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LIGHTLIKE SUBMANIFOLDS OF INDEFINITE QUATERNION KAEHLER MANIFOLDS

Abstract. We define and study both screen QR-lightlike and screen CR-lightlike submanifolds of an indefinite quaternion Kaehler manifold. We show that screen QR-lightlike submanifolds include quaternion submanifolds and screen CR-lightlike submanifolds include quaternion as well as screen real lightlike submanifolds. We also give some examples of screen QR-lightlike and screen CR-lightlike submanifolds.

1. Introduction

QR-submanifolds of a quaternion Kaehler manifold were introduced by A. Bejancu in [1]. Since then many papers appeared on these submanifolds.

On the other hand, the geometry of lightlike submanifolds has shown an increasing development since K.L.Duggal- A.Bejancu defined lightlike submanifolds in [4] and [3]. In their book [5], they introduced CR-lightlike submanifolds and showed that CR-lightlike submanifolds are always proper,i.e., they don't contain invariant and totally real lightlike submanifolds. Therefore, in [7], K.L.Duggal and the present author introduced screen CR-lightlike submanifold which contains invariant and screen real lightlike submanifolds. Lightlike real hypersurfaces of an indefinite quaternion Kaehler manifold studied in [9]. In [10], we defined QR-lightlike submanifold as a generalization of lightlike real hypersurfaces and showed that QR-lightlike submanifolds are also always proper.

In the present paper, we introduce screen QR-lightlike and screen CR-lightlike submanifolds and investigate fundamental properties of such lightlike submanifolds of indefinite quaternion Kaehler manifolds. We study the integrability conditions for the distributions which are involved in the definition of these submanifolds. We also study totally umbilical screen

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QR-lightlike and screen CR-lightlike submanifolds. Finally we show that there exist no proper totally umbilical screen CR-lightlike submanifolds in positively or negatively curved indefinite quaternion Kaehler manifolds and give several examples.

2. Preliminaries

Let $\bar{M}$ be a $4m$−dimensional manifold $(m > 1)$ and g be semi-Riemann metric on $\bar{M}$. Then $\bar{M}$ is called an indefinite quaternion Kaehler manifold (or, semi-Riemann quaternion) if there exists a 3-dimensional vector bundle of tensors of type (1,1) with local basis Hermitian structures $\bar{J}_1$,$\bar{J}_2$ and $\bar{J}_3$ (that is, $g(\bar{J}_a X, \bar{J}_a Y) = g(X,Y)$, $a = 1,2,3$ and $X, Y \in \Gamma(T\bar{M})$) satisfying

$$\bar{J}_1 o \bar{J}_2 = -\bar{J}_2 o \bar{J}_1 = \bar{J}_3 \tag{2.1}$$

and

$$\begin{aligned} \bar{\nabla}_X \bar{J}_1 &= r(X)\bar{J}_2 - q(X)\bar{J}_3 \\ \bar{\nabla}_X \bar{J}_2 &= -r(X)\bar{J}_1 + p(X)\bar{J}_3 \\ \bar{\nabla}_X \bar{J}_3 &= q(X)\bar{J}_1 - p(X)\bar{J}_2 \end{aligned} \tag{2.2}$$

for all vector fields X tangent to $\bar{M}$, where p, q, r are local sections of $\wedge^1(TM)$. and $\bar{\nabla}$ is Levi-Civita connection (see [8]). For sake of shortness, instead of (2.2) we use

$$\bar{\nabla}_X \bar{J}_a = \sum_{b=1}^{3} Q_{ab}(X)\bar{J}_b, a = 1,2,3 \tag{2.3}$$

where Q_{ab} are certain 1−forms locally defined on $\bar{M}$ such that $Q_{ab}+Q_{ba} = 0$.

An indefinite quaternionic space form is a connected indefinite quaternion Kaehler manifold of constant quaternionic sectional curvature and its denoted by $\bar{M}(c)$. The curvature tensor of $\bar{M}(c)$ is given by ([8])

$$\begin{aligned} \bar{R}(X,Y)Z = \frac{c}{4}\Big\{ & g(Z,Y)X - g(X,Z)Y \\ & + \sum_{a=1}^{3} g(Z,\bar{J}_a Y)\bar{J}_a X - g(Z,\bar{J}_a X)\bar{J}_a Y + 2g(X,\bar{J}_a Y)\bar{J}_a Z \Big\} \end{aligned} \tag{2.4}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$.

From now on, we follow [5] for the notation and formulas used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a *lightlike submanifold* if it is a lightlike manifold w.r.t. the metric g induced from $\bar{g}$ and the radical distribution $Rad(TM)$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, i.e., $TM = Rad\,(TM) \perp S(TM)$.

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$. It follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$ [5, page 144]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then,

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$
$$T\bar{M}|_M = S(TM) \perp [Rad(TM) \oplus ltr(TM)] \perp S(TM^{\perp}).$$

Following are four sub cases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$.

Case 1: r-lightlike if $r < min\{m, n\}$.
Case 2: Co-isotropic if $r = n < m$; $S(TM^{\perp}) = \{0\}$.
Case 3: Isotropic if $r = m < n$; $S(TM) = \{0\}$.
Case 4: Totally lightlike if $r = m = n$; $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM), \quad V \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(ltr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $ltr(TM)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X, N), \tag{2.8}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X, W), \tag{2.9}$$

$\forall X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Denote the projection of TM on $S(TM)$ by $\bar{P}$. Then, by using (2.5), (2.7)-(2.9) and taking account that $\bar{\nabla}$ is a metric connection, we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.10}$$

$$\bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X). \tag{2.11}$$

We set

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.13}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. By using above equations we obtain

$$\bar{g}(h^l(X, PY), \xi) = g(A^*_\xi X, PY), \tag{2.14}$$

$$\bar{g}(h^*(X, PY), N) = g(A_N X, PY), \tag{2.15}$$

$$\bar{g}(h^l(X, \xi), \xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.16}$$

In general, the induced connection ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.7) we get

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y). \tag{2.17}$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$. We denote curvature tensors of $\bar{M}$ and M by $\bar{R}$ and R respectively. The Gauss equation for M, $\forall X, Y, Z \in \Gamma(TM)$, is given by

$$\begin{aligned}\bar{R}(X, Y)Z = {} & R(X, Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) \\ & + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z) \\ & - (\nabla_Y h^s)(X, Z) + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)).\end{aligned} \tag{2.18}$$

3. Screen QR-lightlike submanifolds

In this section, we introduce a new class, called screen quaternion-real (**SQR**) lightlike submanifolds of an indefinite quaternion Kaehler manifold and investigate the geometry of such submanifolds.

DEFINITION 1. Let $(M, g, S(TM))$be a lightlike submanifold of an indefinite quaternion Kaehler manifold $(\bar{M}, \bar{g})$. We say that M is a SQR-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

i): There exist real non-null vector subbundles L and $L^\perp$ of $S(TM^\perp)$ such that

$$S(TM^\perp) = L \perp L^\perp, \ \bar{J}_a(L) \subset S(TM), \ \bar{J}_a(L^\perp) = L^\perp. \tag{3.1}$$

ii): $RadTM$ is invariant with respect to $\bar{J}_a$,i.e.,$\bar{J}_a(Rad(TM)) = Rad(TM)$, $a = 1, 2, 3$.

It follows that $ltr(TM)$ is also invariant with respect to $\bar{J}_a$, $a = 1, 2, 3$, that is

$$\bar{J}_a(ltr(TM)) = ltr(TM), \tag{3.2}$$

Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold. Put $D'_{ap} = \bar{J}_a(L_p)$ and $dimL_p = s, p \in M$. Then D'_{1p}, D'_{2p} and D'_{3p} are mutually orthogonal vector bundle of M. We consider $D'_p = D'_{1p} \oplus D'_{2p} \oplus D'_{3p}$. Then we obtain $3s$ dimensional distribution globally defined on M. Also we have

$$\bar{J}_a(D'_{ap}) = L,\ \bar{J}(D'_{bp}) = D'_{cp} \tag{3.3}$$

for each $p \in M\, a = 1, 2, 3$, where (a, b, c) is cyclic permutation of $(1, 2, 3)$. We consider

$$D = RadTM \perp D_0 \tag{3.4}$$

which is orthogonal complementary to D' in TM. It is easy to check that D_0 is an invariant non-degenerate distribution. Then, we obtain that D is also invariant with respect to $\bar{J}_a$. We call D and D' the quaternion and anti-quaternion distribution, respectively. Thus, we have

$$TM = D \oplus D' \tag{3.5}$$

and

$$tr(TM) = ltr(TM) \perp L \perp L^{\perp}. \tag{3.6}$$

We say that M is a proper screen QR-lightlike submanifold of $\bar{M}$ if $D_0 \neq \{0\}$ and $D' \neq \{0\}$.

Note the following special features:

1. Condition ii) implies that $dim(RadTM) = 4r \geq 4$.
2. For proper M, $dim(D_0) \geq 4m$ and $dim(D') \geq 3$.
3. There exist no screen QR-lightlike hypersurface.

Let R^{4m}_{4q}, $(m > 1, q > 1)$ be a semi-Euclidean space. Then, the canonical complex structures $\bar{J}_1, \bar{J}_2, \bar{J}_3$ of R^{4m}_{4q} and the Hermitian metric $\bar{g}$ are given by

$$\begin{aligned}
&\bar{J}_1(x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m)\\
&\qquad\qquad = (-y_1, x_1, -w_1, z_1, \ldots, -y_m, x_m, -w_m, z_m)\\
&\bar{J}_2(x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m)\\
&\qquad\qquad = (-z_1, w_1, x_1, -y_1, \ldots, -z_m, w_m, x_m, -y_m)\\
&\bar{J}_3(x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m)\\
&\qquad\qquad = (-w_1, -z_1, y_1, x_1, \ldots, -w_m, -z_m, y_m, x_m)
\end{aligned}$$

and

$$\begin{aligned}
&\bar{g}((x_1, y_1, z_1, w_1, \ldots, x_m, y_m, z_m, w_m), (u_1, v_1, t_1, s_1, \ldots, u_m, v_m, t_m, s_m))\\
&\quad = -\sum_{i=1}^{q}(x_iu_i + y_iv_i + z_it_i + w_is_i) + \sum_{a=q+1}^{m}(x_au_a + y_av_a + z_at_a + w_as_a).
\end{aligned}$$

EXAMPLE 1. Consider in R_4^{12} the submanifold M given by the equations:

$$x_1 = x_{11},\ x_2 = x_{12},\ x_9 = -x_3,\ x_{10} = -x_4, x_8 = constant.$$

Then the tangent bundle TM is spanned by

$$Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{11}}, \quad Z_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_{12}}, \quad Z_3 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_9},$$
$$Z_4 = \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_{13}}, \quad Z_5 = \frac{\partial}{\partial x_5}, \quad Z_6 = \frac{\partial}{\partial x_6}, \quad Z_7 = \frac{\partial}{\partial x_7}.$$

Hence M is a 4-lightlike submanifold with $RadTM = Span\{Z_1, Z_2, Z_3, Z_4\}$ and $RadTM$ is invariant with respect to canonical almost complex structures of $\bar{J}_a$ of R_4^{12}.We consider the vector field $W = \frac{\partial}{\partial x_8}$ of $S(TM^{\perp})$. Then we can obtain that $\bar{J}_1 W = -Z_7, \bar{J}_2 W = -Z_6, \bar{J}_3 W = -Z_5$. Hence D' is spanned by $\{Z_5,\ Z_6,\ Z_7\}$. We also obtain that lightlike transversal bundle spanned by

$$N_1 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{11}}\right\}, \quad N_2 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_{12}}\right\},$$
$$N_3 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_9}\right\}, \quad N_4 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_{10}}\right\},$$

which is invariant with respect to $\bar{J}_a, a = 1, 2, 3$. Thus M is a screen QR-lightlike submanifold of R_{12}^4, with $D = RadTM = \{Z_1, Z_2, Z_3, Z_4\}$,$D' = span\{Z_5, Z_6, Z_7.\}$

PROPOSITION 3.1. *A screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold is a quaternion lightlike submanifold if and only if* $D' = \{0\}$.

Proof. Let M be a quaternion lightlike submanifold of an indefinite quaternion Kaehler manifold. Then we can easily check that $RadTM$ is invariant with respect to $\bar{J}_a$. Therefore $ltr(TM)$ is also invariant with respect to $\bar{J}_a$. Hence $\bar{J}_a(S(TM^{\perp})) = S(TM^{\perp})$, thus $L = \{0\}$. Conversely, let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold$\bar{M}$. such that $D' = \{0\}$. Then $\bar{J}_a(TM) = TM$. Hence M is a quaternion lightlike submanifold.

For co-isotropic, isotropic and totally lightlike submanifolds we have the following:

PROPOSITION 3.2. *Any screen QR-coisotropic or isotropic or totally lightlike submanifolds of* $\bar{M}$ *is a quaternion lightlike submanifold.*

Proof. Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. If M is coisotropic then $S(TM^{\perp}) = \{0\}$ implies

$L = \{0\}$.Thus we have $TM = D$. Hence M is a quaternion lightlike submanifold. Similarly the assertions for isotropic, totally lightlike M can be proved.

REMARK 1. In [7], Duggal and the present author introduced screen real submanifold of an indefinite Kaehler manifold as follows: Let M be a lightlike submanifold of an indefinite Kaehler manifold, then M is called screen real if $\bar{J}(S(TM)) \subset S(TM^{\perp})$ and $\bar{J}(Rad(TM)) = RadTM$, where $\bar{J}$ is the almost complex structure of an indefinite Kaehler manifold. According to this definition and the definition of screen QR-lightlike submanifold, one can conclude that a screen real lightlike submanifold of an indefinite quaternion Kaehler manifold is not a screen QR-lightlike submanifold due to $\bar{J}_a(D'_b) = D'_c \subset TM$.

EXAMPLE 2. Consider in R^{12}_4 a submanifold given by the equations:

$$x_9 = x_1 \cos\alpha - x_3 \sin\alpha, \quad x_{10} = x_2 \cos\alpha - x_4 \sin\alpha,$$
$$x_{11} = x_1 \sin\alpha + x_3 \cos\alpha, \quad x_{12} = x_2 \sin\alpha + x_4 \cos\alpha, \alpha \in (0, \tfrac{\pi}{2}).$$

Then TM is spanned by

$$Z_1 = \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_9} + \sin\alpha \frac{\partial}{\partial x_{11}},$$
$$Z_2 = \frac{\partial}{\partial x_2} + \cos\alpha \frac{\partial}{\partial x_{10}} + \sin\alpha \frac{\partial}{\partial x_{12}},$$
$$Z_3 = \frac{\partial}{\partial x_3} - \sin\alpha \frac{\partial}{\partial x_9} + \cos\alpha \frac{\partial}{\partial x_{11}},$$
$$Z_4 = \frac{\partial}{\partial x_4} - \sin\alpha \frac{\partial}{\partial x_{10}} + \cos\alpha \frac{\partial}{\partial x_{12}},$$
$$Z_5 = \frac{\partial}{\partial x_5}, \qquad Z_6 = \frac{\partial}{\partial x_6},$$
$$Z_7 = \frac{\partial}{\partial x_7}, \qquad Z_8 = \frac{\partial}{\partial x_8}.$$

Hence M is a lightlike submanifold with $RadTM = span\{Z_1, Z_2, Z_3, Z_4\}$. Hence $RadTM = TM^{\perp} \subset TM$,i.e., M is a coisotropic 8-dimensional submanifold of R^{12}_4. Then $S(TM^{\perp}) = \{0\}$ and lightlike transversal bundle is spanned by

$$N_1 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_9} + \sin\alpha \frac{\partial}{\partial x_{11}}\right\},$$
$$N_2 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_2} + \cos\alpha \frac{\partial}{\partial x_{10}} + \sin\alpha \frac{\partial}{\partial x_{12}}\right\},$$

$$N_3 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_4} - \sin\alpha\,\frac{\partial}{\partial x_9} + \cos\alpha\,\frac{\partial}{\partial x_{11}}\right\},$$

$$N_4 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_4} - \sin\alpha\,\frac{\partial}{\partial x_{10}} + \cos\alpha\,\frac{\partial}{\partial x_{12}}\right\}.$$

It is easy to see that $RadTM$ and $D_0 = span\{Z_5, Z_6, Z_7, Z_8\}$ are invariant. Hence $TM = RadTM \perp D_0$ is invariant. Thus M is a quaternion lightlike submanifold.

Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold. We denote the projection morphism of TM to the quaternion distribution D by S and choose a local field of orthonormal frames $\{v_1, \ldots, v_s\}$ on the vector bundle L in $S(TM^{\perp})$. Then we have the local orthonormal frames

$$\{E_{11}, \ldots, E_{1s}, E_{21}, \ldots, E_{2s}, E_{31}, \ldots, E_{3s}\} \tag{3.7}$$

where $E_{11} = \bar{J}_1(v_1)$. Thus any vector field Y tangent to M can be written locally as

$$Y = SY + \sum_{b=1}^{3} \omega_{bi}(Y)\, E_{bi} \tag{3.8}$$

where $\omega_{bi}(Y) = g(Y, E_{bi})$. Thus applying $\bar{J}_a$ to (3.8) we obtain

$$\bar{J}_a Y = \bar{J}_a SY + \sum_{b=1}^{3} \omega_{bi}(Y)\, E_{ci} - \omega_{ci}(Y)\, E_{bi} - \omega_{ai}(Y)\, v_i. \tag{3.9}$$

For any vector field $V \in \Gamma(S(TM^{\perp}))$ we put

$$\bar{J}_a V = B_a V + C_a V,\ a = 1, 2, 3 \tag{3.10}$$

where $B_a V \in \Gamma(D')$ and $C_a V \in \Gamma(L^{\perp})$.

Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. Taking account of the definition of screen QR-lightlike submanifold and using (2.3), (2.7), (3.9) and (3.10) we have

$$\begin{aligned} \nabla_X \bar{J}_a Y = {} & Q_{ab}(X)\bar{J}_b Y + Q_{ac}(X)\bar{J}_c Y + \bar{J}_a S\nabla_X Y - \omega_{bi}\nabla_X Y)E_{ci} \\ & + \omega_{ci}(\nabla_X Y)E_{bi} + B_a h^s(X, Y), \end{aligned} \tag{3.11}$$

$$h^l(X, \bar{J}_a Y) = \bar{J}_a h^l(X, Y), \tag{3.12}$$

$$h^s(X, \bar{J}_a Y) = \omega_{ai}(\nabla_X Y)v_i + C_a h^s(X, Y) \tag{3.13}$$

for any $X, Y \in \Gamma(D)$.

THEOREM 3.1. *Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold. Then the following conditions are equivalent* *with each other:*

1. $h^s(X, \bar{J}_a Y) = h^s(\bar{J}_a X, Y), a \in \{1,2,3\}, X, Y \in \Gamma(D)$
2. $h^s(X, \bar{J}_a Y) = 0$
3. D is integrable.

Proof. **(1) ⇒ (2):** Since $\bar{J}_a = \bar{J}_b o \bar{J}_c$, we have

$$\begin{aligned} h^s(X, \bar{J}_a Y) = h^s(X, (\bar{J}_b o \bar{J}_c) Y) &= h^s(X, \bar{J}_b(\bar{J}_c Y)) \\ &= h^s(\bar{J}_b X, \bar{J}_c Y) \\ &= h^s(\bar{J}_c o \bar{J}_b X, Y) \\ &= -h^s(\bar{J}_a X, Y), \end{aligned}$$

hence we obtain $h^s(X, \bar{J}_a Y) = 0$.
(2) ⇒ (3): Let us suppose $h^s(X, \bar{J}_a Y) = 0$. Then from (3.13), we obtain $\omega_{ai}(\nabla_X Y) v_i = 0$, hence $\omega_{ai}([X,Y]) = 0$,i.e,$[X,Y] \in \Gamma(D)$.
(3)⇒(1): If D is integrable, from (3.13) we have $\omega_{ai}([X,Y]) = h^s(X, \bar{J}_a Y) - h^s(\bar{J}_a X, Y) = 0$, hence $h^s(X, \bar{J}_a Y) = h^s(\bar{J}_a X, Y)$.

LEMMA 3.1. *Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold. Then we have*

$$\bar{g}(h^s(X, E_{ai}), v_j) = g(A_{v_i} X, E_{aj}) \tag{3.14}$$

and

$$\begin{aligned} g(A_{v_j} E_{ai}, X) &= g(A_{v_i} E_{aj}, X) \\ &\quad - g(D^l(E_{aj}, v_i), X) + g(D^l(E_{ai}, v_j), X) \end{aligned} \tag{3.15}$$

for $X \in \Gamma(D)$ and $E_{ai} \in \Gamma(D')$.

Proof. From (2.7), (2.1) and (2.3), we have

$$\bar{g}(h^s(E_{ai}, X), v_j) = -\bar{g}(\bar{\nabla}_X v_i, \bar{J}_a v_j).$$

By using (2.9) we derive

$$\bar{g}(h^s(E_{ai}, X), v_j) = g(A_{v_i} X, E_{aj}).$$

On the other hand from (3.14) and (2.10) we obtain

$$g(A_{v_j} E_{ai}, X) = g(A_{v_i} X, E_{aj}) + g(X, D^l(E_{ai}, v_j)).$$

Using again (3.14) we have

$$g(A_{v_j} E_{ai}, X) = \bar{g}(h^s(X, E_{aj}), v_i) + g(X, D^l(E_{ai}, v_j)).$$

Since h^s is symmetric, we derive

$$g(A_{v_j} E_{ai}, X) = \bar{g}(h^s(X, E_{aj}), v_i) + g(X, D^l(E_{ai}, v_j)).$$

Thus taking account (2.10) in this equation, we get (3.15).

THEOREM 3.2. *Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. Then the distribution D' is integrable if and only if*

$$\bar{g}(D^s(E_{ai}, \bar{J}_a N), v_j) = \bar{g}(D^s(E_{aj}, \bar{J}_a N), v_i)$$

$$B_{aj}(X) = 0$$

and

$$D_{aj}(N) = 0$$

for $X \in \Gamma(D_0)$, where $B_{aj}(X) = g(\nabla^*_{E_{ai}} E_{aj}, X)$ and $D_{aj}(N) = g(A_{\bar{J}_c N} E_{ai}, E_{aj})$.

Proof. From (2.1), (2.3), (2.10) and (3.15) we have

$$\begin{aligned} g([E_{ai}, E_{aj}], X) &= \bar{g}(\bar{\nabla}_{E_{ai}} E_{aj}, X) - \bar{g}(\bar{\nabla}_{E_{aj}} E_{ai}, X) \\ &= -\bar{g}(\bar{\nabla}_{E_{ai}} v_j, \bar{J}_a X) + \bar{g}(\bar{\nabla}_{E_{aj}} v_i, \bar{J}_a X) \\ &= g(A_{v_j} E_{ai}, \bar{J}_a X) - g(A_{v_i} E_{aj}, \bar{J}_a X) \\ &= 0 \end{aligned} \tag{3.16}$$

for $X \in \Gamma(D_0)$. In a similar way, we get

$$\bar{g}([E_{ai}, E_{aj}], N) = \bar{g}(D^s(E_{ai}, \bar{J}_a N), v_j) - \bar{g}(D^s(E_{aj}, \bar{J}_a N), v_i). \tag{3.17}$$

On the other hand, since $E_{bj} = \bar{J}_c \bar{J}_a v_j$, from (2.1), (2.7) and (2.12) we obtain

$$g([E_{ai}, E_{bj}], X) = g(\nabla^*_{E_{ai}} E_{aj}, \bar{J}_c X) - g(\nabla^*_{E_{bj}} E_{bi}, \bar{J}_c X) \tag{3.18}$$

for $X \in \Gamma(D_0)$. In a similar way,

$$\bar{g}([E_{ai}, E_{bj}], N) = g(A_{\bar{J}_c N} E_{ai}, E_{aj}) - g(A_{\bar{J}_c N} E_{bj}, E_{bi}). \tag{3.19}$$

Thus, from (3.16), (3.17), (3.18) and (3.19), we obtain the assertion of theorem.

Now, we will investigate necessary and sufficient conditions on distributions D and D' to be parallel.

THEOREM 3.3. *Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. Then D defines a totally geodesic foliation if and only if $h^s(X, Y)$ has no components in L for $X, Y \in \Gamma(D)$.*

Proof. From (2.7) and (2.3) we obtain

$$\begin{aligned} g(\nabla_X Y, E_{ai}) &= \bar{g}(\bar{\nabla}_X Y, E_{ai}) \\ &= -\bar{g}(\bar{J}_a \bar{\nabla}_X Y, v_i) \end{aligned}$$

for $X, Y \in \Gamma(D)$. Hence we have

$$g(\nabla_X Y, E_{ai}) = g((\nabla_X \bar{J}_a) Y - \bar{\nabla}_X \bar{J}_a Y, v_i).$$

Now, by using (2.3) and (2.7) we obtain

$$g(\nabla_X Y, E_{ai}) = \bar{g}(h(X, \bar{J}_a Y), v_i)$$

which proves our assertion.

For D', we have the following theorem.

THEOREM 3.4. *Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. Then D' defines a totally geodesic foliation if and only if $A_V X$ has no components on D, where $X \in \Gamma(D')$ and $V \in \Gamma(S(TM^{\perp}))$.*

Proof. Using (2.3), (2.7), (2.10), (3.9), (3.10) and taking the tangential part we obtain

$$\begin{aligned}\nabla_{E_{ai}} E_{aj} = {} & Q_{ab}(E_{ai}) E_{bj} + Q_{ac}(E_{ai}) E_{cj} \\ & - \bar{J}_a S A_{v_j} E_{ai} - \omega_{bi}(A_{v_j} E_{ai}) E_{ci} \\ & + \omega_{ci}(A_{v_j} E_{ai}) E_{bi} + B_a \nabla^s_{E_{ai}} v_j. \end{aligned} \tag{3.20}$$

In similar way we have

$$\begin{aligned}\nabla_{E_{ai}} E_{bj} = {} & -Q_{cb}(E_{ai}) E_{cj} + Q_{ab}(E_{aa}) E_{aj} \\ & - \bar{J}_b S A_{v_j} E_{ai} + \omega_{ci}(A_{v_j} E_{ai}) E_{bi} \\ & + B_b \nabla^s_{E_{ai}} v_j, \end{aligned} \tag{3.21}$$

then proof of the theorem follows from (3.20) and (3.21).

Using Yano-Kon terminology [11] we say that screen QR-lightlike submanifold M is a lightlike product if D and D' are its totally geodesic foliations. Thus from Theorem 3.3. and Theorem 3.4. we have the following corollary.

COROLLARY 3.1. *Let M be a screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$.Then M is a lightlike product if and only if the following conditions are satisfied:*

1. *$A_V X$ has no components in D, $\forall X \in \Gamma(D'), V \in \Gamma(L)$.*
2. *$h^s(X, Y)$ has no components in L, $\forall X, Y \in \Gamma(D)$.*

In the rest of this section we consider totally umbilical screen QR-lightlike submanifolds. First we recall the definition of totally umbilical lightlike submanifolds.

DEFINITION 2. [6] The lightlike submanifold M is called a totally umbilical lightlike submanifold if $H_L \in \Gamma(ltr(TM))$, $H_S \in \Gamma(S(TM^{\perp}))$ which satisfy

$$h^l(X, Y) = g(X, Y) H_L \tag{3.22}$$

and

$$h^s(X,Y) = g(X,Y)H_S \tag{3.23}$$

for any $X, Y \in \Gamma(TM)$.

THEOREM 3.5. *Let M be a totally umbilical screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. Then the induced connection on M is a metric connection.*

Proof. It is well known that the induced connection ∇ on an r-lightlike submanifold is a metric connection if and only if h^l vanishes identically on M [5]. From (2.3), (2.7), (3.9) and taking the lightlike transversal part we obtain

$$h^l(X, \bar{J}_aY) = \bar{J}_a h^l(X,Y), \forall X, Y \in \Gamma(D_0),$$

since M is totally umbilical we get

$$g(X, \bar{J}_aY)H_L = g(X,Y)\bar{J}_a H_L.$$

Thus, interchanging role of X and Y in this equation and subtracting we have

$$g(X, \bar{J}_aY)H_L = 0,$$

hence, we derive $H_L = 0$ due to D_0 is a non-degenerate distribution. Thus we obtain $h^l = 0$.

REMARK 2. We note that above theorem is not true for any r-lightlike submanifold. Therefore it is important property of totally umbilical screen QR-lightlike submanifolds.

THEOREM 3.6. *Let M be a totally umbilical screen QR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. If $dim(L) > 1$ then M is totally geodesic.*

Proof. From the previous theorem, we have $H_L = 0$. So $H_S = 0$ is enough to show that M is totally geodesic. Using (3.13), we have

$$g(X, \bar{J}_aY)H_S = \omega_{ai}(\nabla_X Y)v_i + C_a h^s(X,Y) \tag{3.24}$$

for $X, Y \in \Gamma(D_0)$. Hence we obtain

$$2g(X, \bar{J}_aY)H_S = \omega_{ai}([X,Y])v_i. \tag{3.25}$$

For $E_{ai}, E_{aj} \in \Gamma(D')$, in a similar way, we derive

$$g(E_{ai}, E_{aj})H_S = \omega_{ai}(A_{v_j})v_i + C_a \nabla^s_{E_{ai}} v_j. \tag{3.26}$$

Now suppose $dim(L) > 1$,since L is non-degenerate, it has orthonormal basis. Thus we can choose vector fields E_{aj},E_{ai},$i \neq j$ such that they are orthogonal, then (3.26) becomes

$$\omega_{ak}(A_{v_j}E_{ai})v_k + C_a \nabla^s_{E_{ai}} v_k = 0, k \in \{1, \ldots, dim(L)\}. \tag{3.27}$$

Hence we have

$$\omega_{ak}(A_{v_j}E_{ai}) = 0.$$

Using (2.10) we obtain

$$\bar{g}(h^s(E_{ai}, E_{ak}), v_j) = 0,$$

hence we conclude

$$H_S \in \Gamma(L^{\perp}). \tag{3.28}$$

For $X, Y \in \Gamma(D_0)$, if $[X,Y] \in \Gamma(D)$, then from (3.25) we derive $H_S = 0$. If $[X,Y] \in \Gamma(D')$ then from (3.25) we obtain

$$H_S \in \Gamma(L). \tag{3.29}$$

Then considering (3.28) and (3.29) we derive $H_S = 0$, i.e, M is totally geodesic.

In the end of this section we present an example for totally umbilical screen QR-lightlike submanifolds.

EXAMPLE 3. Consider a submanifold M, in R_4^{12} with the equations:

$$x_9 = x_1 \sin\alpha - x_2 \cos\alpha \qquad x_{10} = x_1 \cos\alpha + x_2 \sin\alpha$$
$$x_{11} = x_3 \sin\alpha + x_4 \sin\alpha \qquad x_{12} = -x_3 \cos\alpha + x_4 \sin\alpha$$
$$x_5 = \sqrt{1 - x_6^2 - x_7^2 - x_8^2}.$$

The tangent bundle of M is spanned by

$$\xi_1 = \frac{\partial}{\partial x_1} + \sin\alpha \frac{\partial}{\partial x_9} + \cos\alpha \frac{\partial}{\partial x_{10}}, \quad \xi_2 = \frac{\partial}{\partial x_2} - \cos\alpha \frac{\partial}{\partial x_9} + \sin\alpha \frac{\partial}{\partial x_{10}},$$
$$\xi_3 = \frac{\partial}{\partial x_3} + \sin\alpha \frac{\partial}{\partial x_{11}} - \cos\alpha \frac{\partial}{\partial x_{12}}, \quad \xi_4 = \frac{\partial}{\partial x_4} + \sin\alpha \frac{\partial}{\partial x_{11}} + \sin\alpha \frac{\partial}{\partial x_{12}},$$
$$Z_1 = -x_6 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_6}, \qquad Z_2 = -x_7 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_7},$$
$$Z_3 = -x_8 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_8}.$$

We see that M is a 4-lightlike submanifold and $RadTM = span\{\xi_1, \xi_2, \xi_3, \xi_4\}$. It is easy to see $RadTM$ is invariant with respect to canonical complex structures $\bar{J}_1, \bar{J}_2, \bar{J}_3$. Screen transversal bundle $S(TM^{\perp})$ is spanned by

$$W = x_5 \frac{\partial}{\partial x_5} + x_6 \frac{\partial}{\partial x_6} + x_7 \frac{\partial}{\partial x_7} + x_8 \frac{\partial}{\partial x_8}.$$

By direct calculations, we have

$$U_1 = \bar{J}_1 W = Z_1 - \frac{x_8}{x_5} Z_2 + \frac{x_7}{x_5} Z_3,$$

$$U_2 = \bar{J}_2 W = \frac{x_8}{x_5} Z_1 + Z_2 - \frac{x_6}{x_5} Z_3,$$

$$U_3 = \bar{J}_3 W = \frac{-x_7}{x_5} Z_1 + \frac{x_6}{x_5} Z_2 + Z_3.$$

Hence $D' = span\{U_1, U_2, U_3\}$. Thus M is a screen QR-lightlike submanifold. On the other hand, the lightlike transversal bundle spanned by

$$N_1 = \frac{1}{2}\left\{ -\frac{\partial}{\partial x_1} + \sin\alpha \frac{\partial}{\partial x_9} + \cos\alpha \frac{\partial}{\partial x_{10}} \right\}$$

$$N_2 = \frac{1}{2}\left\{ -\frac{\partial}{\partial x_2} - \cos\alpha \frac{\partial}{\partial x_9} + \sin\alpha \frac{\partial}{\partial x_{10}} \right\}$$

$$N_3 = \frac{1}{2}\left\{ -\frac{\partial}{\partial x_3} + \sin\alpha \frac{\partial}{\partial x_{11}} - \cos\alpha \frac{\partial}{\partial x_{12}} \right\}$$

$$N_4 = \frac{1}{2}\left\{ -\frac{\partial}{\partial x_4} + \cos\alpha \frac{\partial}{\partial x_{11}} + \sin\alpha \frac{\partial}{\partial x_{12}} \right\}.$$

Hence lightlike transversal is also invariant. By direct calculations, we have

$$\bar{\nabla}_X \xi_1 = \bar{\nabla}_X \xi_2 = \bar{\nabla}_X \xi_3 = \bar{\nabla}_X \xi_4 = \bar{\nabla}_X N_1 = \bar{\nabla}_X N_2 = \bar{\nabla}_X N_3 = \bar{\nabla}_X N_4 = 0$$

for any $X \in \Gamma(TM)$. On the other hand we have

$$\bar{\nabla}_{U_1} U_1 = \bar{\nabla}_{U_2} U_2 = \bar{\nabla}_{U_3} U_3 = -W$$

$$\bar{\nabla}_{U_1} U_2 = x_8 \frac{\partial}{\partial x_5} + x_7 \frac{\partial}{\partial x_6} - x_6 \frac{\partial}{\partial x_7} - x_5 \frac{\partial}{\partial x_8}$$

$$\bar{\nabla}_{U_1} U_3 = -x_7 \frac{\partial}{\partial x_5} + x_8 \frac{\partial}{\partial x_6} + x_5 \frac{\partial}{\partial x_7} - x_6 \frac{\partial}{\partial x_8}$$

$$\bar{\nabla}_{U_2} U_3 = x_6 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_6} + x_8 \frac{\partial}{\partial x_7} - x_7 \frac{\partial}{\partial x_8}.$$

By using (2.7) we obtain

$$h^l = 0, h^s(U_1, U_2) = h^s(U_2, U_1) = h^s(U_1, U_3) = h^s(U_3, U_1) = h^s(X, \xi) = 0$$

and

$$h^s(U_1, U_1) = g(U_1, U_1) H_S$$

$$h^s(U_2, U_2) = g(U_2, U_2) H_S$$

$$h^s(U_3, U_3) = g(U_3, U_3) H_S$$

for $X \in \Gamma(TM)$, $Y \in \Gamma(RadTM)$, where $H_S = -W$, hence M is totally umbilical screen QR-lightlike submanifold.

4. Screen CR-lightlike submanifolds

In section 3, we have seen that a screen real lightlike submanifold is not a screen QR-lightlike submanifold. In this section we will introduce another class of lightlike submanifolds of an indefinite quaternion Kaehler manifold, namely, screen CR-lightlike submanifolds which include screen real lightlike submanifolds as well as quaternion lightlike submanifolds.

DEFINITION 3. Let $(M, g, S(TM))$ be a lightlike submanifold of an indefinite quaternion Kaehler manifold $(\bar{M}, \bar{g}.)$ We say that M is a screen CR-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

1. There exist real non-null distributions D_0 and D' over $S(TM)$ such that

$$S(TM) = D_0 \oplus D',\ \bar{J}_a(D_0) = D_0,\ \bar{J}_a(D') \subset S(TM^\perp),\ a = 1, 2, 3. \tag{4.1}$$

2. $RadTM$ is invariant with respect to $\bar{J}_a$,i.e., $\bar{J}_a(RadTM) = RadTM$, $a = 1, 2, 3$.

It follows that $ltr(TM)$ is also invariant with respect to $\bar{J}_a$, i.e.,

$$\bar{J}_a(ltr(TM) = ltr(TM). \tag{4.2}$$

We denote the orthogonal complementary distribution to $\bar{J}_a D'$ in $S(TM^\perp)$ by μ. We note that D_0 and μ are non-degenerate. By the definition of a screen CR-lightlike submanifold we have

$$TM = D \perp D', \tag{4.3}$$

where

$$D = RadTM \perp D_0. \tag{4.4}$$

We say that M is a proper screen CR-lightlike submanifold of $\bar{M}$ if $D_0 \neq 0$ and $D' \neq 0$. From (4.1), (4.2) and (4.4), for $X \in \Gamma(TM)$ we can write

$$\bar{J}_a X = \phi_a X + F_a X, \tag{4.5}$$

where $\phi X \in \Gamma(D)$ and $F_a X \in \Gamma(\bar{J}_a D')$. Any vector field $V \in \Gamma(S(TM^\perp))$ we put

$$\bar{J}_a V = t_a V + f_a V, \tag{4.6}$$

where $t_a V \in \Gamma(D')$ and $f_a V \in \Gamma(\mu)$.

EXAMPLE 4. Consider in R_4^{16} the submanifold M given by the equations

$$x_1 = x_{13},\ x_2 = x_{14} \qquad x_3 = x_{15},\ x_4 = x_{16}$$
$$x_{11} = \sqrt{1 - x_9^2} \quad x_{10} = constant,\ x_{12} = constant.$$

The tangent bundle of M is spanned by

$$Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{13}}, Z_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_{14}}, Z_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_{15}},$$
$$Z_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_{16}}, Z_5 = \frac{\partial}{\partial x_5}, Z_6 = \frac{\partial}{\partial x_6},$$
$$Z_7 = \frac{\partial}{\partial x_7}, Z_8 = \frac{\partial}{\partial x_8}, Z_9 = \frac{\partial}{\partial x_9} - \frac{x_9}{\sqrt{1 - x_9^2}} \frac{\partial}{\partial x_{11}}.$$

Hence M is a 4-lightlike submanifold with $RadTM = span\{Z_1, Z_2, Z_3, Z_4\}$ and it is invariant with respect to $\bar{J}_1$,$\bar{J}_2$,$\bar{J}_3$. Moreover we can see $D_0 = \{Z_5, Z_6, Z_7, Z_8\}$ is also invariant. It is easy to see $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, W_1 = \bar{J}_1 Z_9, W_2 = \bar{J}_2 Z_9, W_3 = \bar{J}_3 Z_9\}$ is linearly independent. So $span\{W_1, W_2, W_3\} = \bar{J}_a(D') = S(TM^{\perp})$. Finally we obtain the lightlike transversal bundle spanned by

$$N_1 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{13}}\right\}, \qquad N_2 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_{14}}\right\},$$
$$N_3 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_{15}}\right\}, \qquad N_4 = \frac{1}{2}\left\{-\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_{16}}\right\}.$$

Thus we conclude that M is a proper screen CR-lightlike submanifold of R_{16}^4.

PROPOSITION 4.1. *A screen CR-lightlike submanifold of an indefinite quaternion Kaehler manifold is a screen real lightlike submanifold (resp. quaternion lightlike) if and only if $D_0 = \{0\}$ (resp.$D' = \{0\}$).*

Proof. Let M be a screen real lightlike submanifold of an indefinite quaternion Kaehler manifold. Then it is clear, the radical distribution is invariant subspace. Since M is screen real lightlike submanifold we have $D_0 = \{0\}$. Conversely, let M be a screen CR-lightlike submanifold such that $D_0 = \{0\}$. Then we have $S(TM) = D'$, since $\bar{J}_a(D') \subset S(TM^{\perp})$ we obtain that M is a screen real lightlike submanifold. The other assertion of the proposition can be proved as above discussion.

REMARK 3. We note that any co-isotropic, isotropic or totally lightlike screen CR-lightlike submanifold of an indefinite quaternion Kaehler manifold is a quaternion lightlike submanifold. We consider, for example, co-isotropic submanifold M, then $TM = S(TM) \perp RadTM$ and $S(TM^{\perp}) = \{0\}$), therefore D' is undefined, so we have $TM = RadTM \perp D_0$, i.e., it is invariant with respect to $\bar{J}_a, a = 1, 2, 3$.

THEOREM 4.1. *Let M be a screen CR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. Then*

1. *D' is integrable if and only if*

$$A_{\bar{J}_a U}V = A_{\bar{J}_a V}U, \forall U, V \in \Gamma(D').$$

2. *D is integrable if and only if*

$$h^s(X, \bar{J}_a Y) = h^s(\bar{J}_a X, Y), \forall X, Y\Gamma(D).$$

Proof. From (2.3), (2.7), (2.9), (4.5), (4.6) and taking tangential parts we have

$$-A_{\bar{J}_a U}V = \phi_a \nabla_V U + t_a h^s(U, V) \tag{4.7}$$

for $U, V \in \Gamma(D')$. Hence we obtain

$$A_{\bar{J}_a U}V - A_{\bar{J}_a V}U = \phi_a[U, V],$$

thus we get the first assertion. In a similar way, by using (2.3), (2.7), (2.9), (4.5), (4.6) and taking the screen transversal parts we obtain

$$h^s(X, \bar{J}_a Y) = F_a \nabla_X Y + f_a h^s(X, Y) \tag{4.8}$$

for $X, Y \in \Gamma(D)$, hence we obtain the second assertion of theorem.

We now consider totally umbilical screen CR-lightlike submanifolds. First we have:

COROLLARY 4.1. *Let M be a totally umbilical screen CR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. Then the induced connection on M is metric connection.*

The proof is similar to that of Theorem 3.5 from section 3, so we omit it here.

LEMMA 4.1. *Let M be a totally umbilical proper screen CR-lightlike submanifold of an indefinite quaternion Kaehler manifold $\bar{M}$. Then we have*

$$H_S \in \Gamma(\bar{J}_a D'). \tag{4.9}$$

Proof. From (4.8) and (3.23), for $X = Y \in \Gamma(D_0)$ we obtain

$$F_a \nabla_X Y = 0,\ g(X, X) f_a H_S = 0.$$

Since D_0 is non-degenerate we have, at least,a spacelike or timelike vector field, thus $f_a H_S = 0$, which shows us $H_S \in \Gamma(\bar{J}_a D')$.

THEOREM 4.2. *Let M be a totally umbilical proper screen CR-lightlike of an indefinite quaternion Kaehler manifold. Then*

1. *M is totally geodesic or*
2. *the distribution D' is one dimensional.*

Proof. From (4.7), (3.23) and (2.10) we obtain

$$g(X,X)\bar{g}(H_S,\bar{J}_aY) = g(X,Y)\bar{g}(H_S,\bar{J}_aX) \tag{4.10}$$

for $X,Y \in \Gamma(D')$. From (4.9) we have

$$g(Y,Y)\bar{g}(H_S,\bar{J}_aX) = g(X,Y)\bar{g}(H_S,\bar{J}_aY). \tag{4.11}$$

Thus we have

$$\bar{g}(H_S,\bar{J}_aX) = \frac{g(X,Y)^2}{g(X,X)g(Y,Y)}\bar{g}(H_S,\bar{J}_aX). \tag{4.12}$$

Since D' and $S(TM^{\perp})$ are non-degenerate, (4.10) and (4.12) imply $H_S = 0$ or X and Y are linearly depend. Thus we have proved the theorem.

Now, we give an example for a totally umbilical screen CR-lightlike submanifold.

EXAMPLE 5. Let M be the submanifold of R^{16}_4 given in Example 4. Then we have

$$\bar{\nabla}_X Z_1 = \bar{\nabla}_X Z_2 = \bar{\nabla}_X Z_3 = \bar{\nabla}_X Z_4 = 0,$$
$$\bar{\nabla}_X Z_5 = \bar{\nabla}_X Z_6 = \bar{\nabla}_X Z_7 = \bar{\nabla}_X Z_8 = 0$$

for any $X \in \Gamma(TM)$ and

$$\bar{\nabla}_{Z_9} Z_9 = \frac{x_9}{1-x_9^2}Z_9 - \frac{1}{\sqrt{1-x_9^2}}W_2.$$

Using Gauss equation we have

$$h^s(X,Z_1) = h^s(X,Z_2) = h^s(X,Z_3) = h^s(X,Z_4) = 0,$$
$$h^s(X,Z_5) = h^s(X,Z_6) = h^s(X,Z_7) = h^s(X,Z_8) = 0$$

and

$$h^l = 0,\ h^s(Z_9,Z_9) = g(Z_9,Z_9)H_S,$$

where $H_S = -\frac{1-x_9^2}{\sqrt{1-x_9^2}}W_2$, thus M is a totally umbilical and it has a metric connection.

THEOREM 4.3. *There exist no proper totally umbilical screen CR-lightlike submanifold in positively or negatively curved indefinite quaternion Kaehler manifolds.*

Proof. We suppose that M is a proper totally umbilical screen CR-lightlike submanifold of $\bar{M}$ with $K_M(X,Y) \neq 0$ for any $X,Y \in \Gamma(TM)$. By direct calculations we have

$$\bar{R}(X,Y)\bar{J}_1Z - \bar{J}_1\,\bar{R}(X,Y)Z = \gamma(X,Y)\bar{J}_2Z - \beta(X,Y)\bar{J}_3Z \tag{4.13}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$, where $\gamma(X,Y) = dQ_{12}(X,Y) + (Q_{23} \wedge Q_{31})(X,Y)$ and $\beta(X,Y) = dQ_{31}(X,Y) + (Q_{12} \wedge Q_{23})(X,Y)$. Thus we have

$$-\bar{g}(\bar{R}(X,Y)X,Y) + \bar{g}(\bar{R}(X,Y)\bar{J}_1X, \bar{J}_1Y) = 0 \tag{4.14}$$

for $X \in \Gamma(D_0)$, $Z = \bar{J}_1X \in \Gamma(D_0)$ and $Y \in \Gamma(D')$. By using (2.18), (3.22) and (3.23) we have

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)\bar{J}_1X, \bar{J}_1Y) = \bar{g}(H_S, \bar{J}_1Y)\{&-g(\nabla_XY, \bar{J}_1X) - g(Y, \nabla_X\bar{J}_1X)\\ &+ g(\nabla_YX, \bar{J}_1X) + g(X, \nabla_Y\bar{J}_1X)\}.\end{aligned}$$

Then from (2.7) we get

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)\bar{J}_1X, \bar{J}_1Y) = \bar{g}(H_S, \bar{J}_1Y)\{&-\bar{g}(\bar{\nabla}_XY, \bar{J}_1X) - \bar{g}(Y, \bar{\nabla}_X\bar{J}_1X)\\ &+ \bar{g}(\bar{\nabla}_YX, \bar{J}_1X) + \bar{g}(X, \bar{\nabla}_Y\bar{J}_1X)\}.\end{aligned}$$

Since $\bar{\nabla}$ is a metric connection we obtain $\bar{g}(\bar{R}(X,Y)\bar{J}_1X, \bar{J}_1Y) = 0$. Then, using (4.14) we have

$$K_M(X,Y) = \bar{g}(\bar{R}(X,Y)X,Y) = 0,$$

which is a contradiction and the proof is complete.

CONCLUDING REMARKS. We note that among QR-lightlike, screen QR-lightlike and screen CR-lightlike submanifolds there exist no inclusion relations, because a real lightlike hypersurface is a QR-lightlike submanifold (See: [10]), it is not a screen QR-lightlike and a screen CR-lightlike submanifold. On the other hand, a screen real lightlike submanifold is a screen CR-lightlike submanifold, of course it is neither screen QR-lightlike nor QR-lightlike. Finally, invariant lightlike submanifolds lie in the intersection of the screen CR-lightlike and screen QR-lightlike submanifold.

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