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ON THE INVOLUTE AND EVOLUTE CURVES OF THE TIMELIKE CURVE IN MINKOWSKI 3-SPACE

Abstract. In this study, we have generalized the involute and evolute curves of the timelike curve in Minkowski 3-Space. Firstly, we have shown that, the length between the timelike curve α and the spacelike curve β is constant. Furthermore, the Frenet-Serret frame of the involute curve β has been found as dependent on curvatures of the curve α. We have determined the involute curve β is planar in which conditions. Secondly, we have found transformation matrix between the evolute curve β and the curve α. Finally, we have computed the curvatures of the evolute curve β.

1. Preliminaries

Let $IR^3 = \{(x_1, x_2, x_3)|x_1, x_2, x_3 \in IR\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in IR^3. The Lorentz scalar product of x and y is defined by

$$\langle x, y\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3,$$

$IE_1^3 = \left(R^3, \langle x, y\rangle_L\right)$ is called 3-dimensional Lorentzian space, Minkowski 3-Space or 3-dimensional semi-euclidean space. The vector x in IE_1^3 is called a spacelike vector, null vector or a timelike vector if $\langle x, x\rangle_L > 0$ or $x = 0$, $\langle x, x\rangle_L = 0$ or $\langle x, x\rangle_L < 0$, respectively. For $x \in IE_1^3$, the norm of the vector x defined by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$, and x is called a unit vector if $\|x\|_L = 1$. For any x, $y \in IE_1^3$, Lorentzian vectoral product of x and y is defined by

$$x \wedge_L y = \left(x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1\right).$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. Then $T(s), N(s)$ and $B(s)$ are tangent, the principal normal and the binormal vector of the curve $\alpha(s)$, respectively. Depending on the causal character of the curve α, we have the following Frenet-Serret formulas:

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If α is a spacelike curve with a spacelike principal normal N;

$$T' = \kappa N,\ N = -\kappa T + \tau B,\ B' = \tau N, \tag{1.1}$$

$$\langle T,T\rangle_L = \langle N,N\rangle_L = 1, \langle B,B\rangle_L = -1, \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0.$$

If α is a spacelike curve with a timelike principal normal N;

$$T' = \kappa N,\ N = \kappa T + \tau B,\ B' = \tau N, \tag{1.2}$$

$$\langle T,T\rangle_L = \langle B,B\rangle_L = 1, \langle N,N\rangle_L = -1, \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0.$$

If α is a timelike curve and finally;

$$T' = \kappa N,\ N = \kappa T + \tau B,\ B' = -\tau N \tag{1.3}$$

$$\langle T,T\rangle_L = -1, \langle B,B\rangle_L = \langle N,N\rangle_L = 1, \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0,$$

[see 2]. If the timelike curve α is non-unit speed, then

$$\kappa(t) = \frac{\left\| \alpha'(t) \wedge_L \alpha''(t) \right\|_L}{\left\| \alpha'(t) \right\|_L^3},\ \ \tau(t) = \frac{\det\left(\alpha'(t), \alpha''(t), \alpha'''(t)\right)}{\left\| \alpha'(t) \wedge_L \alpha''(t) \right\|_L^2}. \tag{1.4}$$

If timelike curve α is unit speed, then

$$\kappa(s) = \left\| \alpha''(s) \right\|_L,\ \ \tau(s) = \left\| B'(s) \right\|_L. \tag{1.5}$$

2. The involute of the timelike curve

DEFINITION 2.1. Let timelike unit speed timelike curve $\alpha : I \longrightarrow E_1^3$ and the curve $\beta : I \longrightarrow E_1^3$ be given. For $\forall s \in I$, then the curve β is called the involute of the curve α, if the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\beta(s)$ to the curve β and

$$\langle T^*(s), T(s)\rangle_L = 0. \tag{2.1}$$

Let the Frenet-Serret frames of the curves α and β be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. In this case, the causal characteristics of the Frenet-Serret frames of the curves α and β must be of the form.

$$\{T \text{ timelike, } N \text{ spacelike, } B \text{ spacelike}\}$$

and

$$\{T^* \text{ spacelike, } N^* \text{ timelike, } B^* \text{ spacelike}\}.$$

THEOREM 2.1. *Let the curve β be involute of the curve α and let k be a constant real number. Then*

$$\beta(s) = \alpha(s) + (k - s)T(s). \tag{2.2}$$

Proof. The curve $\beta(s)$ may be given as

$$\beta(s) = \alpha(s) + u(s)T(s). \tag{2.3}$$

If we take the derivative Eq. (2.3), then we have

$$\beta'(s) = \left(1 + u'(s)\right) T(s) + u(s)\kappa(s)N(s).$$

Since the curve β is involute of the curve α, $\langle T^*(s), T(s)\rangle_L = 0$. Then, we get

$$1 + u'(s) = 0 \text{ or } u(s) = k - s. \tag{2.4}$$ ■

Thus we get

$$\beta(s) - \alpha(s) = (k - s)\,T(s). \tag{2.5}$$ ■

COROLLARY 2.2. *The distance between the curves β and α is $|k - s|$.*

Proof. If we take the norm in Eq. (2.5), then we get

$$\|\beta(s) - \alpha(s)\|_L = |k - s|\,. \tag{2.6}$$ ■

THEOREM 2.3. *Let the curve β be involute of the the curve α, then*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \left(\left|\kappa^2 - \tau^2\right|\right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -\kappa & 0 & -\tau \\ -\tau & 0 & -\kappa \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Proof. If we take the derivative Eq.(2.5), we can write

$$\beta'(s) = (k - s)\kappa(s)N(s)$$

and

$$\left\|\beta'(s)\right\|_L = |(k - s)\kappa(s)|\,.$$

Furthermore, we get

$$T^*(s) = \frac{\beta'(s)}{\|\beta'(s)\|_L} = \frac{(k - s)\kappa(s)}{|(k - s)\kappa(s)|}N(s).$$

From the last equation, we must have

$$T^*(s) = N(s) \text{ or } T^*(s) = -N(s)\,.$$

We assume that $T^*(s) = N(s)$. Let's denote the coordinate function on IR by x. Then, for $\forall s \in IR$, $x(s) = s$, we get

$$\beta'(s) = (k - s)\kappa(s)N(s),$$
$$\beta' = (k - x)\kappa N.$$

Thus, we have

$$\beta'' = -\kappa N + (k-x)\kappa' N + (k-x)\kappa(\kappa T + \tau B)$$
$$\beta'' = (k-x)\kappa^2 T + \left((k-x)\kappa' - \kappa\right) N + (k-x)\kappa\tau B.$$

Hence, we have

$$\beta' \wedge_L \beta'' = (k-x)^2\kappa^2\left(-\tau T - \kappa B\right)$$

and

$$\left\|\beta' \wedge_L \beta''\right\|_L = |k-x|^2\kappa^2\sqrt{|\kappa^2-\tau^2|}.$$

Furthermore, we get

$$B^* = \frac{\beta' \wedge_L \beta''}{\left\|\beta' \wedge_L \beta''\right\|} = \frac{(k-x)^2.\kappa^2.\left(-\tau T - \kappa B\right)}{(k-x)^2.\kappa^2.\sqrt{|\kappa^2-\tau^2|}} = \frac{-\tau T - \kappa B}{\sqrt{|\kappa^2-\tau^2|}}.$$

Since $N^* = B^* \wedge_L T^*$, then we obtain

$$N^* = \frac{-\kappa T - \tau B}{\sqrt{|\kappa^2-\tau^2|}}.$$

∎

THEOREM 2.4. *Let the curve β be involute of the the curve α. Let the curvature and torsion of the curve β be κ^* and τ^*, respectively. Then*

$$\kappa^*(s) = \frac{\sqrt{|(\kappa^2-\tau^2)(s)|}}{|k-s|.\kappa(s)}, \quad \tau^*(s) = \frac{\kappa(s)\tau'(s) - \kappa'(s)\tau(s)}{|k-s|.\kappa(s).\left(\tau^2-\kappa^2\right)}.$$

Proof. From Eq. (1.3) and Eq. (1.4), we have

$$\kappa^*(s) = \frac{|k-s|^2\kappa^2(s)}{|k-s|^3.\kappa^3(s)} = \frac{\sqrt{|\kappa^2(s)-\tau^2(s)|}}{\kappa(s).|k-s|}$$

and

$$\begin{aligned}
\beta''' &= \left[-\kappa^2 T + (k-x)2\kappa\kappa' T + (k-x)\kappa^2(\kappa N)\right] \\
&\quad + \left[-\kappa' - \kappa' + (k-x)\kappa''\right] N \\
&\quad + \left[-\kappa + (k-x)\kappa'\right)\right] (\kappa T + \tau B) \\
&\quad + \left[-\kappa\tau + (k-x)\kappa'\tau + (k-x)\kappa\tau'\right] B \\
&\quad + \left[(k-x)\kappa\tau\right](-\tau N) \\
&= \left(-2\kappa^2 + 3(k-x)\kappa\kappa'\right) T \\
&\quad + \left((k-x)\kappa^3 - 2\kappa' + (k-x)\kappa'' - (k-x)\kappa\tau^2\right) N \\
&\quad + \left(-2\kappa\tau + 2(k-x)\kappa'\tau + (k-x)\kappa\tau'\right) B.
\end{aligned}$$

Furthermore, since

$$\tau^*(s) = \frac{\det\left(\beta'(s), \beta''(s), \beta'''(s)\right)}{\left\|\beta'(s) \wedge_L \beta''(s)\right\|_L^2},$$

we have

$$\begin{aligned}
\Delta &= -(k-x)^2\kappa^2 \begin{bmatrix} \kappa & \tau \\ -2\kappa^2+3(k-x)\kappa\kappa' & -2\kappa\tau+2(k-x)\kappa'\tau+(k-x)\kappa\tau' \end{bmatrix} \\
&= -(k-x)^2\kappa^2\left[-2\kappa^2\tau+2(k-x)\kappa'\kappa\tau+(k-x)\kappa^2\tau'+2\kappa^2\tau-3(k-x)\kappa\kappa'\tau\right] \\
&= -(k-x)^2\kappa^3\left[(k-x)\kappa\tau'-(k-x)\kappa'\tau\right] \\
&= (k-x)^3\kappa^3\left(\kappa'\tau-\kappa\tau'\right),
\end{aligned}$$

$$\Delta = \det\left(\beta', \beta'', \beta'''\right).$$

Hence, we get

$$\tau^*(s) = \frac{\kappa^3.(k-s)^3\left(\kappa(s)\tau'(s) - \kappa'(s)\tau(s)\right)}{\kappa^4\left|k-s\right|^4(\tau^2(s) - \kappa^2(s))},$$

$$\tau^*(s) = \frac{\kappa(s)\tau'(s) - \kappa'(s)\tau(s)}{\kappa(s)\left|k-s\right|(\tau^2(s) - \kappa^2(s))}. \qquad \blacksquare$$

From the last equation, we have the following corollaries:

COROLLARY 2.5. *If the curve α is planar, then its involute curve β is also planar.*

COROLLARY 2.6. *If the curvature $\kappa \neq 0$ and the torsion $\tau \neq 0$ of the curve α are constant, then the involute curve β is planar, i.e., if the curve α is an ordinary helix, then its involute curve β is planar.*

COROLLARY 2.7. *If the curvature $\kappa \neq 0$ and the torsion $\tau \neq 0$ of the curve α are not constant but $\frac{\tau}{\kappa}$ is constant, then the involute curve β is planar, i.e. if the curve α is a general helix, then its the involute curve β is planar.*

THEOREM 2.8. *Suppose that the curve $\alpha : I \longrightarrow E_1^3$ with arc-length parameter are given. Then, the locus of the centre of the curvature of the curve α is the unique involute of the curve α which lies on the plane of the curve α.*

Proof. The locus of the centre of the curvature of the curve α is

$$C(s) = \alpha(s) - \frac{1}{\kappa(s)}N(s).$$

If we take the derivative in the above equation, then we have

$$\frac{dC}{ds} = T - \left(\frac{1}{\kappa}\right)' N - \frac{1}{\kappa}(\kappa T), \kappa \neq 0$$
$$= T - \left(\frac{1}{\kappa}\right)' N - \frac{1}{\kappa}\kappa T$$
$$C'(s) = -\left(\frac{1}{\kappa}\right)' N$$
$$\left\langle C', T\right\rangle_L = \left\langle -\left(\frac{1}{\kappa}\right)' N, T\right\rangle_L$$
$$\left\langle C', T\right\rangle_L = 0$$
$$\left\langle C'(s), T(s)\right\rangle_L = 0.$$

Therefore, the involute C of the timelike curve α is the locus of the centre of the curvature. Is the curve C planar ? If the torsion of the curve C is denoted by τ^*, then

$$\tau^*(s) = \frac{\left(\kappa'\tau - \kappa\tau'\right)(s)}{\kappa(s)\,|k - s|\,(\tau^2(s) - \kappa^2(s))}.$$

If we take $\tau = 0$, then we have

$$\tau^*(s) = 0$$

Thus, the curve C is planar. ■

3. The evolute of the timelike curve

DEFINITION 3.1. Let the unit speed curve α and the curve β with the same interval be given. For $\forall s \in I$, the tangent at the point $\beta(s)$ to the curve β passes through the point $\alpha(s)$ and

$$\langle T^*(s), T(s)\rangle_L = 0.$$

Then, β is called the evolute of the curve α. Let the Frenet-Serret frames of the curves α and β be

THEOREM 3.1. *Let the curve β be the evolute of the unit speed timelike curve α, then*

$$\beta(s) = \alpha(s) - \frac{1}{\kappa(s)}N(s) + \frac{1}{\kappa(s)}\left[\tan\left(\varphi(s) + c\right)\right] B(s), \tag{3.1}$$

where $c \in IR$ and $\varphi(s) + c = \int \tau(s)ds$. Furthermore, in the normal plane of *the point* $\alpha(s)$ *the measure of directed angle between* $\beta(s)-\alpha(s)$ *and* $N(s)$ *is* $\varphi(s)+c$.

Proof. The tangent of the curve β at the point $\beta(s)$ is the line constructed by the vector $T^*(s)$. Since this line passes through the point $\alpha(s)$, the vector $\beta(s)-\alpha(s)$ is perpendicular to the vector $T(s)$. Then

$$\beta(s)-\alpha(s)=\lambda N(s)+\mu B(s). \tag{3.2}$$

If we take the derivative of Eq. (3.2), then we have

$$\beta'(s)=\alpha'(s)+\lambda' N+\lambda(\kappa T+\tau B)+\mu' B(s)+\mu(-\tau N),$$

$$\beta'(s)=(1+\lambda\kappa)T+\left(\lambda'-\mu\tau\right)N+\left(\lambda\tau+\mu'\right)B. \tag{3.3}$$

According to the definition of the evolute, since $\langle T^*(s),T(s)\rangle=0$, from Eq. (3.3), we get

$$\lambda=-\frac{1}{\kappa}\ , \tag{3.4}$$

and

$$\beta'=\left(\lambda'-\mu\tau\right)N+\left(\lambda\tau+\mu'\right)B. \tag{3.5}$$

From the Eq. (3.2) and Eq. (3.5), the vector field $\beta'$is parallel to the vector field $\beta-\alpha$. Then we have

$$\frac{\lambda'-\mu\tau}{\lambda}=\frac{\lambda\tau+\mu'}{\mu}.$$

After that, we have

$$\tau=\frac{\lambda'\mu-\lambda\mu'}{\lambda^2+\mu^2},$$

$$\tau=-\frac{\left(\frac{\mu}{\lambda}\right)'}{1+\left(\frac{\mu}{\lambda}\right)^2}.$$

If we take the integral the last equation, we get

$$\varphi(s)+c=-\arctan\left(\frac{\mu(s)}{\lambda(s)}\right).$$

Hence, we find

$$\mu(s)=-\lambda(s)\tan\left(\varphi(s)+c\right). \tag{3.6}$$

If we substitute Eq. (3.4) and Eq. (3.6) into Eq. (3.2), we have

$$\beta(s) = \alpha(s) - \frac{1}{\kappa(s)}N(s) + \frac{1}{\kappa(s)}\left[\tan\left(\varphi(s) + c\right)\right]B(s),$$

$$\beta(s) = M(s) + \frac{1}{\kappa(s)}\tan\left[\varphi(s) + c\right]B(s).$$

Then, we obtain an evolute curve for each $c \in IR$. Since

$$\left\langle \overrightarrow{M(s)\beta(s)}, \overrightarrow{M(s)\ \alpha(s)} \right\rangle_L = 0,$$

in the Lorentzian triangle which have corners $\beta(s)$, $M(s)$ and $\alpha(s)$ the angle M is right angle in the Lorentzian mean. In the same triangle, the tangent of the angle $\alpha(s)$ is

$$\frac{\frac{1}{\kappa(s)}\tan\left[\varphi(s) + c\right]}{\frac{1}{\kappa(s)}} = \tan\left[\varphi(s) + c\right]. \tag{3.7}$$

Then, the measure of the angle between the vectors $\beta(s) - \alpha(s)$ and $V_2(s)$ is $\varphi(s) + c$. ∎

THEOREM 3.2. *Let the spacelike curve $\beta : I \longrightarrow E_1^3$ be evolute of the unit speed time curve $\alpha : I \longrightarrow E_1^3$. If the Frenet-Serret vector fields of the curve β are T^* (spacelike), N^* (timelike), B^* (spacelike), then*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} 0 & \cos(\varphi + c)quad{-}\sin(\varphi + c) \\ -1 & 0 \quad quad \quad 0 \\ 0 & \sin(\varphi + c)quad\ \cos(\varphi + c) \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{3.8}$$

Proof. Since the Frenet-Serret vector fields of the curve β are T^*, N^*, B^* and

$$\beta = \alpha - \rho N + \rho \tan\left(\varphi + c\right)B,$$

we have

$$\begin{aligned} \beta'(s) &= \alpha' - \rho' N - \rho\left(\kappa T + \tau B\right) \\ &\quad + \left[\rho' \tan\left(\varphi + c\right)B + \rho\varphi' \sec^2\left(\varphi + c\right)B + \rho\tan\left(\varphi + c\right)\left(-\tau N\right)\right] \\ &= \left(1 - \rho\kappa\right)T + \left(-\rho' - \rho\tau\tan\left(\varphi + c\right)\right)N \\ &\quad + \left[\left(-\rho\tau + \rho\varphi'\right) + \rho'\tan\left(\varphi + c\right) + \rho\varphi'\tan^2\left(\varphi + c\right)\right]B \\ &= \left(-\rho' - \rho\tau\tan\left[\varphi + c\right]\right)N + \left(\rho'\tan\left(\varphi + c\right) + \rho\tau\tan^2\left(\varphi + c\right)\right)B \\ &= \left[-\rho' - \rho\tau\tan(\varphi + c)\right]\left[N - \tan\left(\varphi + c\right)B\right] \end{aligned}$$

(3.9) $$\beta'(s) = \left[\frac{-\rho' - \rho\tau\tan(\varphi+c)}{\cos(\varphi+c)}\right]\left[\cos(\varphi+c)N - \sin(\varphi+c)B\right].$$

If we take the norm in the Eq. (3.9), then we obtain

$$\left\|\beta'(s)\right\|_L = \frac{\left|\rho' + \rho\tau\tan(\varphi+c)\right|}{\left|\cos(\varphi+c)\right|}.$$

Since $T^* = \frac{\beta'}{\|\beta'\|_L}$, then we get

(3.10) $$T^* = \cos(\varphi+c)N - \sin(\varphi+c)B.$$

Therefore, we have obtained Eq. (3.9). The curve β is not a unit speed curve. If we take the derivative of Eq. (3.10) with respect to s, we find

$$\begin{aligned}(T^*)' &= \left(\tau - \varphi'\right)\left[B\cos(\varphi+c) + N\sin(\varphi+c)\right] + \kappa T\cos(\varphi+c)\\ &= \kappa T\cos(\varphi+c).\end{aligned}$$

Since $T' = \left\|\alpha'\right\|_L \kappa N$ we have

$$(T^*)' = \left\|\beta'\right\|_L \kappa^* N^*.$$

Thus

$$\left\|\beta'\right\|_L \kappa^* N^* = \kappa\cos(\varphi+c)T.$$

Since the vectors N^* and T have the unit length, we get $N^* = -T$ or $N^* = T$. Since $B^* = N^* \wedge_L (-T^*)$, we have

(3.11) $$B^* = \sin(\varphi+c)N + \cos(\varphi+c)B.$$

Thus, the proof is completed. ■

THEOREM 3.3. *Let $\beta : I \longrightarrow E_1^3$ be the evolute of the unit speed curve $\alpha : I \longrightarrow E_1^3$. Let the Frenet vector fields, curvature and torsion of the curve β be T^*, N^*, B^*, κ^* and τ^*, respectively. Then*

$$|\kappa^*| = \frac{\kappa^3\left|\cos^3(\varphi+c)\right|}{\left|\kappa\tau\sin(\varphi+c) - \kappa'\cos(\varphi+c)\right|},$$

$$|\tau^*| = \frac{\kappa^3\left|\sin(\varphi+c)\right|\cos^2(\varphi+c)}{\left|\kappa\tau\sin(\varphi+c) - \kappa'\cos(\varphi+c)\right|}.$$

Proof. Since N^* and T have unit length, then taking norm equility $\left\|\beta'\right\|_L \kappa^* N^* = \kappa\cos(\varphi+c)T$, we can write

$$\left\|\beta'\right\|_L |\kappa^*| = \kappa\left|\cos(\varphi+c)\right|.$$

Therefore, we have

$$|\kappa^*| = \frac{\kappa\,|\cos(\varphi + c)|}{\|\beta'\|_L}, \tag{3.12}$$

$$|\kappa^*| = \kappa\,|\cos(\varphi + c)| : \frac{\left|\rho' + \rho\tau\tan(\varphi + c)\right|}{|\cos(\varphi + c)|},$$

$$|\kappa^*| = \frac{\kappa^3\left|\cos^3(\varphi + c)\right|}{|\kappa\tau\sin(\varphi + c) - \kappa'\cos(\varphi + c)|}.$$

If we take the derivative Eq. (3.11) with respect to s, then we have

$$\begin{aligned}(B^*)' &= \left(\varphi' - \tau\right)[N\cos(\varphi + c) - B\sin(\varphi + c)] + \kappa T\sin(\varphi + c)\\ &= \kappa T\sin(\varphi + c).\end{aligned}$$

Since $(B^*)' = \left\|\beta'\right\|_L \tau^* N^*$, we get

$$\left\|\beta'\right\|_L \tau^* N^* = \kappa T\sin(\varphi + c).$$

Since $N^* = -T$, we find that

$$|\tau^*| = \frac{\kappa\,|\sin(\varphi + c)|}{\|\beta'\|}, \tag{3.13}$$

$$|\tau^*| = \kappa\,|\sin(\varphi + c)| : \frac{\left|\rho' + \rho\tau\tan(\varphi + c)\right|}{|\cos(\varphi + c)|},$$

$$|\tau^*| = \frac{\kappa^3\,|\sin(\varphi + c)|\cos^2(\varphi + c)}{|\kappa\tau\sin(\varphi + c) - \kappa'\cos(\varphi + c)|}. \qquad \blacksquare$$

THEOREM 3.4. *Let $\beta : I \longrightarrow E_1^3$ be the evolute of the unit speed curve $\alpha : I \longrightarrow E_1^3$. Let the curvature and torsion of the curve β be κ^* and τ^*, respectively. Then*

$$\left|\frac{\tau^*}{\kappa^*}\right| = |\tan(\varphi + c)|. \tag{3.14}$$

Furthermore, we denote by $\beta^{(1)}$ and $\beta^{(2)}$, the evolute curves obtained by using c_1 and c_2 instead of c, respectively. The tangents of the curves $\beta^{(1)}$ and $\beta^{(2)}$ at the points $\beta^{(1)}(s)$ and $\beta^{(2)}(s)$ intersect at the point $\alpha(s)$. The measure of the angle between the tangents is $c_1 - c_2$.

Proof. The Eq. (3.14) is obtained easily by using Eq. (3.12) and Eq. (3.13), i.e.,

$$\left|\frac{\tau^*}{\kappa^*}\right| = \frac{\kappa\,|\sin(\varphi + c)|}{\|\beta'\|_L} : \frac{\kappa\,|\cos(\varphi + c)|}{\|\beta'\|_L} = |\tan(\varphi + c)|.$$

The measure of the angle between the vectors $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ and $V_2(s)$, and between the vectors $\overrightarrow{\alpha(s)\beta^{(2)}(s)}$ and $N(s)$ are $\varphi(s)+c_1$ and $\varphi(s)+c_2$, respectively. The vector $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ is parallel to the tangent of the curve $\beta^{(1)}$ at the point $\beta^{(1)}(s)$. The vector $\overrightarrow{\alpha(s)\beta^{(2)}(s)}$ is parallel to the tangent of the curve $\beta^{(2)}$ at the point $\beta^{(2)}(s)$. Furthermore, since $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$, $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ and $\overrightarrow{N}$ are perpendicular to the vector $T(s)$, these three vectors are planar. Then, the measure of the angle between the tangents of the curves $\beta^{(1)}$ and $\beta^{(2)}$ at the points $\beta^{(1)}(s)$ and $\beta^{(2)}(s)$ is

$$\varphi(s) + c_1 - (\varphi(s) + c_2) = c_1 - c_2.$$

So, the proof is completed. ■

THEOREM 3.5. *Suppose that, two different evolutes of the timelike curve α are given. Let the points on the evolutes of the curve α corresponding to the point P be P_1 and P_2. Then the angle $\widehat{P_1 P P_2}$ is constant.*

Proof. Let the evolutes of the curve α be β and γ. Let the arc-length parameters of the α, β and γ be s, $s^*$and $\widehat{s}$, respectively. Let the curvatures of the curves α, β and γ be k, k^* and $\widehat{k}$, respectively. And let the Frenet vectors of the curves α, β and γ be $\{T, N, B\}, \{T^*, N^*, B^*\}$ and $\left\{\widehat{T}, \widehat{N}, \widehat{B}\right\}$. Then

$$T = N^*, T = \widehat{N}. \tag{3.15}$$

Since the curves β and γ are evolute, then

$$\langle T, T^*\rangle_L = \left\langle T, \widehat{T}\right\rangle_L = 0. \tag{3.16}$$

Therefore, if $f(s) = \left\langle T^*, \widehat{T}\right\rangle_L$, then we have

$$\begin{aligned}
(f)'(s) &= \left\langle (T^*)', \widehat{T}\right\rangle_L + \left\langle T^*, \left(\widehat{T}\right)'\right\rangle_L \\
&= \left\langle \kappa^* N^* \frac{ds^*}{ds}, \widehat{T}\right\rangle_L + \left\langle T^*, \widehat{\kappa}\widehat{N}\frac{d\widehat{s}}{ds}\right\rangle_L \\
&= \kappa^*\frac{ds^*}{ds}\left\langle N^*, \widehat{T}\right\rangle_L + \widehat{\kappa}\frac{d\widehat{s}}{ds}\left\langle T^*, \widehat{N}\right\rangle_L \\
&= \kappa^*\frac{ds^*}{ds}\left\langle T, \widehat{T}\right\rangle_L + \widehat{\kappa}\frac{d\widehat{s}}{ds}\langle T^*, N^*\rangle_L \\
&= \kappa^*\frac{ds^*}{ds}.0 + \widehat{\kappa}\frac{d\widehat{s}}{ds}.0 \\
(f)'(s) &= 0.
\end{aligned}$$

Therefore, we have $f(s)=\theta=$constant. Hence, $m\left(\widehat{P_1PP_2}\right)=m\left(T^*,\widehat{T}\right)=\theta=$constant. ∎

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