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ON gβ-REGULAR AND gβ-NORMAL SPACES

Abstract. The aim of this paper is to introduce and study two new classes of spaces, called gβ-regular and gβ-normal spaces. The concept of gβ-regularity and gβ-normality are separation properties obtained by utilizing gβ-closed sets. Recall that a subset A of a space (X,τ) is called gβ-closed if $\beta\,\mathrm{Cl}(A) \subseteq U$ where $A \subseteq U$ and U is open in X.

1. Introduction and preliminaries

Throughout this paper we consider spaces on which no separation axiom are assumed unless explicitly stated. The topology of a space (by space we always mean a topological space) is denoted by τ and (X,τ) will be replaced by X if there is no chance of confusion. For $A \subseteq X$, the closure and interior of A in X are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively.

A subset A of a topological space X is called is called semi-open [11] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. Further it is said to be β-open [1] or semi-preopen [3] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a semi-open (resp. β-open) set is called a semi-closed (resp. β-closed). The semi-closure [5] (resp. β-closure [2]) of A, denoted by $S\,\mathrm{Cl}(A)$ (resp. $\beta\,\mathrm{Cl}(A)$) is the intersection of all semi-closed (resp. β-closed) sets containing A. It is said to be semi-generalized closed, briefly sg-closed [4], if $S\,\mathrm{Cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open in X. It is said to be generalized β-closed [15] or generalized semi-preclosed [6], briefly gβ-closed, if $\beta\,\mathrm{Cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is open in X. Further, the complement of gβ-closed set is said to be gβ-open [15].

In 1970, Levine [12] introduced generalized closed set. A subset A of a topological space (X,τ) is called generalized closed, briefly g-closed if $\mathrm{Cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is open in (X,τ).

This notion has been studied extended in recent years by many topologists. Ganguly et al. [7] generalized the usual notations of regularity

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and normality by replacing "closed-set" by "g-closed" set in the definitions, thus obtaining the notions of g-regularity and g-normality. Further Ganster et. al [8] introduced the concept of semi-g-regular and semi-g-normal spaces. where disjoint "sg-closed" sets can be separated by disjoint semi-open sets. The aim of our paper is to introduce and investigate the notions of generalized β-regular, briefly gβ-regular and generalized β-normal, briefly gβ-normal spaces which generalizes the concept of semi-g-regular and semi-g-normal spaces by utilizing the concept of gβ-closed set and replacing "sg-closed" set by "gβ-closed" set.

REMARK 1. Every g-closed and sg-closed sets are gβ-closed but not conversely as can be seen in Examples 3.3 and diagram on page 38 of [6].

2. gβ-Regular spaces

DEFINITION 1. A topological space (X, τ) is said to be $(*)$ semi-pre regular [14] if for each β-closed set $A \subseteq X$ and each point $x \in X$ such that $x \notin A$, there exist disjoint β-open sets U and V of X such that $x \in U$ and $A \subseteq V$.

DEFINITION 2. A topological space (X, τ) is said to be gβ-regular if for each gβ-closed set A and each point $x \in X$ such that $x \notin A$, there exist disjoint β-open sets $U, V \subseteq X$ such that $A \subseteq V$ and $x \in U$.

DEFINITION 3. A topological space (X, τ) is said to be semi-pre $T_{1/2}$ [6] if every gβ-closed set is β-closed.

REMARK 2. It is shown in [6] that a space (X, τ) is semi-pre $T_{1/2}$ if and only if every singleton of X is closed or β-open. Also a space (X, τ) is said to be β-T_2 [13] if distinct points of X can be separated by disjoint β-open sets.

LEMMA 1. *A space (X, τ) is gβ-regular if and only if (X, τ) is semi-pre regular and semi-pre $T_{1/2}$.*

Proof. Suppose that (X, τ) is gβ-regular. Then clearly (X, τ) is semi-pre regular. Now, let $A \subseteq X$ be gβ-closed. For each $x \notin A$, there exists a β-open set V_x containing x such that $V_x \cap A = \emptyset$. If $V = \cup\{V_x : x \notin A\}$, then V is a β-open set and $V = X \setminus A$. Hence A is β-closed. Converse is obvious.

REMARK 3. There exist a topological space which is semi-pre regular but not gβ-regular. Following is the example:

EXAMPLE 1. Let $X = \{a, b, c, d\}$ and let $T = \{\emptyset, X, \{a, b\}\}$ be the topology on X. Now (X, T) is semi-pre regular but not gβ-regular since $\{a, b, c\}$

is gβ-closed but for $d \notin \{a, b, c\}$, there does not exist any pair of disjoint β-open sets containing d and $\{a, b, c\}$.

Our next result characterizes semi-pre regular spaces. We shall call the subset $A \subseteq X$ as β-clopen in (X, τ) if A is both β-open and β-closed. Also $\beta\,\mathrm{Cl}(V)$ is always β-clopen for any β-open set V [9].

THEOREM 1. *For a topological space (X, τ), the following are equivalent:*

(i) *(X, τ) is gβ-regular.*
(ii) *Every gβ-open set U is a union of β-clopen sets.*
(iii) *Every gβ-closed set A is intersection of β-clopen sets.*

Proof. (i)⇒(ii). Let U be gβ-open and $x \in U$. If $A = X \setminus U$, then A is gβ-closed. By assumption, there exist disjoint β-open subsets W_1 and W_2 of X such that $x \in W_1$ and $A \subseteq W_2$. If $V = \beta\,\mathrm{Cl}(W_1)$, then V is β-clopen and $V \cap A \subseteq V \cap W_2 = \emptyset$. It follows that $x \in V \subseteq U$. Thus U is a union of β-clopen sets.

(ii)⇔(iii). Obvious.

(iii)⇒(i). Let A be gβ-closed and let $x \notin A$. By assumption, there exists a β-clopen set V such that $A \subseteq V$ and $x \notin V$. If $U = X \setminus V$, then U is a β-open set containing x and $U \cap V = \emptyset$. Thus (X, T) is gβ-regular.

Gβ-open sets give rise to various separation properties which are as follows:

DEFINITION 4. A topological space (X, τ) is said to be

(i) gβ-T_0 space if for each pair of distinct points of X, there exists a gβ-open set containing one point but not the other.
(ii) $(\beta, g\beta)$-R_0 space if $\beta\,\mathrm{Cl}(\{x\}) \subseteq U$ whenever U is gβ-open and $x \in U$.

REMARK 4. Let $X = \{a, b, c\}$ and let $T = \{\emptyset, X, \{a\}\}$ be the topology in X. Then the space is not gβ-regular. Also it is neither β-T_2 nor $(\beta, g\beta)$-R_0. Also note that every β-T_2 space is semi-pre $T_{1/2}$, and every semi-pre $T_{1/2}$ space is gβ-T_0.

THEOREM 2. *Every gβ-regular space is both β-T_2 and $(\beta, g\beta)$-R_0.*

Proof. Let (X, τ) be gβ-regular and let $x, y \in X$ such that $x \neq y$. By Lemma 1, $\{x\}$ is either β-open or closed. If $\{x\}$ is β-open, hence gβ-open, then $\{x\}$ is β-clopen by Theorem 1. Thus $\{x\}$ and $X \setminus \{x\}$ are separating β-open sets. If $\{x\}$ is closed, hence β-closed, then $X \setminus \{x\}$ is β-open and so, by Theorem 1, the union of β-clopen sets. Hence there is a β-clopen set $V \subseteq X \setminus \{x\}$ containing y. This proves that (X, τ) is β-T_2. By Theorem 1, it follows immediately that (X, τ) also has to be $(\beta, g\beta)$-R_0.

3. gβ-Normal spaces

DEFINITION 5. A topological space (X, τ) is said to be

(i) gβ-normal if for every pair of disjoint gβ-closed sets A and B of X, there exist disjoint β-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.
(ii) Strongly β-normal [9] if for each pair $A, B \subseteq X$ of disjoint β-closed sets, there exist disjoint β-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

REMARK 5. It is obvious that gβ-normal space is strongly β-normal. There exist a topological space which is strongly β-normal but not gβ-normal as can be seen in Example 1.

THEOREM 3. *For a topological space (X, τ), the following are equivalent:*

(i) (X, τ) is gβ-normal.
(ii) For every gβ-closed set A and every gβ-open set U containing A, there is a β-clopen set V such that $A \subseteq V \subseteq U$.

Proof. (i)⇒(ii). Let A be gβ-closed and U be gβ-open with $A \subseteq U$. Now, we have $A \cap (X \setminus U) = \emptyset$, hence there exist disjoint β-open sets W_1 and W_2 such that $A \subseteq W_1$ and $X \setminus U \subseteq W_2$. If $V = \beta\,\mathrm{Cl}(W_1)$, then V is a β-clopen set satisfying $A \subseteq V \subseteq U$.

(ii)⇒(i). Obvious.

DEFINITION 6. (i) A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra β-continuous [10] if the inverse image of every closed set in (Y, σ) is β-open in (X, τ).
(ii) A space (X, τ) is called weakly gβ-normal if disjoint gβ-closed set can be separated by disjoint closed sets.
(iii) A function $f : (X, \tau) \to (Y, \sigma)$ is said to be always gβ-closed if the image of each gβ-closed set in (X, τ) is gβ-closed in (Y, σ).

THEOREM 6. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective contra β-continuous always gβ-closed function and (Y, σ) is weakly gβ-normal, then (X, τ) is gβ-normal.*

Proof. Suppose that $A_1, A_2 \subseteq X$ are gβ-closed and disjoint. Since f is always gβ-closed and injective, $f(A_1), f(A_2) \subseteq Y$ are gβ-closed and disjoint. Since (Y, σ) is weakly gβ-normal, $f(A_1)$ and $f(A_2)$ can be separated by disjoint closed sets $B_1, B_2 \subseteq Y$. Morever as f is contra β-continuous, A_1 and A_2 can be separated by disjoint β-open sets $f^{-1}(B_1)$ and $f^{-1}(B_2)$. Thus (X, τ) is gβ-normal.

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