= (0 \wedge x \wedge y)' \wedge (x \vee y) =
\end{aligned}
$$

$$= 0' \wedge (x \vee y) = 1 \wedge (x \vee y) =$$
$$= x \vee y = x + y$$

for all $x, y \in L$.

Conversely, assume $(R, +, \cdot, ', 0, 1)$ to be a QN-algebra satisfying $1 + 1 = 1$. Put $x \vee y := (x'y')'$ for all $x, y \in R$. According to Lemma 2.2, $(R, \vee, \cdot, ', 0, 1)$ is an ortholattice. Because of

$$\begin{aligned} x + y &= ((1+1)'xy)'(x'y')' = (1'xy)'(x'y')' = (0xy)'(x'y')' = 0'(x'y')' \\ &= 1(x'y')' = (x'y')' = x \vee y \end{aligned}$$

for all $x, y \in R$, $(R, +, \cdot, ', 0, 1)$ is an ortholattice. ∎

THEOREM 2.4. *The Boolean quasirings are exactly the* QN*-algebras* $(R, +, \cdot, ', 0, 1)$ *satisfying* $1 + 1 = 0$.

Proof. First let $(R, +, \cdot, ', 0, 1)$ be a Boolean quasiring. Define $x \vee y := (x'y')'$ for all $x, y \in R$. According to Theorem 1.4, $(R, \vee, \cdot, ', 0, 1)$ is an ortholattice and $1 + 1 = 1' = 0$. Moreover,

$$((1 \vee 1)'x)'(x'y')' = (0'xy)'(x \vee y) = (1xy)'(x \vee y) = (xy)'(x \vee y) = x + y$$

for all $x, y \in R$.

Conversely, assume $(R, +, \cdot, ', 0, 1)$ to be a QN-algebra satisfying $1+1 = 0$. Define $x \vee y := (x'y')'$ for all $x, y \in R$. According to Lemma 2.2, $(R, \vee, \cdot, ', 0, 1)$ is an ortholattice. Put $x \oplus y := (x \vee y)(xy)'$ for all $x, y \in R$. According to Theorem 1.4, $(R, \oplus, \cdot, ', 0, 1)$ is a Booloean quasiring. Now

$$\begin{aligned} x \oplus y &= (x \vee y)(xy)' = (x'y')'(xy)' = (xy)'(x'y')' = (1xy)'(x'y')' = \\ &= (0'xy)'(x'y')' = ((1+1)'xy)'(x'y')' = x + y \end{aligned}$$

for all $x, y \in R$ and hence $(R, +, \cdot, ', 0, 1)$ is a Boolean quasiring. ∎

3. Mutations of QN-algebras

DEFINITION 3.1. *Let* $\mathcal{R} = (R, +, \cdot, ', 0, 1)$ *be a* QN*-algebra and* $a \in R$. *Then the algebra* $\mathcal{R}_a := (R, +_a, \cdot, ', 0, 1)$ *with* $x +_a y := (a'xy)'(x'y')'$ *for all* $x, y \in R$ *is called the* a-mutation *of* $\mathcal{R}$.

We can now prove a theorem analogous to Theorem 3 of [2].

THEOREM 3.2. *Let* $\mathcal{R} = (R, +, \cdot, ', 0, 1)$ *be a* QN*-algebra and* $a, b \in R$. *Then the following hold:*

(i) $1 +_a 1 = a$.

(ii) $\mathcal{R}_a$ *is a* QN-*algebra.*

(iii) $\mathcal{R}_1$ *is an ortholattice.*

(iv) $\mathcal{R}_0$ *is a Boolean quasiring.*

(v) $\mathcal{R}_{1+1} = \mathcal{R}$.

(vi) $(\mathcal{R}_a)_b = \mathcal{R}_b$.

(vii) $\{\mathcal{R}_c \mid c \in R\}$ *is the set of all* QN-*algebras with base set* R *having the same multiplication and the same unary operation as* $\mathcal{R}$.

(viii) $\mathcal{R}$ *and* $\mathcal{R}_a$ *admit the same congruences.*

Proof. (i) $1 +_a 1 = (a'11)'(1'1')' = (a')'(1')' = a1 = a$.

(ii) $((1 +_a 1)'xy)'(x'y')' = (a'xy)'(x'y')' = x +_a y$ for all $x, y \in R$.

(iii) According to (ii), $\mathcal{R}_1$ is a QN-algebra and according to (i), $1 +_1 1 = 1$ and hence $\mathcal{R}_1$ is an ortholattice according to Theorem 2.3.

(iv) According to (ii), $\mathcal{R}_0$ is a QN-algebra and according to (i), $1 +_0 1 = 0$ and hence $\mathcal{R}_0$ is an Boolean quasiring according to Theorem 2.4.

(v) $x +_{1+1} y = ((1 + 1)'xy)'(x'y')' = x + y$ for all $x, y \in R$.

(vi) Since $\mathcal{R}$ is a QN-algebra, the same is true for $\mathcal{R}_a = (R, +_a, \cdot, ', 0, 1)$ according to (ii), and $x +_a y = (a'xy)'(x'y')'$ for all $x, y \in R$. Since $\mathcal{R}_a$ is a QN-algebra, the same is true for $(\mathcal{R}_a)_b = (R, (+_a)_b, \cdot, ', 0, 1)$ according to (ii), and $x(+_a)_b y = (b'xy)'(x'y')' = x +_b y$ for all $x, y \in R$.

(vii) Let $\mathcal{S} = (R, \oplus, \cdot, ', 0, 1)$ be a QN-algebra. Then $x \oplus y = ((1 \oplus 1)'xy)'(x'y')' = x +_{1\oplus 1} y$ for all $x, y \in R$ and hence $\mathcal{S} = \mathcal{R}_{1\oplus 1}$.

(viii) $\mathcal{R}$ and $\mathcal{R}_a$ admit the same congruences as $(R, \cdot, ')$. ■

Finally, we describe the correspondence stated in Theorem 1.4 in a simple way by means of mutations:

THEOREM 3.3. *The mappings* $\mathcal{R} \mapsto \mathcal{R}_0$ *and* $\mathcal{R} \mapsto \mathcal{R}_1$ *coincide with the mappings* $\mathbf{R}$ *and* $\mathbf{L}$ *introduced in Theorem 1.4, respectively.*

Proof. If $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ is an ortholattice then it is a QN-algebra and

$$x \vee_0 y = (0' \wedge x \wedge y)' \wedge (x' \wedge y')' = (1 \wedge x \wedge y)' \wedge (x \vee y) = (x \wedge y)' \wedge (x \vee y)$$

for all $x, y \in L$. Hence $\mathcal{L}_0 = \mathbf{R}(\mathcal{L})$. If, conversely, $\mathcal{R} = (R, +, \cdot, ', 0, 1)$ is a Boolean quasiring then it is a QN-algebra and

$$x +_1 y = (1'xy)'(x'y')' = (0xy)'(x'y')' = 0'(x'y')' = 1(x'y')' = (x'y')'$$

for all $x, y \in L$ and hence $\mathcal{R}_1 = \mathbf{L}(\mathcal{R})$. ■

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