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ŁUKASIEWICZ RESIDUATION ALGEBRAS WITH INFIMUM

Abstract. Łukasiewicz residuation algebras with an underlying ordered structure of meet semilattice (or iŁR-algebras) are studied. These algebras are the algebraic counterpart of the $\{\to, \wedge\}$-fragment of Łukasiewicz's many-valued logic. An equational basis for this class of algebras is shown. In addition, the subvariety of $(n+1)$–valued iŁR-algebras for $0 < n < \omega$ is considered. In particular, the structure of the free finitely generated $(n+1)$–valued iŁR-algebra is described. Moreover, a formula to compute its cardinal number in terms of n and the number of free generators is obtained.

1. Preliminares

B. Bosbach ([5, 6]) undertook the investigation of a class of residuated structures that were related to but considerably more general than Brouwerian semilattices and the algebras associated with $\{\to, \wedge\}$-fragment of Łukasiewicz's many valued logic.

In a manuscript by J. Büchi and T. Owens ([8]) devoted to a study of Bosbach's algebras, written in the mid–seventies, the commutative members of this equational class were given the name *hoops*. More precisely, they are algebras $\langle A, \to, \cdot, 1\rangle$ of type $(2,2,0)$ that satisfy:

(H1) $\langle A, \cdot, 1\rangle$ is a commutative monoid,
(H2) $x \to x = 1$,
(H3) $x \to (y \to z) = (x \cdot y) \to z$,
(H4) $x \cdot (x \to y) = y \cdot (y \to x)$.

An important subclass of the variety of hoops is the variety of *Wajsberg hoops*, so named and studied by W. Blok and I. Ferreirim in [3]. These

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algebras were defined as hoops that satisfy the additional identity:

(T) $(x \to y) \to y = (y \to x) \to x$,

and they constitute the $\{\cdot, \to, 1\}$-subreducts of Wajsberg algebras.

On the other hand, J. Berman and W. Blok ([2]) investigated the $\{\to, 1\}$-subreducts of hoops which they called *hoop residuation algebras.* It seems worth mentioning that the algebras which verify (H1), (H2), (H3) and the following two axioms:

(H5) $x \to 1 = 1$,
(H6) $x \to y = 1$ and $y \to x = 1$ imply $x = y$,

are known as *pocrims* and the $\{\to, 1\}$-subreducts of them are precisely the BCK-algebras; hoop residuation algebras are therefore BCK-algebras.

It was conjectured by A. Wroński and proved by Ferreirim ([11]) that hoop residuation algebras form a variety that can be defined by any axiomatization of BCK-algebras together with the axiom

(Hra) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$.

An important subvariety of this variety is that of $\{\to, 1\}$-subreducts of Wajsberg hoops which in [2], were called Łukasiewicz residuation algebras (or ŁR-algebras, for short). It is well-known that in these algebras the relation $\leq$ defined by $x \leq y$ if and only if $x \to y = 1$ is a partial order on A and $x \leq 1$ for every $x \in A$. In addition, $(A, \leq)$ is a join semilattice where $x \vee y = (x \to y) \to y$ is the supremum of the elements x and y.

On the other hand, a bounded ŁR-algebra (or ŁR^0-algebra) is an algebra $\langle A, \to, 0, 1\rangle$ where the reduct $\langle A, \to, 1\rangle$ is an ŁR-algebra and 0 is the least element for $\leq$.

We shall denote by ***ŁRA*** and ***ŁRA***^{0} the varieties of ŁR-algebras and ŁR^0-algebras respectively. In [17] (see also [19]), it was proved that the variety ***ŁRA***^{0} coincides with that of Wajsberg algebras which are MV-algebras, up to term equivalence (see [9]).

Let $A \in$***ŁRA*** or $A \in$***ŁRA***^{0}. Then, if S is a subalgebra of A, we shall write $S \lhd A$. Besides, if $X \subseteq A$, we shall represent by $[X]$ the subalgebra generated by X. For the concepts on universal algebra we direct the reader to the bibliography quoted in [7].

2. ŁR-algebras with infimum

In [10], W. Cornish defined the commutative BCK-algebras with supremum which were studied by T. Traczyk in [20]. In this section, we introduce a new class of Łukasiewicz residuation algebras and we show that this notion coincides with the $\{\to, \wedge, 1\}$−subreducts of MV-algebras. Furthermore, we establish the relationship between the dual notion of commutative BCK-algebras with supremum and the new ones.

DEFINITION 2.1. An iŁR-algebra is an algebra $\langle A, \rightarrow, \wedge, 1\rangle$ of type $(2,2,0)$ where the reduct $\langle A, \rightarrow, 1\rangle \in$ **ŁRA** and the following identity is verified:

(Ł1) $(x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

In what follows we shall denote by ***iŁRA*** the variety of iŁR-algebras.

Notice that if A is a Wajsberg algebra or $A \in$ ***ŁRA***^{0} and we define $x \wedge y = \sim (\sim x \vee \sim y)$ or $x \wedge y = ((x \rightarrow 0) \vee (y \rightarrow 0)) \rightarrow 0$ respectively, then $\langle A, \wedge, \rightarrow, 1\rangle \in$ ***iŁRA***.

THEOREM 2.1. *iŁR-algebras are exactly the* $\{\rightarrow, \wedge, 1\}$*−subreducts of MV-algebras.*

Proof. The $\{\rightarrow, \wedge, 1\}$−subreducts of MV-algebras are clearly iŁR-algebras, since MV-algebras satisfy $x \wedge y = x \cdot (x \rightarrow y)$ and therefore, $(x \wedge y) \rightarrow z = (x \cdot (x \rightarrow y)) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

For the converse, observe that every iŁR-algebra is downward directed. Indeed, it follows easily from (L1) that $x \wedge y$ is a lower bound of each pair x, y of elements in the algebra with respect to the definable order ($x \leq y$ iff $x \rightarrow y = 1$). Since iŁR-algebras are special BCK-algebras with an additional binary operation $\wedge$, taking into account [20] we conclude that every iŁR-algebra is a meet-semilattice where the greatest lower bound operation will be denoted by $\sqcap$. On the other hand, from the results established in [11], any downward directed BCK-algebra can be naturally embedded into a bounded one that satisfies all the same identities, namely an ultraproduct of its principal order-filters which are subalgebras. The ultraproduct is lattice-ordered and it is easy to see that the embedding preserves the undistinguished operation $\sqcap$. Besides, the ultraproduct satisfies the identity $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ because it holds in all ŁR-algebra. So the ultraproduct is an MV-algebra, up to term equivalence. Thus, the $\{\rightarrow, 1\}$−reduct of an iŁR-algebra embeds in an MV-algebra with preservation of the implicit operation $\sqcap$. To see that every iŁR-algebra embeds in an MV-algebra, it is therefore enough to show that in any iŁR-algebra, $\wedge$ and $\sqcap$ coincide. Indeed, since $\sqcap$ is the greatest lower bound and $\wedge$ is a lower bound, we have $x \wedge y \leq x \sqcap y$. For the reverse inequality, recall that $(x \sqcap y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$ holds in every MV-algebra, so in an iŁR-algebra, both $\wedge$ and $\sqcap$ satisfy (L1). Since these two laws have the same right hand side, every iŁR-algebra satisfies (1) $(x \sqcap y) \rightarrow z = (x \wedge y) \rightarrow z$. Substituting $x \wedge y$ for z in (1), we obtain $(x \sqcap y) \rightarrow (x \wedge y) = 1$ and so, $x \sqcap y \leq x \wedge y$. Hence, $x \wedge y = x \sqcap y$. Thus, iŁR-algebras are exactly the $\{\rightarrow, \wedge, 1\}$−subreducts of MV-algebras. ∎

In particular, from the above theorem we conclude

PROPOSITION 2.1. *Let $\langle A, \rightarrow, \wedge, 1\rangle$ be an algebra of type $(2,2,0)$. Then the following conditions are equivalent:*

(i) *$\langle A, \rightarrow, 1\rangle$ is an ŁR-algebra such that every pair of elements has a common lower bound where $x \wedge y$ is the infimum of the elements x, y,*
(ii) *$\langle A, \rightarrow, \wedge, 1\rangle$ is an iŁR-algebra.*

Proposition 2.1 justifies the name of ŁR-algebras with infimum given to iŁR-algebras.

It is well known that MV-algebras constitute the algebraic counterpart of the infinite-valued logic ω-LPC of Łukasiewicz, and that the only connective really used in the algebraization process is $\rightarrow$. It follows on general grounds (see [4, Cor. 2.12]) that the various subreducts of MV-algebras that retain $\rightarrow$ algebraize the corresponding fragments of ω-LPC. Then, by Theorem 2.1 we can conclude that iŁR-algebras constitute the algebraic counterpart of the $\{\rightarrow, \wedge\}$–fragment of ω-LPC.

An axiomatization for this calculus can be obtained from the one given by Wozniakowska in [21] for ω-LPC. Taking into account this paper, the $\{\rightarrow, \wedge\}$–fragment is captured by adopting the detachment rule, the substitution rule and the following set of axioms:

(A0) $x \rightarrow (y \rightarrow x)$,
(A1) $((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow ((y \rightarrow x) \rightarrow (y \rightarrow z))$,
(A2) $((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x))$,
(A3) $(x \wedge y) \rightarrow x$,
(A4) $(x \wedge y) \rightarrow y$,
(A5) $(x \rightarrow y) \rightarrow ((x \rightarrow z)) \rightarrow (x \rightarrow (y \wedge z)))$.

In what follows our attention is focused on the subvariety of ***iŁRA*** consisting of $(n+1)$–valued iŁR-algebras (or iŁR$_{n+1}$–algebras) for $0 \leq n < \omega$, which we shall denote by $\boldsymbol{i\text{Ł}RA_{n+1}}$.

DEFINITION 2.2. An iŁR$_{n+1}$-algebra for $0 \leq n < \omega$ is an iŁR-algebra which satisfies the identity

(Ł2) $(x \rightarrow^n y) \vee x = 1$,

where $x \rightarrow^0 y = y$ and $x \rightarrow^{i+1} y = x \rightarrow (x \rightarrow^i y)$ for $i < \omega$.

Now we shall indicate some properties of $\boldsymbol{i\text{Ł}RA_{n+1}}$ in order to determine the free finitely generated algebras.

(P1) Let $\text{Ł}_{n+1} = \{e^0, e^1, \ldots, e^n\}$ with $1 = e^0 > e^1 > \ldots > e^n = 0$. Then, $\vec{\text{Ł}}_{n+1} = \langle \text{Ł}_{n+1}, \rightarrow, \wedge, 1\rangle \in \boldsymbol{i\text{Ł}RA_{n+1}}$, where

$$e^i \to e^j = \begin{cases} 1, & \text{if } i \geq j, \\ e^{j-i}, & \text{otherwise} \end{cases} \quad \text{and} \quad e^i \wedge e^j = e^{max\{i,j\}}.$$

Besides, if $S \lhd \vec{Ł}_{n+1}$ and $|S| > 1$, then $S \simeq \vec{Ł}_{t+1}$ for $1 \leq t \leq n$.

From Theorem 2.1 and well kown result we obtain

(P2) The variety $\boldsymbol{iŁRA}_{\mathbf{n+1}}$ is generated by $\vec{Ł}_{n+1}$ and its subalgebras.

On the other hand, it holds

(P3) $\vec{Ł}_{n+1}$ is a quasiprimal algebra for $0 \leq n < \omega$.
Indeed, taking into account [16, Theorem 2] it follows immediately that $\boldsymbol{iŁRA}_{\mathbf{n+1}}$ is an arithmetical variety. Besides, every non trivial subalgebra of $\vec{Ł}_{n+1}$ is simple. Then by a result due to Pixley (see [7, Section 10]) we have that $\vec{Ł}_{n+1}$ is quasiprimal.

3. Free $\mathbf{iŁR_{n+1}}$-algebras

In what follows, we shall denote by $\mathcal{L}_{n+1}(c)$ the (n+1)-valued iŁR-algebra with a set G of free generators, such that $|G| = c$ where c is a cardinal number. The notion of free iŁR$_{n+1}$-algebra is defined in the usual way and since iŁR$_{n+1}$-algebras are equationally definable, for any cardinal number $c > 0$ the free algebra $\mathcal{L}_{n+1}(c)$ exists and it is unique up to isomorphism.

In 1982, J. Berman ([1]) and later on A. V. Figallo and J. Tolosa ([14]) obtained the free iŁR$_{n+1}$-algebra in the case that $n = 2$, independently.

The aim of this paper is to determine the structure of $\mathcal{L}_{n+1}(m)$ and the formula which provides $|\mathcal{L}_{n+1}(m)|$ for every pair n, m such that $0 < n$, $m < \omega$.

Now, as $\vec{Ł}_{n+1}$ is a quasiprimal algebra with the property that every subalgebra of it has no automorphisms other than the identity map, then by well known results of universal algebra we have that

$$\mathcal{L}_{n+1}(m) = \prod_{i=1}^{n} \vec{Ł}_{i+1}^{\,\alpha_{m,i}}, \tag{1}$$

where

$$\alpha_{m,i} = | \{ f \in \vec{Ł}_{i+1}^{\,G} : [f(G)] = \vec{Ł}_{i+1} \} | \tag{2}$$

and $\vec{Ł}_{i+1}^{\,G}$ denotes the set of all the mappings from G into $Ł_{i+1}$.

On the other hand, observe that

$$[f(G)] = \vec{Ł}_{i+1} \quad \text{implies} \quad e^i \in f(G). \tag{3}$$

Indeed, if $e^i \notin f(G)$ then there is $e^t = min\, f(G)$ with $0 \leq t < i$ and since $[e^t, e^0] = \{x \in \vec{Ł}_{i+1} : e^t \leq x \leq e^i\} \lhd \vec{Ł}_{i+1}$, then $[f(G)] \subseteq [e^t, e^0]$ and therefore, $[f(G)] \subset \vec{Ł}_{i+1}$.

Besides, from the theory of Wajsberg algebras (see [19]) it follows that

(4) any subalgebra of $\vec{Ł}_{i+1}$ that contains e^i must be of the form $\{e^0, e^j, e^{2j}, \ldots, e^{kj}\}$ for any j with $i = jk$ for some integer k.

Now, we are going to compute $\alpha_{m,i}$.

For every i, $1 \leq i \leq n$ let us consider the set

$$\mathcal{F}_{m,i} = \{f \in \vec{Ł}_{i+1}^{\,G} \colon e^i \in f(G)\}.$$

Then we have that

$$| \mathcal{F}_{m,i} | = (i+1)^m - i^m. \tag{5}$$

On the other hand, taking into account (3) and (4), it is simple to check that

$$| \mathcal{F}_{m,i} | = | \bigcup_{j|i} \{f \in \vec{Ł}_{j+1}^{\,G} \colon [f(G)] = \vec{Ł}_{j+1}\} | . \tag{6}$$

Therefore, from (6) and (2) we obtain

$$| \mathcal{F}_{m,i} | = \sum_{j|i} \alpha_{m,j} = \alpha_{m,i} + \sum_{j|i,\, j\neq i} \alpha_{m,j}. \tag{7}$$

Finally, from (5) and (7) we conclude that

$$\alpha_{m,i} = (i+1)^m - i^m - \sum_{j|i,\, j\neq i} \alpha_{m,j}. \tag{8}$$

Hence we have shown the main result of this paper which is the following

THEOREM 3.1. *Let $\mathcal{L}_{n+1}(m)$ be a free $iŁR_{n+1}$-algebra with m free generators. Then its cardinality is given by the following formula:*

$$|\mathcal{L}_{n+1}(m)| = \prod_{i=1}^{n} (i+1)^{\ \alpha_{m,i}}$$

where

$$\alpha_{m,i} = (i+1)^m - i^m - \sum_{j|i,\, j\neq i} \alpha_{m,j}.$$

REMARK 3.1. From Theorem 3.1 we obtain that

$$\mathcal{L}_{n+1}(m) = \mathcal{L}_n(m) \times (\vec{Ł}_{n+1})^{\ \alpha_{m,n}}$$

which makes clear the recursive structure of the free algebras in $\boldsymbol{iŁRA}_{\mathbf{n+1}}$. Note that for $n = 1$ we have

$$\mathcal{L}_2(m) = (\vec{Ł}_2)^{\alpha_{m,1}} = (\vec{Ł}_2)^{2m-1}.$$

On the other hand, for $m = 1$ we get

$$\alpha_{1,1} = 1 \quad \text{and} \quad \alpha_{1,i} = 0, \quad \text{for } i > 1.$$

Hence,

$$\mathcal{L}_{n+1}(1) = \mathcal{L}_1(1) = (\vec{\text{Ł}}_2)^1 \quad \text{for all } n.$$

Finally, for $n = 2$ we have

$$\mathcal{L}_3(m) = (\vec{\text{Ł}}_2)^{2^m-1} \times (\vec{\text{Ł}}_3)^{3^m-2^{m+1}+1},$$

and then it follows

$$|\mathcal{L}_3(m)| = 2^{2^m-1} \cdot 3^{3^m-2^{m+1}+1}.$$

These formulas were obtained both in [1] and [14].

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